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Nonlinear Mathematical Model of the Rotor Motion in a Thin Hydrodynamic Gap

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***Abstract*—**The article presents two mathematical models of the interaction between a rotating shaft and an incompressible fluid. The mathematical model includes both the journal bearings and the axially traversed hydrodynamic sealing gaps of hydraulic machines. A method is shown for the identification of additional effects of the fluid acting on the rotor of the machine, both for a linear and a non-linear model. The interaction is expressed by matrices of mass, stiffness and damping.

***Keywords*—**CFD modeling, hydrodynamic gap, matrices of mass, stiffness and damping, nonlinear mathematical model.

I. Mathematical Model

THE mathematical model of a journal bearing and a hydrodynamic sealing gap can be derived from the Navier Stokes equations and the continuity equation for an incompressible fluid. It is based on the interaction of a rotating shaft and a liquid. Zero pressure gradients are assumed in the radial direction. The velocity component gradients in the circumferential direction are negligible in comparison to the gradient in radial direction. The resultant force, caused by the liquid, acting on the shaft surface [1], may be then expressed as:

$$F_i = \int_S \sigma_{ij} m_j dS \,;\; \boldsymbol{F}^T = (F_1, F_2). \tag{1}$$

Assuming linearity, this force can be determined depending on the displacement vector $\boldsymbol{u}$ by (1), see [1], [2]:

$$\boldsymbol{F} = -(\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{B}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u}), \boldsymbol{u}^T = \big(x(t), y(t)\big). \tag{2}$$

The matrices of mass, stiffness, and damping are antisymmetric and it holds:

$$\boldsymbol{M} = \begin{Vmatrix} M & m \\ -m & M \end{Vmatrix} \,;\; \boldsymbol{B} = \begin{Vmatrix} B & b \\ -b & B \end{Vmatrix} \,;\; \boldsymbol{K} = \begin{Vmatrix} K & k \\ -k & K \end{Vmatrix}. \tag{3}$$

Knowing the left side of (3), for example from an experiment or a calculation using CFD methods, the effects of additional matrices **M**, **B**, **K** can be determined from (2). For this purpose, it is preferable to quote the forces in the radial and circumferential direction, see Fig. 1, so that the following applies:

$$\boldsymbol{f}^T = \big(F_r, F_\varphi\big) \,;\; \boldsymbol{f} = \boldsymbol{R}^T\boldsymbol{F}, \tag{4}$$

whereR is a matrix of elementary rotations.

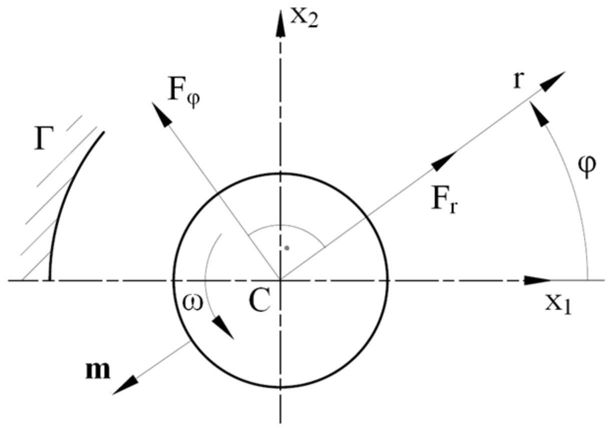

Fig. 1 Coordinate system with designation of the forces

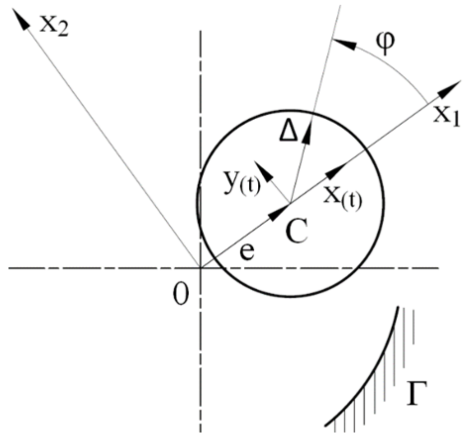

Fig. 2 Coordinate system for double displacement

Mathematical model (2) applies only to shaft vibration near the zero point, i.e. for small e, x, y. For large values of eccentricity e, high speed of the shaft and assuming that $|\Delta| \ll e$, a non-linear model can be derived in the form:

