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Cyclostationary Gaussian Linearization for Analyzing Nonlinear System Response under Sinusoidal Signal and White Noise Excitation

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***Abstract*—**A cyclostationary Gaussian linearization method is formulated for investigating the time average response of nonlinear system under sinusoidal signal and white noise excitation. The quantitative measure of cyclostationary mean, variance, spectrum of mean amplitude, and mean power spectral density of noise are analyzed. The qualitative response behavior of stochastic jump and bifurcation are investigated. The validity of the present approach in predicting the quantitative and qualitative statistical responses is supported by utilizing Monte Carlo simulations. The present analysis without imposing restrictive analytical conditions can be directly derived by solving non-linear algebraic equations. The analytical solution gives reliable quantitative and qualitative prediction of mean and noise response for the Duffing system subjected to both sinusoidal signal and white noise excitation.

***Keywords*—**Cyclostationary, Duffing system, Gaussian linearization, sinusoidal signal and white noise.

I. Introduction

THE investigation of the dynamic behavior of stochastic nonlinear systems has attracted numerous researchers in different fields of science and engineering [1], [2]. For a nonlinear system subjected to the combined sinusoidal signal and white noise excitation, the long-time asymptotic statistics of stable response is cyclostationary [2]. A cyclostationary process is a non-stationary process; however, the statistical properties are periodic in time. Cyclostationary processes have many important applications in systems science and engineering, for example, response of rotating machines subjected to randomly fluctuating flow velocity [3], floating crane oscillation subjected to sea wave excitation [4], and signal processing in engineering systems [5]. For a nonlinear system subjected to both sinusoidal signal and white noise excitation, in principle, the associated cyclostationary density response can be obtained by solving the Fokker-Planck-Kolmogorov equation. However, in practice, no known exact density has been derived so far for the dynamic response. Therefore, for finding the non-stationary solution, it strongly depends on approximate methods and numerical methods. There are several approximate analytical methods including Gaussian linearization (Closure) method [4], [6]-[9], perturbation method [10], [11], stochastic averaging method [12]-[14], eigenfunction expansion method [1], etc., which have developed and extended for obtaining the response behavior. In numerical methods, there are several schemes including Monte Carlo simulation, Cell mapping methods, path integral method, etc [15]-[17]. A literature survey on the research problem shows that the existing analytical methods, mostly applied to Duffing oscillator, usually require restrictive conditions on the order of magnitude of parameters and/or excitation in applications. For example, stochastic averaging method is usually applied to lightly damped system. A second-order closure method is developed for the random response which is small compared with the mean response [9]. Adiabatic approximation method is developed for density response under slowly external sinusoidal excitation [2]. The restrictive conditions make the publishing results of more interest to qualitative interpretation of physical phenomenon than quantitative engineering applications.

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In engineering applications, Gaussian linearization method, among the approximation methods, has been the mostly employed for analyzing the response behavior of general non-linear stochastic systems [18], [19]. Despite of some subtle differences [19], stochastic linearization, statistical linearization, equivalent linearization, or stochastic equivalent linearization have been used in agreement as a standard method to utilize Gaussian density in analyzing statistical response of non-linear dynamic systems subjected to stochastic external excitation. Recently, a two-stage optimal Gaussian linearization method to incorporate the merits of standard method and SPEC-alternative was proposed [19]. In analyzing variance response of different nonlinear oscillators by the improved Gaussian linearization method, the improvement of accuracy has been investigated. For the Duffing oscillator subjected to sinusoidal signal and stochastic excitation, the standard Gaussian linearization approach has been extended for analyzing statistical response [6]-[9]. By employing the Gaussian linearization method, the sinusoidal mean and time-varying variance response can be obtained numerically by solving differential equations. However, the accuracy in the analyzing of long-time asymptotic cyclostationary response has not been focused and investigated. Moreover, an effective analytical method has not been developed for analyzing certain important response properties such as stochastic jump and bifurcation.

In this paper, an extension of Gaussian linearization for the prediction of cyclostationary response of nonlinear system is proposed. Firstly, a Gaussian linearization model is derived for nonlinear system. Next, a cyclostationary formulation of the

Gaussian linearization model is developed. Then, time-invariant algebraic equations are derived for the time-domain cyclostationary formulation. For analyzing spectrum response, the frequency-domain formulations of sinusoidal signal and noise responses are derived. Duffing system is employed for the numerical comparison of the analytical solution and Monte Carlo simulation. Finally, the applications and performance of the present approach are concluded.

