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Explicit Chain Homotopic Function to Compute Hochschild Homology of the Polynomial Algebra

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***Abstract*—**In this paper, an explicit homotopic function is constructed to compute the Hochschild homology of a finite dimensional free k-module V. Because the polynomial algebra is of course fundamental in the computation of the Hochschild homology *HH* and the cyclic homology *CH* of commutative algebras, we concentrate our work to compute *HH* of the polynomial algebra, by providing certain homotopic function.

***Keywords*—**Exterior algebra, free resolution, free and projective modules, Hochschild homology, homotopic function, symmetric algebra.

I. Introduction

In order to interpret index theorems for non-commutative Banach algebras, [2] developed cyclic homology as a non-commutative variant of the de Rham cohomology which is illustrated very well in [1]. In different other sources, cyclic homology appeared as the primitive part of the Lie algebra homology of matrices by [4]. The cyclic homology of an algebra A consists of a family of abelian groups $C_n(A), n \geq 0$ which are in characteristic zero, the homology of the quotient of the Hochschild complex by the action of the finite cyclic groups. Thus cyclic homology is a variant of Hochschild homology in such a way. Loday [3] worked on cyclic homology and Hochschild homology and provided different aspects of uses of these kinds of homologies. The example of polynomial algebra is very important in HH in the sense of commutative algebras, because polynomial algebras can be underlying algebra of differential graded models that can be used to perform computations. Also, Hochschild homology computations of polynomial algebra can be generalized to smooth algebras and symmetric algebras because polynomial algebra is a symmetric algebra of a finite dimensional free k-module.

Loday [3] proved that the Hochschild homology of polynomial algebra is the module khäler differentials (the module of differential forms); He introduced a commutative differential graded algebra with a certain product there to get an isomorphism commutative differential graded algebras.

Here we are trying to have different approach to get the same result by constructing an explicit homotopic function to get a free resolution that is necessary to find the Tor functor there. This technique can be carried out to calculate Hochschild homology of different kind of algebras.

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II. Preliminaries

Definition 1 [5]. Let $\boldsymbol{K}$ be a commutative ring, and $\boldsymbol{M}$ be an $\boldsymbol{A}$-bimodule of an associative (not necessarily commutative) $\boldsymbol{K}$-algebra $\boldsymbol{A}$. We define the Hochschild complex $\boldsymbol{CH_*(A,M)}$ as the sequence of maps

$$\dots \xrightarrow{b} M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \dots \xrightarrow{b} M \otimes A \xrightarrow{b} M \quad \dots$$

where the module $\boldsymbol{M \otimes A^{\otimes n}}$ is in degree $\boldsymbol{n}$. The Hochschild boundary map $\boldsymbol{b}: \boldsymbol{M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1}}$ is given by

$$b(m \otimes a_1 \otimes a_2 \otimes a_3 \otimes \dots \otimes a_n) = m a_1 \otimes a_2 \otimes a_3 \otimes \dots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes a_2 \otimes a_3 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n + (-1)^n a_n m \otimes a_1 \otimes a_2 \otimes \dots \otimes a_{n-1} \tag{1}$$

for $m \in M$ and $a_i \in A$ for all $i = 1,2,\dots,n$. The homology groups of the Hochschild complex $CH_n(A,M)$ are called the Hochschild homology groups $HH_n(A,M)$.

Definition 2 [3]. For $\boldsymbol{A}$ unital and commutative algebra, let $\boldsymbol{\Omega^1_{A/K}}$ be the $\boldsymbol{A}$-module of Kahler differentials. It is generated by the $\boldsymbol{k}$-linear symbols $\boldsymbol{da}$ for $\boldsymbol{a \in A}$ (so $\boldsymbol{d(\lambda a + \mu b) = \lambda\, d\, a + \mu db}$, where $\boldsymbol{\lambda, \mu \in k}$ and $\boldsymbol{a, b \in A}$) with the relation $\boldsymbol{du = 0}$, for $\boldsymbol{u \in k}$

Definition 3. Let $\boldsymbol{A}$ be a unital and commutative algebra. Then $\boldsymbol{A}$-module of differential $\boldsymbol{n}$-forms $\boldsymbol{\Omega^n_{A/K}}$ is, by definition, the exterior product $\boldsymbol{\Omega^n_{A/K} = \wedge^n_A \Omega^1_{A/K}}$. The exterior is spanned by the elements $\boldsymbol{a_0 da_1 \wedge da_2 \wedge \dots \wedge d a_n}$ for $\boldsymbol{a_i \in A}$, that can be written as $\boldsymbol{a_0 da_1 da_2 \dots d a_n}$.