$$\boldsymbol{F} = -\left(\begin{Vmatrix} K & k \\ -k & K \end{Vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{Vmatrix} -\tilde{K}(e+x)y & -\tilde{k}y^2 \\ \tilde{k}(e+x)^2 & \tilde{K}(e+x)y \end{Vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + + \begin{Vmatrix} B & b \\ -b & B \end{Vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} + \begin{Vmatrix} -\tilde{B}y^2 & \tilde{b}(e+x)y \\ \tilde{b}(e+x) & -\tilde{B}(e+x)^2 \end{Vmatrix} \begin{vmatrix} \dot{x} \\ \dot{y} \end{vmatrix} + + \begin{Vmatrix} M & 0 \\ 0 & M \end{Vmatrix} \begin{vmatrix} \ddot{x} \\ \ddot{y} \end{vmatrix}\right). \tag{5}$$

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The mathematical model can be applied both to journal bearings that are not traversed by the fluid in the axial direction and to hydrodynamic sealing gaps axially traversed by the fluid, depending on the pressure gradient. The shape of the lining (stator section) of both of these elements is shown for example in Figs. 3 and 4.

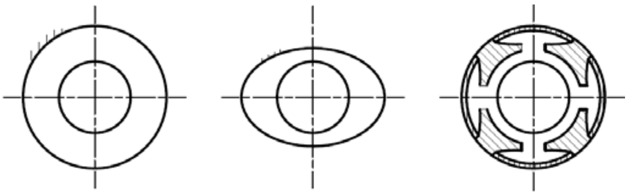

Fig. 3 Examples of the shape of the journal bearing lining

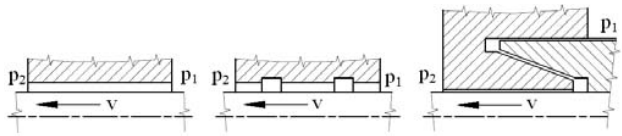

Fig. 4 Examples of the hydrodynamic sealing gaps

II. Identification of the Additional Effect Matrices

A. Linear model

The identification is based on the fact that the kinematics of the motion is prescribed for the center of the shaft. Contrary to [1], [2] is the mathematical model widen of the effect of eccentricity *e* [5]. The principle of the identification is simple when the shaft performs a rotary motion with angular speed *ω* around the center *C*, see

Fig. 1 together with precession angular speed *Ω*, specified by relations:

$$\varphi = \Omega t;\ \ \boldsymbol{v}^T = (cos\varphi, sin\varphi);\ \ \boldsymbol{u} = \Delta\boldsymbol{v};\ \ \boldsymbol{u}^T = (x, y). \tag{6}$$

If the following matrix $\boldsymbol{\Gamma}$ is used then the mathematical model (2) can be written in the form; see [5], [7]:

$$\boldsymbol{\Gamma} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix};\ \ \boldsymbol{F} = (\Omega^2\boldsymbol{M} - \boldsymbol{K} - \Omega\boldsymbol{B}\boldsymbol{\Gamma})\boldsymbol{u}. \tag{7}$$

When vector F is expressed depending on f, we can write:

$$\boldsymbol{F} = \boldsymbol{P}\boldsymbol{v}\,;\ \ \boldsymbol{P} = \begin{Vmatrix} F_r & -F_\varphi \\ F_\varphi & F_r \end{Vmatrix}. \tag{8}$$

Considering (6) and substituting (8) into (7) we obtain (9):

$$\boldsymbol{P} = (\Omega^2\boldsymbol{M} - \boldsymbol{K} - \Omega\boldsymbol{B}\boldsymbol{\Gamma})\Delta. \tag{9}$$