II. Gaussian Linearization of Nonlinear System

Consider a stable nonlinear system under sinusoidal signal and white noise excitation is given by;

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}x_2\\ -b_1x_1-b^Nx_1^n-b_2x_2\end{bmatrix}+\begin{bmatrix}0\\ a\end{bmatrix}\sin(\omega_0 t)+\begin{bmatrix}0\\ 1\end{bmatrix}w(t), \tag{1a}$$

where b_1, b_2, and b^N are constants, n= 3,5,7.., and $w(t)$ is a zero-mean Gaussian white noise process with intensity

$$E[w(t)w(s)]=2q\delta(t-s). \tag{1b}$$

For the application of Gaussian linearization on the nonlinear system, the evaluation of expectation by utilizing Gaussian density is defined as $E_G[\cdot]$. The mean and covariance of states, utilizing Gaussian density, can be expressed, respectively, as $m_i(t)=E_G[x_i]$ and $h_{ij}(t)=E_G[(x_i-m_i(t))(x_j-m_j(t))]$, for i=1, 2. In employing the standard Gaussian linearization approach, it is noted that the Gaussian density is assumed to be independent of the linearization coefficients. By following the standard Gaussian linearization approach [18], [19], the nonlinear function in (1a) can be approximated by a linear function through the minimization of mean-square error to yield;

$$x_1^n=c(t)+d(t)(x_1-m_1(t)), \tag{2a}$$

where

$$c(t)=E_G[x_1^n] \tag{2b}$$

and

$$d(t)=nE_G[x_1^{n-1}]. \tag{2c}$$

Thus, the Gaussian linearization model of (1) is derived to yield;

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -b_1-b^N(nE_G[x_1^{n-1}]) & -b_2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ -b^NE_G[x_1^n]+b^N(nE_G[x_1^{n-1}])m_1(t)\end{bmatrix}+\begin{bmatrix}0\\ a\end{bmatrix}\sin(\omega_0 t)+\begin{bmatrix}0\\ 1\end{bmatrix}w(t). \tag{3}$$

By performing ensemble average of (3), a mean propagation equation can be derived as;

$$\begin{bmatrix}\dot{m}_1\\ \dot{m}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -b_1 & -b_2\end{bmatrix}\begin{bmatrix}m_1\\ m_2\end{bmatrix}+\begin{bmatrix}0\\ -b^NE_G[x_1^n]\end{bmatrix}+\begin{bmatrix}0\\ a\end{bmatrix}\sin(\omega_0 t). \tag{4}$$

By subtracting (4) from (3) and defining $x_{n,i}=x_i-m_i$, a linearization model of noise response is given as;

$$\begin{bmatrix}\dot{x}_{n,1}\\ \dot{x}_{n,2}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -b_1-b^N(nE_G[x_1^{n-1}]) & -b_2\end{bmatrix}\begin{bmatrix}x_{n,1}\\ x_{n,2}\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}w(t). \tag{5}$$

By forming a covariance matrix and utilizing (5), a covariance propagation equation is derived to yield;

$$\begin{aligned}\dot{h}_{11}&=2h_{12}(t),\\ \dot{h}_{12}&=h_{22}(t)+h_{11}(-b_1-b^N(nE_G[x_1^{n-1}]))-b_2h_{12}(t),\\ \dot{h}_{22}&=2h_{12}(-b_1-b^N(nE_G[x_1^{n-1}]))-2b_2h_{22}(t)+2q.\end{aligned} \tag{6}$$

For the nonlinear system, the Gaussian linearization model has been derived as given by (3), (4), and (6). The non-stationary mean and covariance response can be obtained by numerical solution of the simultaneous ordinary differential equations. In order to investigate the important behavior of cyclostationary response, one needs to derive a cyclostationary formulation of Gaussian linearization model.

III. Cyclostationary Formulation

The response of the nonlinear system in general includes subharmic and superharmonic oscillations. For obtaining the cyclostationary formulation, the dominant cyclostationary response of states is assumed to be a sinusoidal function. Thus, the dominant mean response can be expressed as a sinusoidal function of input signal frequency,

$$m_i(t)=m_i^0\sin(w_0t+q_i), \tag{7}$$

where m_i^0 are amplitudes of means and q_i are phases of means. For the cyclostationary response of covariance, the $h_{ij}(t)$ can be expressed as a combination of mean of covariance and residue oscillation,

$$h_{ij}(t)=\langle h_{ij}(t)\rangle+h_{ij}^r(t), \tag{8}$$

where $\langle\rangle$ is a time average operation. By assuming that the amplitude of residue oscillation is smaller than the mean of variance, the cyclostationary covariance $h_{ij}(t)$ can be approximated as $\langle h_{ij}(t)\rangle$.

In order to specify cyclostationary derivations, the evaluation of expected value $E_G[\cdot]$ is denoted as $E_{Gc-s}[\cdot]$. By utilizing (7) in (4), the cyclostationary mean response is derived to yield;

$$\omega_0\begin{bmatrix} m_1^0\cos(\omega_0 t+\theta_1) \\ m_2^0\cos(\omega_0 t+\theta_2)\end{bmatrix}=\begin{bmatrix} m_2^0\sin(\omega_0 t+\theta_2) \\ -b_1 m_1^0\sin(\omega_0 t+\theta_1)-b_2 m_2^0\sin(\omega_0 t+\theta_2)\end{bmatrix}+\begin{bmatrix} 0 \\ -b^N E_{Gc-s}[x_1^n]+a\sin(\omega_0 t)\end{bmatrix}. \quad (9)$$