Example 1 [3]. Let V be a free module over k and let $A = S(V)$ be the symmetric algebra of V. If V is finite dimensional with basis $x_1, x_2, \dots, x_n$, then one gets the polynomial algebra $S(V) = k[x_1, x_2, \dots, x_n]$. Then we get a great result that $S(V) \otimes V \cong \Omega^1_{S(V)/K}, a \otimes v \mapsto a\, d(v)$. To see the proof of the given isomorphism, then please look to [3], page 26).

Definition 4. Let $\boldsymbol{A}$ be a $\boldsymbol{k}$-algebra. Then the opposite algebra of $\boldsymbol{A}$ is denoted by $\boldsymbol{A^{op}}$ and the product of $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\boldsymbol{A^{op}}$ is given by $\boldsymbol{a.b = ab}$ (where the product is in the algebra $\boldsymbol{A}$). In addition to, the algebra $\boldsymbol{A^e}$ is the enveloping algebra and defined as $\boldsymbol{A^e = A \otimes A^{op}}$.

Definition 5 [6]. Given a module $\boldsymbol{M}$, a projective (free) resolution of $\boldsymbol{M}$ is an infinite exact sequence of modules

$$\boldsymbol{M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0}$$

with all the projective (free) modules $\boldsymbol{M_i}$, where $\boldsymbol{0 \leq i \leq n}$

Definition 6. Let $\boldsymbol{R}$ be a ring. For $\boldsymbol{A}$ is a right $\boldsymbol{R}$-module and $\boldsymbol{B}$ is a left $\boldsymbol{R}$-module, we construct a projective resolution of $\boldsymbol{A}$ as $\boldsymbol{M_n} \longrightarrow \boldsymbol{M_{n-1}} \longrightarrow \cdots \longrightarrow \boldsymbol{M_1} \longrightarrow \boldsymbol{M_0} \longrightarrow \boldsymbol{A} \longrightarrow \boldsymbol{0}$. Then, the homology of the complex $\boldsymbol{M_n} \otimes_{\boldsymbol{R}} \boldsymbol{B} \longrightarrow \boldsymbol{M_{n-1}} \otimes_{\boldsymbol{R}} \boldsymbol{B} \longrightarrow \cdots \longrightarrow \boldsymbol{M_1} \otimes_{\boldsymbol{R}} \boldsymbol{B} \longrightarrow \boldsymbol{M_0} \otimes_{\boldsymbol{R}} \boldsymbol{B}$ is called the Tor functor and denoted by $\boldsymbol{Tor_n^R}(\boldsymbol{A}, \boldsymbol{B})$.

III. Computing the Hochschild Homology of the Polynomial Algebra

Theorem 1 [3]. If the unital algebra A is projective as a module over k, then for any A-bimodule M there is an isomorphism

$$HH_n(A, M) \cong Tor_n^{A^e}(M, A)$$

Example 2. Let $A = \mathbb{R}[x]$ be the polynomial algebra of one variable x . We try to find its Hochschild homology groups. From the last theorem, we get

$$HH_2(A, A) = Tor_2{}^{A \otimes A^{op}}(A, A) = Tor_2{}^{A \otimes A}(A, A).$$

Now, $A \otimes < x > \otimes A$ is free $A \otimes A$-module where $< x > = \{rx : r \in \mathbb{R}\}$. Let us construct the following sequence

$$0 \leftarrow A \overset{m}{\leftarrow} A \otimes A \overset{d_0}{\leftarrow} A \otimes \ < x > \otimes A \overset{d_1}{\leftarrow}$$

where $m(a \otimes b) = ab$ for $a, b \in A$ and $d_0(c \otimes x \otimes d) = cx \otimes d - c \otimes xd$ for $c, d \in A$.