From here, considering the structure of matrices it is evident that matrices **M**, **B**, **K** are antisymmetric so the term (3) holds and it can be written:

$$F_r = -(K + \Omega b - \Omega^2\mathrm{M}) \tag{10}$$

$$F_\varphi = -(-k + \Omega B - \Omega^2\mathrm{m}). \tag{11}$$

If we know, e.g. from experiments or CFD calculations,the dependence of $F_r(\Omega)$, $F_\varphi(\Omega)$ for the selected values of *Ω*, K, **B**, **M**, **k**, **b**, **m** may be determined from terms (10), (11) based on regression analysis.

An Example - Consider the shape of hydrodynamic sealing gap [4], for example, according to Fig. 5. Calculations [3] are performed using software Fluent, for the following entries:

Physical properties of water are $= 1000\,\mathrm{kg/m^3}$,$\nu = 1 \cdot 10^{-6}\mathrm{m^2/s}$, the pressure gradient between the input and the output is $\Delta\mathrm{p} = 100000\,\mathrm{Pa}$ The results of the calculation for the given parameters in Fig. 5 are shown in Fig. 6.

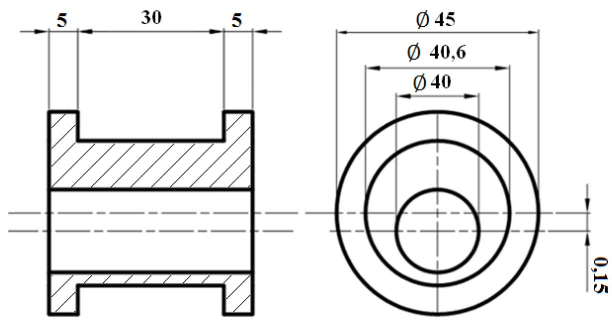

Fig. 5 The geometry of the solved hydrodynamic sealing gap (all dimensions are in mm)

It is obvious from the progression of the forces that in this case it is possible to describe the mathematical model of the hydrodynamic gap with a linear model. The calculations were done for four values of the precession speed *Ω*. The progression of radial $F_r(\Omega)$ and tangential $F_\varphi(\Omega)$ forces was approximated by regression function and the values of **K**, **k**, **B**, **b**, **M**, **m**, listed in Table I were determined from (10) and (11).

TABLE I
Values of K, k, B, b, M, m

ω [rad/s]	**K**	**k**	**B**	**b**	**M**	**m**
62,8319	59057,3	3804,6	457,3	52,6	2,00e-1	0

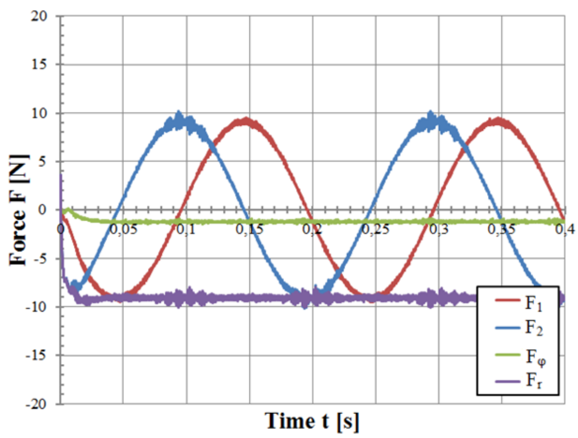

Fig. 6 Progression of forces for variant: ω=62[rad/s] , Ω=31 [rad/s] (e=0,15 ,Δ=0)

B. Non-linear Model

For relatively large eccentricities e and deflections Δ, the nonlinear terms of (5) can be applied. Identification of unknown stiffness, mass and damping is based on the given kinematics of the rotor motion according to the regulation (6). Multiplying (5) by the transformation matrix T and introducing the matrices$\mathrm{U}, \mathrm{A}, \Theta, \mathrm{k_L}, \mathrm{k_N}$, then (15) can be written for Θ, from which both linear and non-linear additional effects can be identified.