The formulation of time average of cyclostationary covariance response can be derived with notation $\langle h_{ii}(t)\rangle=\bar{h}_i$ and with $\bar{h}_{12}=\bar{h}_{21}=0$ in (6) to yield;

$$\begin{aligned}&\bar{h}_2-b_1\bar{h}_1+\bar{h}_1(-b^N(nE_{Gc-s}[x_1^{n-1}]))=0\\&-2b_2\bar{h}_2+2q=0.\end{aligned} \quad (10)$$

For deriving explicit formulation, without loss of generality, n is assigned to 3 to form a Duffing system. The cyclostationary mean response of (9) is simplified to a scalar equation as;

$$\begin{aligned}-\omega_0^2 m_1^0\sin(\omega_0 t+\theta_1)=-b_1 m_1^0\sin(\omega_0 t+\theta_1)-b_2\omega_0 m_1^0\cos(\omega_0 t+\theta_1)-\\ b^N((m_1^0\sin(\omega_0 t+\theta_1))^3+3m_1^0\sin(\omega_0 t+\theta_1)h_1)+a\sin\omega_0 t.\end{aligned} \quad (11)$$

By utilizing (7) in (10), a nonlinear algebraic equation of variance response is obtained as;

$$q/b_2-(b_1+b^N(3((m_1^0\sin(\omega_0 t+\theta_1))^2+\bar{h}_1))\bar{h}_1=0. \quad (12)$$

The cyclostationary formulations of mean response (11) and variance response (12) are derived as time-varying algebraic equations. For the effective analysis of cyclostationary response, one needs to further derive time-invariant formulations of mean and variance response.

IV. Time Average of Cyclostationary Response

For the cyclostationary formulations of (11) and (12), these formulations include functions of $\sin(w_0 t+q_1)$ and $\cos(w_0 t+q_1)$. On multiplying (11) by $\sin(\omega_0 t+\theta_1)$ and $\cos(\omega_0 t+\theta_1)$ and carrying out the time average over a finite period, two time-invariant algebraic equations are derived, respectively, to yield;

$$-\frac{1}{2}\omega_0^2 m_1^0=\frac{1}{2}(-b_1-3b^N\bar{h}_1)(m_1^0)-\frac{3}{8}b^N(m_1^0)^3+\frac{a}{2}\cos\theta_1, \quad (13)$$

$$0=-\frac{1}{2}b_2 m_1^0\omega_0-\frac{a}{2}\sin\theta_1. \quad (14)$$

From (13) and (14), the amplitude gain and phase angle are obtained as;

$$(\frac{m_1^0}{a})^2=\frac{1}{[(-\omega_0^2+b_1+3b^N\bar{h}_1)+3b^N(m_1^0)^2/4]^2+(b_2\omega_0)^2}, \quad (15)$$

$$\theta_1=\tan^{-1}(\frac{b_2\omega_0}{\omega_0^2+(-b_1-3b^N\bar{h}_1)-\frac{3}{4}b^N(m_1^0)^2}). \quad (16)$$

By carrying out the time average of variance equation (12) over a finite period, the amplitude response is derived to yield;

$$(m_1^0)^2=\frac{2q/b_2-2b_1\bar{h}_1-6b^N\bar{h}_1^2}{3b^N\bar{h}_1}. \quad (17)$$

From (15) and (17), two unknown m_1^0 and $\bar{h}_1$ can be solved and consequently, θ_1 can be derived from (16).

The dominant mean amplitude and phase of sinusoidal response and the time average response of cyclostationary variance have been derived from time-domain formulation. The cyclostationary response in frequency domain will be derived in the following section.

V. Frequency-Domain Formulation

For the investigation of frequency response, the response of noise spectrum will be first derived. From the noise propagation equation of (5), a scalar form of input-output model for n=3 can be expressed as;

$$\ddot{x}_n+b_2\dot{x}_n+b_1 x_n+b^N(3h_{11}(t)+3m_1(t)^2)x_n=w(t). \quad (18)$$

By employing $h_{11}(t)=\bar{h}_1$, $m_1(t)=m_1^0\sin(\omega_0 t+\phi_1)$ and substituting them into (18), one has;

$$\ddot{x}_n+b_2\dot{x}_n+b_1 x_n+b^N(3\bar{h}_1+3(m_1^0\sin(\omega_0 t+\theta_1))^2)x_n=w(t). \quad (19)$$

For deriving a frequency response formulation of input frequency, the frequency of input sinusoidal signal ω_0 is replaced by a variable ω in (19). By taking the time average of $(\sin(\omega t+\theta_1))^2$ over a finite period and through Fourier transform, the response of power spectral density is obtained as;

$$\langle S_x(\omega)\rangle=\frac{2q}{(b_2\omega)^2+(b_1+3b^N\bar{h}_1+3b^N(m_1^0)^2/2-\omega^2)^2}. \quad (20)$$