Claim 1. The above sequence is exact.

Proof:

a) We try to prove that $im(d_0) \subset \ker(m)$. To do so, take $cx \otimes d - c \otimes xd \in im(d_0)$. Then

$$m(cx \otimes d - c \otimes xd) = cxd - cxd = 0.$$

Thus,

$$cx \otimes d - c \otimes xd \in \ker(m).$$

So,

$$im(d_0) \subset \ker(m).$$

b) We will prove $\ker(m) \subset im(d_0)$.

Let $h_{-1} = 0$, and we try to find h_i for $i = 0,1,2$ in the following sequences:

$$\begin{array}{l} 0 \overset{0}{\leftarrow} A \overset{m}{\leftarrow} A \otimes A \overset{d_0}{\leftarrow} A \otimes < x > \otimes A \overset{d_1}{\leftarrow} \\ \ \searrow h_{-1} \searrow h_0 \quad \searrow h_1 \qquad\qquad\quad \searrow h_2 \\ 0 \overset{0}{\leftarrow} A \overset{m}{\leftarrow} A \otimes A \overset{d_0}{\leftarrow} A \otimes < x > \otimes A \overset{d_1}{\leftarrow} \end{array}$$

Since h_i for $i = 0,1,2$ is homotopic, then for any $a \in A$, we have $h_{-1}(0(a)) + m(h_0(a)) = a$, so $m(h_0(a)) = a$. That means, we can take $h_0(a) = 1 \otimes a$. Now, if $\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i} \in A \otimes A$ such that $\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i} \in \ker(m)$. We have

$$h_0\left(m\left(\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}\right)\right) + d_0\left(h_1\left(\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}\right)\right) = \sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}.$$

So,

$$h_0(0) + d_0\left(h_1\left(\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}\right)\right) = \sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}.$$

Then

$$d_0\left(h_1\left(\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}\right)\right) = \sum_{i=1}^{n} a_{1,i} \otimes b_{1,i}.$$

That means

$$\sum_{i=1}^{n} a_{1,i} \otimes b_{1,i} \in im(d_0).$$

This completes the proof of $\ker(m) \subset im(d_0)$.

Combining both parts (a) and (b), we get

$$\ker(m) = im(d_0).$$

This completes the proof of claim 1.

Now, let us define h_1 such that

$$h_0 m + d_0 h_1 = \mathbb{1}$$

Let $1 \otimes 1 \in A \otimes A$. To find $h_1(1 \otimes 1)$, we have:

$$h_0(m(1 \otimes 1)) + d_0(h_1(1 \otimes 1)) = 1 \otimes 1.$$

So,

$$h_0(1) + d_0(h_1(1 \otimes 1)) = 1 \otimes 1.$$

That means

$$1 \otimes 1 + d_0(h_1(1 \otimes 1)) = 1 \otimes 1.$$

and so,

$$d_0(h_1(1 \otimes 1)) = 0.$$

Thus, we define $h_1(1 \otimes 1) = 0$. Similarly, $h_1(1 \otimes x^n) = 0$ in $A \otimes < x > \otimes A$ for $n = 0,1,2, \ldots$ Now, we can find $h_1(x \otimes 1)$, such that

$$h_0(m(x \otimes 1)) + d_0(h_1(x \otimes 1)) = x \otimes 1.$$

Thus,

$$h_0(x) + d_0(h_1(x \otimes 1)) = x \otimes 1.$$
$$1 \otimes x + d_0(h_1(x \otimes 1)) = x \otimes 1.$$

That means,

$$d_0(h_1(x \otimes 1)) = x \otimes 1 - 1 \otimes x.$$

Thus, we define $h_1(x \otimes 1) = 1 \otimes x \otimes 1$. Similarly, we can find

$$h_1(x^2 \otimes 1) = x \otimes x \otimes 1 + 1 \otimes x \otimes x,$$

and

$$h_1(x^3 \otimes 1) = x^2 \otimes x \otimes 1 + 1 \otimes x \otimes x^2 + x \otimes x \otimes x.$$

Claim 2. $h_1(x^n \otimes 1) = \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n-i-1}$.