$$\boldsymbol{T} = \begin{Vmatrix} x & y \\ -y & x \end{Vmatrix}; \ \boldsymbol{U} = \begin{Vmatrix} \Delta^2 & 0 \\ 0 & -\Delta^2 \end{Vmatrix} \tag{12}$$

$$\boldsymbol{A} = \begin{Vmatrix} (e+x)y^3 - (e+x)yx^2 & -xy^3 + (e+x)xy \\ (e+x)xy^2 + (e+x)y^2x & y^4 + (e+x)^2x^2 \end{Vmatrix} \tag{13}$$

$$\boldsymbol{\Theta} = \Delta \begin{vmatrix} F_r \\ F_\varphi \end{vmatrix}; \quad \boldsymbol{k}_L = \begin{vmatrix} K - \Omega^2 M + \Omega b \\ k - \Omega^2 m - \Omega B \end{vmatrix}; \ \boldsymbol{k}_N = \begin{vmatrix} \tilde{K} - \Omega\tilde{b} \\ \tilde{k} - \Omega\tilde{B} \end{vmatrix} \tag{14}$$

$$\boldsymbol{\Theta} = -(\boldsymbol{U}\boldsymbol{k}_L + \boldsymbol{A}\boldsymbol{k}_N). \tag{15}$$

The matrix $\boldsymbol{A}$ consists of elements dependent on time and Δ, e, Ω. The basic period of rotor motion is $T_0 = 2\pi/\Omega$. For the precession motion of the rotor when $e = 0$ is assumed, the forcesF_r and $F_\varphi$are dependent on the quadruple of the fundamental frequency with period T_0. Under this assumption, vectors $\boldsymbol{k}_L, \boldsymbol{k}_N$are identified.

The oscillation period of $\mathrm{F_r}$ and $\mathrm{F_t}$ will thus be $\mathrm{T} = (1/4)\mathrm{T_0}$, see Fig. 7. For $\mathrm{e} \neq 0$, this period will be equal to $\frac{1}{2}\mathrm{T_0}$. The results are shown in Figs. 9, 10.

Vector $\boldsymbol{k}_L$corresponds to the linear model; vector $\boldsymbol{k}_N$ is associated with nonlinearity. These vectors can be determined from (15) for the two known values Δ_1, Δ_2using the same selected Ω. If we decompose F_r and F_φ to the stationary (S) and non-stationary (N) part, see Figs. 7-10, it can be written:

$$F_r = F_{rS} + F_{rN} f_r(t)\,; \ F_t = F_{\varphi S} + F_{yN} f_\varphi(t) \tag{16}$$

$$\boldsymbol{k}_L = \frac{\Delta_1}{\Delta_2^2 - \Delta_1^2} \begin{vmatrix} F_{rS1} \\ -F_{\varphi S1} \end{vmatrix} + \frac{\Delta_1 \Delta_2}{\Delta_2^2 - \Delta_1^2} \begin{vmatrix} F_{rS2} \\ -F_{\varphi S2} \end{vmatrix} \tag{17}$$

$$\boldsymbol{k}_N = \frac{1}{\Delta_1(\Delta_2^2 - \Delta_1^2)} \begin{vmatrix} 3F_{rN1} - F_{\varphi S1} \\ -F_{rN1} - F_{\varphi S1} \end{vmatrix} - \frac{1}{\Delta_2(\Delta_2^2 - \Delta_1^2)} \begin{vmatrix} 3F_{rN2} - F_{\varphi S2} \\ -F_{rN2} - F_{\varphi S2} \end{vmatrix}. \tag{18}$$

The expression (18) is determined for different Ω and using regression we will obtain $\mathrm{M}, \mathrm{m}, \mathrm{B}, \mathrm{b}, \mathrm{K}, \mathrm{k}, \tilde{\mathrm{B}}, \tilde{\mathrm{b}}, \tilde{\mathrm{K}}, \tilde{\mathrm{k}}$.Noteworthy is the progression of forces $\mathrm{F_r}$ and $\mathrm{F_\varphi}$in Figs. 9, 10.