The response spectrum of (20) is considered as the time average of power spectral density. The mathematical power spectral density $\langle S_x(\omega)\rangle$ is equivalent to physical power spectral density $\langle G_x(\omega)\rangle$ by utilizing $\langle G_x(\omega)\rangle=2\langle S_x(\omega)\rangle$, for $\omega\geq 0$. By substituting (17) into (20), the time average response of physical power spectral density of noise is expresses as;

$$\langle G_x(\omega)\rangle=2\frac{2q}{(b_2\omega)^2+(q/b_2\bar{h}_1-\omega^2)^2}. \quad (21)$$

Here, it is observed that if the sinusoidal signal is absent, the $\bar{h}_1$ in (21) is obtained from (12) by assigning $m_1^0=0$. Then, the response spectrum of (21) reduces to the same result as obtained by utilizing the standard Gaussian linearization. By taking the derivative of (21) with respect to ω^2 and setting it to zero, one has peak frequency:

$$\omega_{p,n}^2 = \frac{q}{b_2(\bar{h}_1)_{p,n}} - \frac{b_2^2}{2}. \tag{22}$$

The requirement of positive value in the right hand side of (22) gives the existence condition of peak in the response of noise spectrum. From (22), it is observed that the peak frequency is a function of $(\bar{h}_1)_{p,n}$. The $(\bar{h}_1)_{p,n}$ can be derived by substituting (17) into (15) and utilizing $\omega_{p,n}^2$ for ω_0^2 and $(\bar{h}_1)_{p,n}$ for $\bar{h}_1$.

For deriving the frequency response of mean amplitude, the frequency of input sinusoidal signal ω_0 is replaced by a variable ω and (15) is expanded to give a polynomial of ω as;

$$\frac{1}{2}\omega^4 - z_1\omega^2 + z_2 = 0, \tag{23a}$$

where;

$$z_1(m_1^0, \bar{h}_1) = \frac{1}{2}(\frac{3}{2}b^N(m_1^0)^2 + 2(b_1 + 3b^N\bar{h}_1) - b_2^2), \tag{23b}$$

$$z_2(m_1^0, \bar{h}_1) = \frac{1}{2}(\frac{9}{16}(b^N)^2 a^4 + \frac{3}{2}b^N(b_1 + 3b^N\bar{h}_1)a^2 + (b_1 + 3b^N\bar{h}_1)^2) - \frac{1}{2}(\frac{a}{m_1^0})^2. \tag{23c}$$

From (23a), one derives:

$$\omega^2 = z_1 \pm \sqrt{z_1^{\,2} - 2z_2}. \tag{24}$$

Equation (24) gives the relationship between mean amplitude and input frequency. Here, it is noted that (24) reduces to the same result as obtained by utilizing harmonic balance when the noise excitation is absent, i.e. $\bar{h}_1 = 0$. For investigating the behavior of peak response, the peak frequency and the corresponding peak of mean amplitude can be derived from (24). By setting the square root in the right hand side of (24) to zero, one has:

$$\omega_p^{\,2} = z_1((m_1^0)_p, (\bar{h}_1)_p), \tag{25a}$$

$$(z_1((m_1^0)_p, (\bar{h}_1)_p))^2 - 2z_2((m_1^0)_p, (\bar{h}_1)_p) = 0. \tag{25b}$$

By substituting (17) into (25a), one derives:

$$\omega_p^2 = \frac{b_1}{2} + \frac{3}{2}b^N(\bar{h}_1)_p + \frac{q}{2b_2(\bar{h}_1)_p} - \frac{b_2^2}{2}. \tag{26}$$

The requirement of positive value in the right hand side of (26) gives the condition of existence of peak in the response of signal spectrum. The peak of mean amplitude can be derived by substituting (17) into (25b) to yield:

$$(m_1^0)_p^2 = \frac{(b_2^4 - 4(b_1 + 3b^N(\bar{h}_1)_p b_2^2) + \sqrt{(b_2^4 - 4(b_1 + 3b^N(\bar{h}_1)_p b_2^2)^2 + 48b^N b_2^2 a^2}}{6b^N b_2^2}. \tag{27}$$

It is observed from (26) and (27) that the peak frequency and the corresponding peak of mean amplitude are functions of $(\bar{h}_1)_p$. The $(\bar{h}_1)_p$ can be solved by substituting (17) into (15) and utilizing ω_p^2 for ω_0^2 and $(\bar{h}_1)_{p,n}$ for $\bar{h}_1$.

The formulations of frequency response of sinusoidal signal and white-noise excitation have been derived. The existence conditions and the corresponding frequencies in the spectra of peak amplitude of mean and noise response are also derived. In the next section, the validity in the application of the present time-domain and frequency-domain formulations will be numerically simulated.

VI. Numerical Simulations

Since it is lack of exact solution for the nonlinear response, Monte Carlo simulations are employed for validation. In the numerical results, only the response of Duffing system (n= 3) is simulated. The validity of utilizing the Gaussian linearization method in analyzing the cyclostationary response of the Duffing system under sinusoidal signal and white noise excitation is first investigated. By choosing system parameters $b_1 = 1$, $b_2 = 1$, $b^N = 1$ to form a Duffing system, and assigning input signal amplitude $a = 1$, input signal frequency 0.2 Hz and input noise intensity $2q$=5, the cyclostationary mean and variance response, respectively, by employing the Gaussian linearization of (3), (4), and (6) and Monte Carlo method are simulated as shown in Figs. 1 and 2.