Proof. We know that

$$h_0(m(x^n \otimes 1)) + d_0(h_1(x^n \otimes 1)) = x^n \otimes 1.$$

If we replace $h_1(x^n \otimes 1)$ by $\sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n-i-1}$ in the above equation, we find

$$h_0(x^n) + d_0(\sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n-i-1})) = x^n \otimes 1.$$

Then

$$1 \otimes x^n + \sum_{i=0}^{n-1} d_0(x^i \otimes x \otimes x^{n-i-1}) = x^n \otimes 1.$$

But,

$$\sum_{i=0}^{n-1} d_0(x^i \otimes x \otimes x^{n-i-1}) = x \otimes x^{n-1} - 1 \otimes x^n + x^2 \otimes x^{n-2} - x \otimes x^{n-1} + x^3 \otimes x^{n-3} - x^2 \otimes x^{n-2} + \cdots + x^n \otimes 1 - x^{n-2} \otimes x = x^n \otimes 1 - 1 \otimes x^n.$$

This completes the proof of claim 2.

Claim 3. The map d_1 is the zero map.

Proof. Because we are looking to have the following sequence to be exact

$$0 \leftarrow A \stackrel{m}{\leftarrow} A \otimes A \stackrel{d_0}{\leftarrow} A \otimes <x> \otimes A \stackrel{d_1}{\leftarrow},$$

We have $\ker(d_0) = im(d_1)$. Now,

$$h_1(d_0(a \otimes x \otimes b)) = h_1(ax \otimes b - a \otimes xb),$$

for $a, b \in A$. If we put $a = 1$ *and* $b = 1$, we get

$$h_1(d_0(1 \otimes x \otimes 1)) = h_1(x \otimes 1 - 1 \otimes x) = 1 \otimes x \otimes 1.$$

Now, put $a = x$ *and* $b = 1$, then we get

$$h_1(d_0(x \otimes x \otimes 1)) = h_1(x^2 \otimes 1 - x \otimes x) = x \otimes x \otimes 1 + 1 \otimes x \otimes x - h_1(x \otimes x).$$

Replacing $h_1(x \otimes x)$ by $1 \otimes x \otimes x$ will make sense in the above equation, so

$$h_1(x \otimes x) = 1 \otimes x \otimes x.$$

Thus

$$h_1(d_0(x \otimes x \otimes 1)) = x \otimes x \otimes 1.$$

That means the map $h_1 d_0$ is the identity map, so

$$\ker(d_0) = 0.$$

As we said in the beginning of the proof of claim 3 that

$$\ker(d_0) = im(d_1).$$

Thus, $d_1 = 0$, and so that we get this free exact sequence:

$$0 \leftarrow A \stackrel{m}{\leftarrow} A \otimes A \stackrel{d_0}{\leftarrow} A \otimes <x> A \stackrel{d_1}{\leftarrow} 0.$$

Tensoring with A over $A \otimes A$ and removing the first term A, we get

$$\begin{array}{ccccccc} 0 \longleftarrow & (A \otimes A) \otimes A & \stackrel{d_0 \times \mathbb{I}}{\longleftarrow} & (A \otimes <x> \otimes A) \otimes A & \longleftarrow 0 \\ & \cong \downarrow \alpha & & \cong \downarrow \beta & \\ 0 \longleftarrow & A & \stackrel{\psi}{\leftarrow} & <x> \otimes A & \longleftarrow 0 \end{array}$$

where the maps $d_0 \times \mathbb{I}$, α, and β are defined as

$$d_0 \times \mathbb{I}((a \otimes x \otimes b) \otimes c) = (ax \otimes b - a \otimes xb) \otimes c,$$
$$\alpha((a \otimes b) \otimes c) = acb,$$

and

$$\beta((a \otimes x \otimes b) \otimes c) = x \otimes acb.$$

Computing

$$\alpha((d_0 \times \mathbb{I})((a \otimes x \otimes b) \otimes c),$$

We get

$$\alpha((d_0 \times \mathbb{I})((a \otimes x \otimes b) \otimes c) = \alpha((ax \otimes b - a \otimes xb) \otimes c)$$
$$= \alpha((ax \otimes b) \otimes c) - \alpha(a \otimes xb) \otimes c) = (ax)cb - ac(xb)$$
$$= 0.$$