Fig. 9 shows the progress of the forces in the hydrodynamic sealing gap of the hydraulic machines, traversed by the liquid in the rotation axis direction due to the pressure gradient. In terms of the mathematical model, the frequency that corresponds to a double of the fundamental frequency of the shaft rotation is important. Fluid oscillations at this frequency are significantly dampened by the axial flow and thus the pressure gradient. Fig. 9 shows the force progression in the journal bearing, without the influence of the pressure gradient. Their frequency again corresponds to a double of the fundamental frequency, but the relative amplitude of the vibrations due to F_1, F_2 is significantly higher.

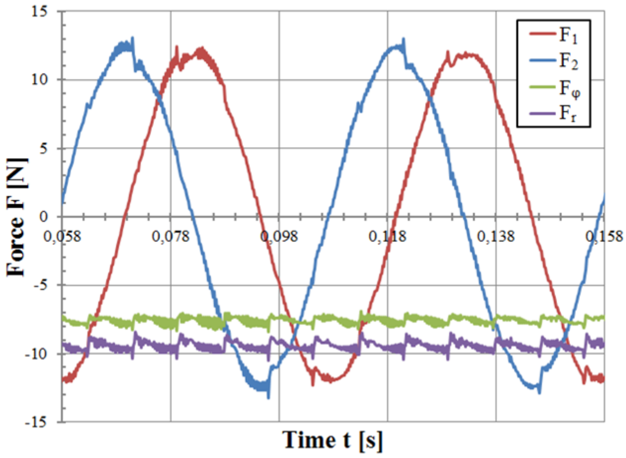

Fig. 7 Progression of forces for variant: Δp = 100000 Pa, single deviation ω=62[rad/s] – Ω=125 [rad/s] (e=0,15)

The pressure gradient significantly influences - increases - the value of K, and improves the conditions of the rotor stability.

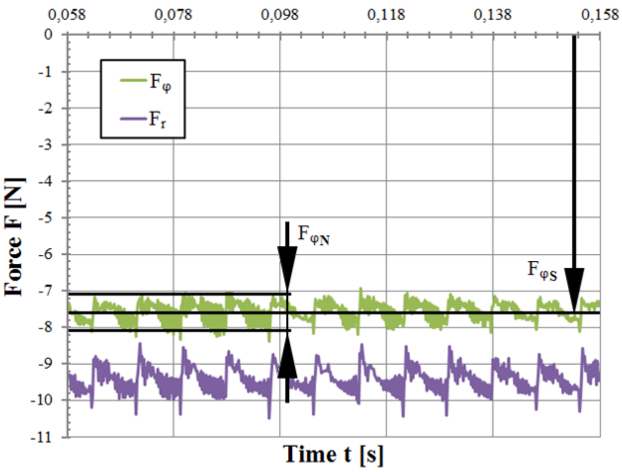

Fig. 8 Progression of forces for variant: Δp = 100000 Pa, ω=62[rad/s] – Ω=125 [rad/s] (e=0,15) only average values

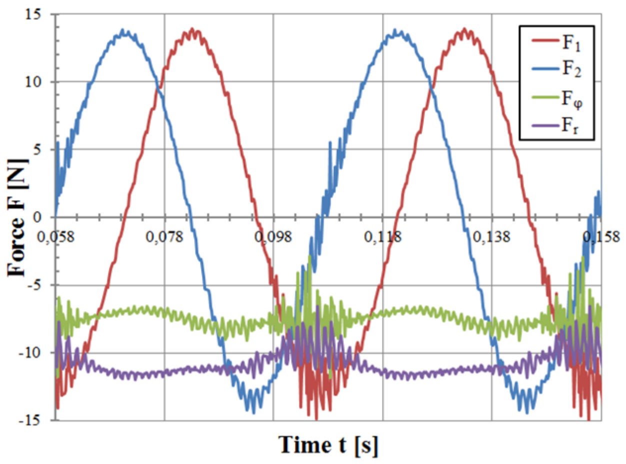

Fig. 9 Progression of forces for variant: Δp = 100000 Pa, ω=62[rad/s] – Ω=125 [rad/s] (e=0,15Δ=0,05)