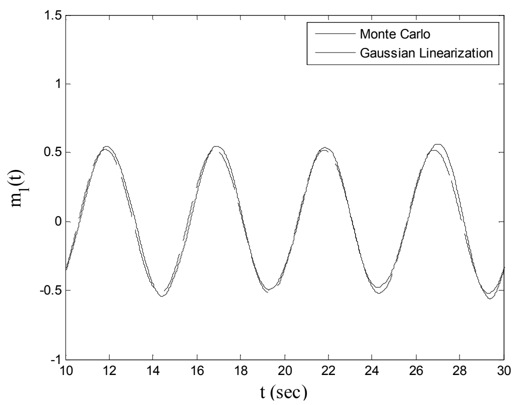

Fig. 1 Cyclostationary mean response of state x_1

In the Monte Carlo simulation, the sample response is passed through a narrow band filter which is designed with central frequency of sinusoidal input frequency and quality factor of 10. From Fig. 1, it is shown that the Gaussian linearization method gives very accurate cyclostationary mean response. Fig. 2 reveals that the response of variance includes the components of twice signal frequency and other harmonic frequencies. By employing the Gaussian linearization for variance response, the predicted cyclostationary variance is underestimated. The error of time average of variance response is about 9.75%. From the results of Gaussian linearization, the variance response can be expressed as $h_{11}(t) = 0.722 + 0.044\sin(2w_0 t + f)$. The sinusoidal amplitude of variance is 6% of the time average of the variance response. As a result, the variance response of Fig. 2 supports the analytical assumption of utilizing $h_{11}(t) = \langle h_{11}(t) \rangle = \bar{h}_1$ in the present formulation.

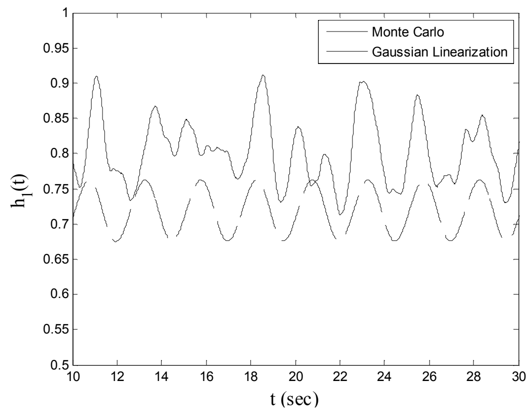

Fig. 2 Cyclostationary variance response of state x_1

Next, the applications of the present approach for analyzing the time-domain cyclostationary response are investigated. By employing the same parameters as in simulating Figs. 1 and 2 but varying input noise intensities, the amplitude of mean response at input sinusoidal frequency and the time average of variance response are simulated by utilizing the Monte Carlo method, Gaussian linearization, and present formulation as shown in Figs. 3 and 4.

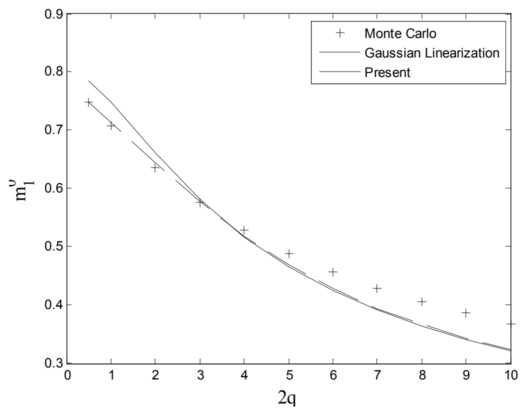

Fig. 3 Cyclostationary amplitude of mean response by different approaches

The simulated figures reveal that the present formulation of (15) and (17) with cyclostationary approximations in the Gaussian linearization predicts accurate cyclostationary amplitude of mean response and time average of variance response as those obtained by employing standard Gaussian linearization approach. In comparing with Monte Carlo results, Fig. 3 shows that the present approach gives higher error in the amplitude of mean response for either weaker or higher noise intensity. For the time average of variance response as shown in Fig. 4, the Gaussian linearization approaches give an underestimation error about 10%. The numerical results in Figs. 3 and 4 reveal that the present algebraic approach can be employed for analyzing the time-domain statistical response.

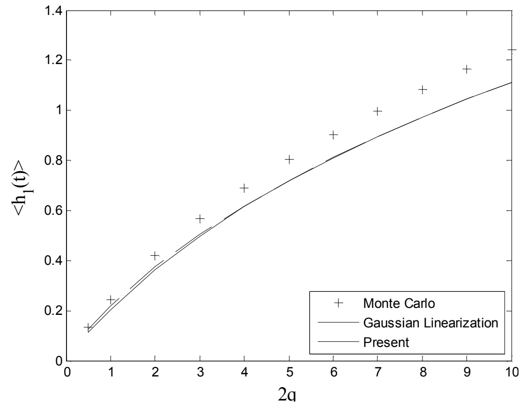

Fig. 4 Cyclostationary time average of variance response obtained by different approaches

In the investigation of signal frequency response, the same system parameters as those in simulating the time-domain response are utilized. With different input noise intensities, the frequency response of mean amplitude by (24) is simulated as shown in Fig. 5. For the simulated Duffing system, the frequency response reveals single-valued amplitude for the noise intensity ranging from 1 to 10. When the noise intensity increases, the peak amplitude decreases and the associated peak frequency shifts to higher frequency.