Since $\alpha((d_0 \times \mathbb{I}) = 0$ and $\alpha((d_0 \times \mathbb{I}) = \psi(\beta)$, we get the result

$$\psi = 0.$$

Also, since the maps α and β are isomorphism maps, then we have that the sequence

$$0 \longleftarrow (A \otimes A) \otimes A \stackrel{d_0 \times \mathbb{I}}{\longleftarrow} (A \otimes <x> \otimes A) \otimes A \longleftarrow 0$$

is the same as the sequence

$$0 \longleftarrow A \stackrel{\psi}{\leftarrow} <x> \otimes A \longleftarrow 0.$$

Finding the homology groups of the sequence

$$0 \longleftarrow A \stackrel{\psi=0}{\longleftarrow} <x> \otimes A \longleftarrow 0.$$

We find,

$$HH_0(A) = A, HH_1(A) = <x> \otimes A = \Omega^1(\mathrm{A})$$

and

$$HH_n(A) = 0 \text{ for } n \geq 2.$$

Example 3. Let $A = \mathbb{R}[x_1, x_2, \dots, x_n]$. We are trying to find the Hochschild homology $HH_n(A)$**.** Now, we show that the following sequence is exact with given an explicit chain homotopic h_i for $i = -1,0,1,2, \dots, n-1$:

$$\begin{array}{l} 0 \stackrel{0}{\leftarrow} A \stackrel{m}{\leftarrow} A \otimes A \stackrel{d_0}{\leftarrow} A \otimes V \otimes A \stackrel{d_1}{\leftarrow} \dots \stackrel{d_{n-1}}{\longleftarrow} A \otimes \Lambda^n V \otimes A \stackrel{0}{\leftarrow} 0 \\ \quad \searrow h_{-1} \searrow h_0 \quad \searrow h_1 \qquad \dots \qquad \searrow h_{n-1} \\ 0 \stackrel{0}{\leftarrow} A \stackrel{m}{\leftarrow} A \otimes A \stackrel{d_0}{\leftarrow} A \otimes V \otimes A \stackrel{d_1}{\leftarrow} \dots \stackrel{d_{n-1}}{\longleftarrow} A \otimes \Lambda^n V \otimes A \stackrel{0}{\leftarrow} 0 \end{array}$$

where

$$V = \langle x_1, x_2, \dots, x_n \rangle = \{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n, m_i \in \mathbb{R}\}$$

and

$$
\begin{aligned}
&d_k\left(a\otimes x_{i_1}\wedge x_{i_2}\wedge \dots \wedge x_{i_{k+1}}\otimes b\right) \\
&\quad = \sum_{j=1}^{k+1}\Big\{(-1)^{j+1} a x_{i_j}\otimes x_{i_1}\wedge \dots \wedge x_{i_{k+1}}\otimes b \\
&\quad + (-1)^{j} a\otimes x_{i_1}\wedge \dots \wedge \hat{x}_{i_j}\wedge \dots \wedge x_{i_{k+1}}\otimes x_{i_j} b\Big\}
\end{aligned}
$$

where the notation $\hat{x}_{i_j}$ denotes that the element is deleted and

$$
\begin{aligned}
&h_{k+1}\left(x_l\otimes x_{i_1}\wedge \dots \wedge x_{i_k}\otimes b\right) \\
&= \begin{cases} 1\otimes x_l\otimes x_{i_1}\wedge \dots \wedge x_{i_k}\otimes b & l > \max\{i_1,\dots,i_k\} \\ 0 & O.W. \end{cases}
\end{aligned}
$$

Claim 1. For $v_1,\dots,v_{k+1}\in\{x_1,x_2,\dots,x_n\}$, $(d_{k-1}\circ d_k)\,(a\otimes v_1\wedge \dots \wedge v_{k+1}\otimes b)=0$.