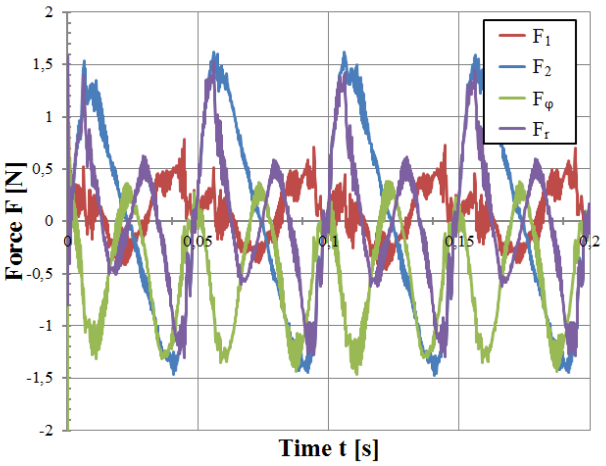

Fig. 10 Progression of forces for variant: Δp = 0 Pa, ω=62[rad/s] – Ω=125 [rad/s] (e=0,15 Δ=0,05)

III. Conclusion

Based on the analytical solution of flow in the journal bearing and generalization of the Reynolds equation [5], [6] a nonlinear model of the interaction between the rotating shaft and the liquid can be defined. This interaction is solved for the flow in hydrodynamic sealing gaps, even considering the pressure gradient. Therefore the solution includes both journal bearings and hydrodynamic gaps in hydraulic machines. This theory was widen of the effect of the hydrophobic surface of the journal bearing lining [9], and the effects of magnetic fields [8]. The mathematical model is defined by:

$$\boldsymbol{F} = -\left(\left\| \begin{matrix} K & k \\ -k & K \end{matrix} \right\| \left| \begin{matrix} x \\ y \end{matrix} \right| + \left\| \begin{matrix} -\tilde{K}(e+x)y & -\tilde{k}y^2 \\ \tilde{k}(e+x)^2 & \tilde{K}(e+x)y \end{matrix} \right\| \left| \begin{matrix} x \\ y \end{matrix} \right| + + \left\| \begin{matrix} B & b \\ -b & B \end{matrix} \right\| \left| \begin{matrix} x \\ y \end{matrix} \right| + \left\| \begin{matrix} -\tilde{B}y^2 & \tilde{b}(e+x)y \\ \tilde{b}(e+x) & -\tilde{B}(e+x)^2 \end{matrix} \right\| \left| \begin{matrix} \dot{x} \\ \dot{y} \end{matrix} \right| + + \left\| \begin{matrix} M & 0 \\ 0 & M \end{matrix} \right\| \left| \begin{matrix} \ddot{x} \\ \ddot{y} \end{matrix} \right| \right). \qquad (19)$$

Proposed within this work is the identification of additional effects from liquids, which are expressed by matrices of mass, damping and stiffness. Matrix elements are determined based on the selected kinematics of the rotor motion by (6). During solution, it is necessary that the size of the time step relative to the geometric configuration is chosen carefully. From the computational modeling, a general conclusion can be declared: The value of the pressure gradient increases the diagonal elements of the matrix of stiffness and has a beneficial effect on rotor stability. For nonzero values of e, the time response of pressure and velocity fields of the liquid is double the value of the angular velocity of precession.

Appendix - Signification

$\boldsymbol{F}$	force
$\boldsymbol{M}, \boldsymbol{B}, \boldsymbol{K}$	matrices of mass, damping and stiffness
$\boldsymbol{u}$	displacement vector
m	outer normal vector
ω	angular velocity of the rotor
Ω	angular velocity of the rotor precession
$\boldsymbol{P}$	pressure
ρ	density
ν	kinematic viscosity
S	area
e	eccentricity
t	time

Acknowledgment

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