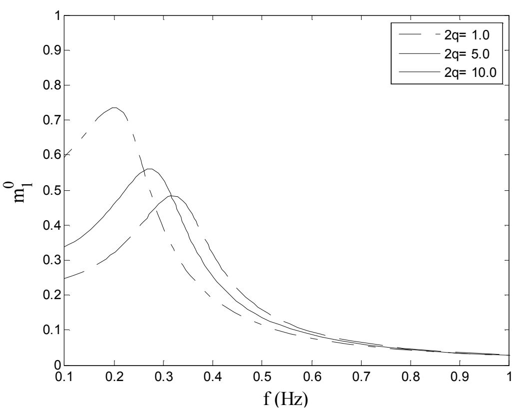

Fig. 5 Frequency response of amplitude of mean with different noise intensities

For the sinusoidal response with 2q=5, the amplitude spectrum of mean response obtained by the present approach is compared with that obtained by Monte Calo simulation as shown in Fig. 6. The present predicted peak frequency is 0.267Hz that is the same frequency as obtained by emloying (26). The present predicted peak frequency is higher than the frequency by Monte Carlo simulation with an error about 9%. The peak frequency of amplitude of mean response with varying noise intensity is simulated and compared with the results by Monte Calo simulation as shown in Fig. 7. From the simulated results with 2q from 0.5 to 10, the error of peak frequency predicted by the present formulation is about 10%.

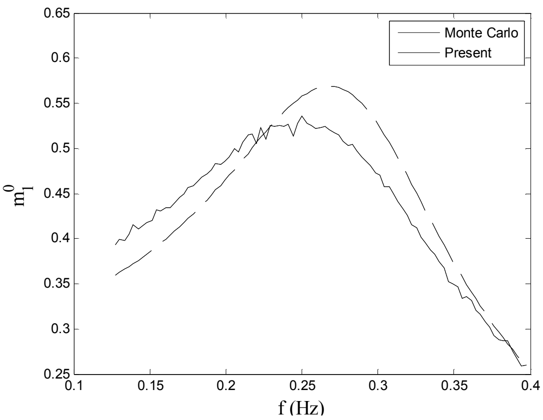

Fig. 6 Amplitude spectrum of mean response with 2q=5 by the present approach and Monte Calo simulation

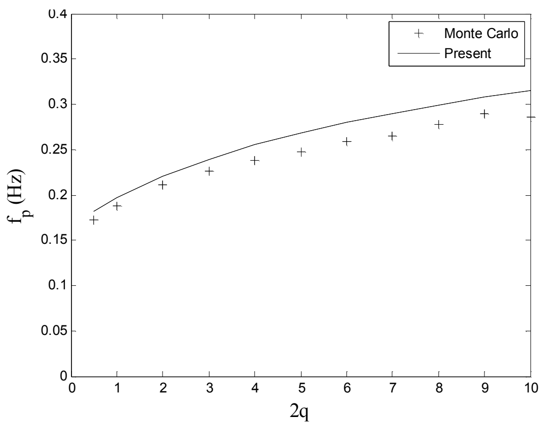

Fig. 7 Peak frequency of amplitude of mean response with varying input noise intensity by the present approach and Monte Calo simulation

For the frequency response of noise spectrum, the mean physical power spectral density under unit-amplitude sinusoidal signal of 0.2 Hz and white-noise excitation with 2q=5 by the Monte Calo, Gaussian linearization, and present approach are simulated as shown in Fig. 8. In Fig. 8, the power spectral density by the Monte Calo simulation is obtained through periodic time average of response spectra. The power spectral density by the Gaussian linearization approach is obtained by simulating (19) and through Welch spectrum estimator. From Fig. 8, it is observed that the present approach by (21) which utilizes cyclostationary approximations in the Gaussian linearization predicts almost the same mean power spectral density as that obtained by standard Gaussian linearization approach. By utilizing the peak frequency to divide the whole frequency range into a high frequency range and a low frequency range, the present approach predicts accurate magnitude of spectrum in the high frequency range but lower magnitude of spectrum in the low frequency range. For the present approach as simulated in Fig. 8, the predicted peak frequency in the power spectral density is 0.272Hz. The peak frequency is the same as that obtained by emloying (22). The present predicted peak frequency is higher than that by Monte Carlo simulation with an error about 26%.

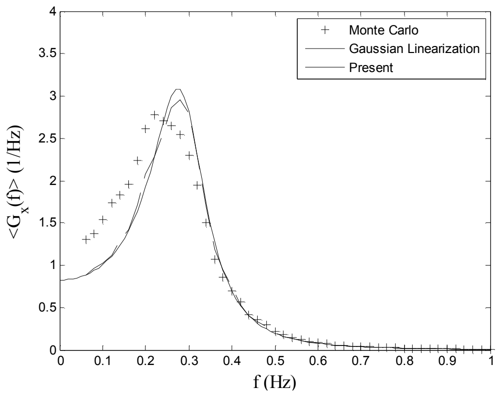

Fig. 8 Mean physical power spectral density of noise response under unit-amplitude sinusoidal signal of 0.2 Hz and white-noise excitation with 2q=5 by different approaches

Finally, the qualitative response behavior of stochastic jump in the Duffing system will be investigated. The parameters of the Duffing system are selected to afford a lightly damped oscillator with weak noise excitation. By choosing $b_1 = 1,\ b_2 = 0.1,\ b^N = 0.3,\ a = 0.2,\ 2q = 0.008,$ a triple-valued frequency response of amplitude of mean can be obtained as shown in Fig. 9. Fig. 9 reveals that in a frequency range between point **a**, at 0.1997Hz, and point **b**, at 0.1878Hz, two stable and one unstable stationary solution are coexistent. The unstable solution is indicated by a dash line connecting points **a** and **b**. The stochastic jump between two stationary stable solutions such as points 2 and 3, which are at the same input frequency, implies the existence of bimodal probability density function in stationary response [10], [12]. For single-valued stable solutions in other frequency range such as point 1, at 0.1751 Hz, and point 4, at 0.2069 Hz, the probability density response is a single peak distribution. The effectiveness of employing the present Gaussian linearization in predicting frequency response of mean is further investigated. The frequency response of phase of mean versus amplitude of mean is simulated as shown in Fig. 10. The numbers 1, 2, 3, 4 and alphabets **a**, **b** indicated in Fig. 10 are corresponding to those in Fig. 9 under the same input frequency. For those points 1, 2, 3, and 4 with frequencies specified in Fig. 9, the corresponding probability density responses are simulated by Monte Carlo method. From the simulated density functions, the phases of mean and amplitudes of mean of the probability-density peaks are indicated in Fig. 10. For the point 1 and point 4 in Fig. 10, the existence of single density peak and the associated coordinates of density peak can be reliably inferred by employing the phase of mean and amplitude of mean as supported by the Monte Carlo simulation. For points 2 and 3 at the same input frequency in Fig. 10, the coexistence of two phases of mean and amplitudes of mean at the same frequency gives the evidence of the existence of bimodal probability density function in stationary response. By considering the predicted phases and amplitudes in the mean response as estimations of coordinates of two peaks in bimodal probability density, the phases of mean and amplitudes of mean on the coordinates of those two peaks can be reliably predicted as supported by Monte Carlo simulation. By employing the present method for estimating the coordinates of two peaks in bimodal probability density, the prediction is reliable for weak noise excitation since the stochastic jumps under combined sinusoidal and white noise excitation can be regarded as random spread of the deterministic jump of the Duffing system.

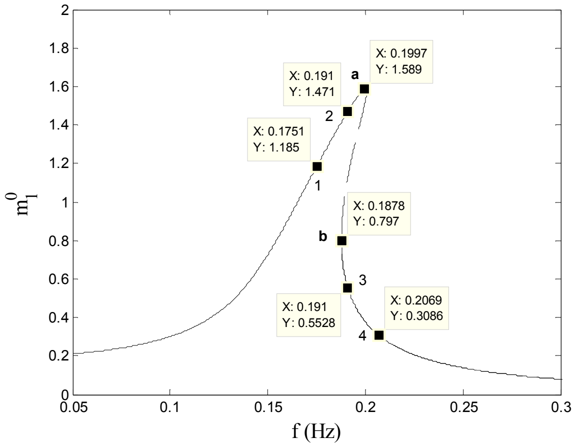

Fig. 9 Multi-valued frequency response of amplitude of mean

VII. Conclusion

An algebraic formulation is developed for predicting the time-average of cyclostationary response of nonlinear system subjected to sinusoidal signal and white noise excitation. Duffing system is selected for validating the validity of the present approach in analyzing nonlinear response. The present formulated algebraic equation predicts accurate time-average of cyclostationary responses as those obtained by numerical solution of differential equations in the standard Gaussian linearization method. For the simulated numerical example, higher error in the amplitude of mean response appears in a region of either weaker or higher noise intensity. For the time average of variance response, the present approach gives an underestimation error about 10%.

For the frequency response, algebraic formulations of the existence conditions of peaks and the corresponding peak frequencies in both spectra of amplitude of mean and noise response are derived and verified. The peak frequencies of amplitude spectrum of mean response and power spectral density of noise response are overestimated with error about 9%, and 26%, respectively. The present approach has been applied for investigating the stochastic jumps of a lightly damped oscillator with weak noise excitation. By analyzing the frequency response of amplitude of mean, the stationary probability density response in either single peak or bimodal distribution under different input sinusoidal signal frequencies can be effectively predicted. The present algebraic solution gives reliable quantitative and qualitative prediction of mean and noise response for the Duffing system subjected to sinusoidal signal and white noise excitation.

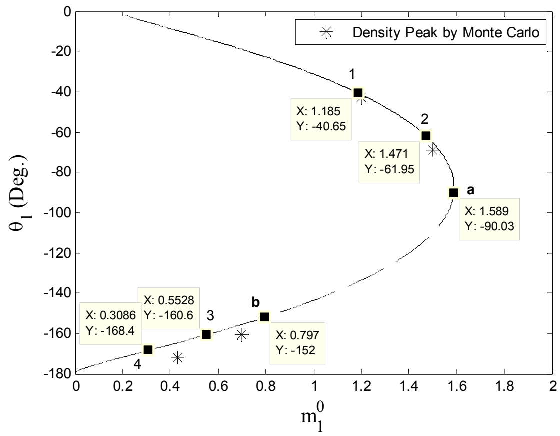

Fig. 10 Frequency response of phase of mean versus amplitude of mean

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References

[1] A. A. Pervozvanskii, *Random Processes in Nonlinear Control Systems*, New York: Academic Press, 1965.
[2] P. Jung, "Periodically driven stochastic systems," *Physics Reports*, vol. 234, nos. 4, 5, pp. 175-295, 1993.
[3] A. Jha and E. Nikolaidis, "Vibration of dynamic systems under cyclostationary excitations," *AIAA J.*, vol. 38, no. 12, pp. 2284-2291, 2000.
[4] K. Ellermann, "On the determination of nonlinear response distributions for oscillators with combined harmonic and random excitation," *Nonlinear Dynamics,* vol. 42, pp. 305-318, 2005.
[5] W. A. Gardner, N. Antonio, and P. Luigi "Cyclostationarity: Half a century of research," *Signal Processing*, vol. 86, no. 4, pp. 639–697, 2006.
[6] A. B. Budgor, "Studies in nonlinear stochastic processes. III. Approximate solutions of nonlinear stochastic differential equations excited by Gaussian noise and harmonic disturbances," *J. Statistical Physics*, vol. 17, no. 1, pp. 21-44, 1977.
[7] A. R. Bulsara, K. Lindenberg, and K. E. Shuler, "Spectral analysis of a nonlinear oscillator driven by random and periodic forces. I. Linearized theory," *J. Statistical Physics*, vol. 27, no. 4, pp. 789-796, 1982.
[8] R. N. Yengar, "A nonlinear system under combined periodic and random excitation," *J. Statistical Physics*, vol. 44, no. 5/6, pp. 907-920, 1986.
[9] A. H. Nayfeh and S. J., Serhan, Response statistics of non-linear systems to combined deterministic and random excitations, *Int. J. Non-Linear Mechanics*, vol. 25, no. 5, pp. 493-505, 1990.
[10] U. V. Wagner, On double crater-like probability density functions of a duffing oscillator subjected to harmonic and stochastic excitations," *Nonlinear Dynamics*, vol. 28, pp. 343-355, 2002.
[11] H. Rong, G. Meng, X. Wang, Xu, W., and T. Fang, Response statistic of non-linear oscillator to combined deterministic and random excitation," *Int. J. Non-Linear Mechanics*, vol. 39, pp. 871-878, 2004.

[12] Z. L. Huang, W. Q. Zhu, Y. Suzuki, "Stochastic averaging of non-linear oscillators under combined harmonic and white-noise excitations," *J. Sound and Vibration*, vol. 238, no. 2, pp. 233-256, 2000.
[13] Y. J. Wu and W. Q. Zhu, "Stochastic averaging of strongly nonlinear oscillators under combined harmonic and wide-band noise excitations," *J. Vibration and Acoustics*, vol.130, 051004, pp. 1-19, 2008.
[14] Z. L. Huang and W. Q. Zhu, "Stochastic averaging of quasi-integrable Hamiltonian systems under combined harmonic and white noise excitations," *Int. J. Non-Linear Mechanics*, vol. 39, pp. 1421-1434, 2004.
[15] J. Yu and Y. Lin, "Numerical path integration of a nonlinear oscillator subject to both sinusoidal and white noise excitations," *5th Int. Conf. Stochastic Structural Dynamics; Advances in stochastic structural dynamics*, Florida, USA, 2003, pp. 525-534.
[16] W. Xu, Q. He, T. Fang, and H. Rong, "Stochastic bifurcation in Duffing system subjected to harmonic excitation and in presence of random noise," *Int. J. Non-Linear Mechanics*, vol. 39, pp. 1473-1479, 2004.
[17] U. V. Wagner and Wedig, W. V. "On the calculation of stationary solutions of multi-dimensional Fokker-Planck equations by orthogonal functions," *Nonlinear Dynamics*, vol. 21, pp. 289-306, 2000.
[18] L. Socha, *Linearization Methods for Stochastic Dynamic Systems*, Berlin: Springer, 2008.
[19] R. J. Chang, "Two-stage optimal stochastic linearization in analyzing of non-linear stochastic dynamic systems," *Int. J. Non-Linear. Mechanics*, vol. 58, pp. 295-304, 2014.