Proof: (Conceptual proof)

1. The term $v_i v_j$ is multiplied by a: If i is first and j is second: $(-1)^j(-1)^{i+1} a v_i v_j\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge v_{k+1}\otimes b$ and for i is second and j is first: $(-1)^i(-1)^j a v_i v_j\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes b$.
2. The term $v_i v_j$ is multiplied by b: If i is first and j is second: $(-1)^{j-1}(-1)^i a\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_i v_j b$ and for i is second and j is first: $(-1)^i(-1)^j a\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_i v_j b$.
3. The term v_i is multiplied by a and v_j by b: If i is first and j is second: $(-1)^{j-1}(-1)^{i+1} a v_i\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge v_{k+1}\otimes v_j b$ and for i is second and j is first: $(-1)^j(-1)^{i+1} a v_i\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_j b$.
4. The term v_i is multiplied by b and v_j by a: If i is first and j is second: $(-1)^j(-1)^i a v_j\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_i b$ and for i is second and j is first: $(-1)^i(-1)^{j+1} a v_j\otimes v_1\wedge \dots \wedge \hat{v}_i\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_i b$.

When we add all the terms in the above possibilities, it is clear that all of them are deleted together, so we get $d_{k-1}\circ d_k=0$.

Claim 2. $h_{k+1}\left(d_k(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)+d_{k+1}\left(h_{k+2}(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)=v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b$.

Proof: (Conceptual proof)

1. If $l>\max\{1,2,\dots,k+1\}$, then

$$
\begin{aligned}
&h_{k+1}\left(d_k(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right) \\
&\quad = h_{k+1}\left[\sum_{j=1}^{k+1}\{(-1)^{j+1} v_l v_j\otimes v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes b\right. \\
&\quad \left. + (-1)^j v_l\otimes v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_j b\}\right] \\
&= \sum_{j=1}^{k+1}\{(-1)^{j+1} v_j\otimes v_l\wedge v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1} \\
&\quad + (-1)^{j+1} 1\otimes v_{k+1}\wedge v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_k\otimes v_l b \\
&\quad + (-1)^j 1\otimes v_l\wedge v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_j b\}
\end{aligned}
$$

in the other hand,

$$
\begin{aligned}
&d_{k+1}\left(h_{k+2}(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right) \\
&\quad = v_l\wedge v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1} \\
&\quad + \sum_{j=1}^{k+1}\{(-1)^j v_j\otimes v_l\wedge v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes b \\
&\quad + (-1)^{j+1} 1\otimes v_l\wedge v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_j b\}
\end{aligned}
$$

It is clear that

$$
\begin{gathered}
h_{k+1}\left(d_k(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)+ \\
d_{k+1}\left(h_{k+2}(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)=v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b.
\end{gathered}
$$

2. If $l\le \max\{1,2,\dots,k+1\}$, then

$$
d_{k+1}\left(h_{k+2}(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)=d_{k+1}(0)=0.
$$

Now,

$$
\begin{gathered}
h_{k+1}\left(d_k(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)= \\
h_{k+1}\left[\sum_{j=1}^{k+1}\{(-1)^{j+1} v_l v_j\otimes v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes b+\right. \\
\left.(-1)^j v_l\otimes v_1\wedge \dots \wedge \hat{v}_j\wedge \dots \wedge v_{k+1}\otimes v_j b\}\right]= \\
(-1)^{k+2} v_l\otimes v_{k+1}\wedge v_1\wedge \dots \wedge v_k\otimes b= \\
(-1)^k(-1)^k v_l\otimes v_1\wedge \dots \wedge v_k\wedge v_{k+1}\otimes b= \\
v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b.
\end{gathered}
$$

Thus,

$$
\begin{gathered}
h_{k+1}\left(d_k(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)+ \\
d_{k+1}\left(h_{k+2}(v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b)\right)= \\
v_l\otimes v_1\wedge v_2\wedge \dots \wedge v_{k+1}\otimes b.
\end{gathered}
$$

From the above claims, we guarantee the following sequence is free resolution of A

$$
0\overset{0}{\leftarrow} A\overset{m}{\leftarrow} A\otimes A\overset{d_0}{\leftarrow}\dots\overset{d_{n-1}}{\longleftarrow} A\otimes\wedge^n V\otimes A\overset{0}{\leftarrow} 0
$$

Tensoring with A over $A\otimes A$ and removing the first term A, we get $(A\otimes A)\otimes A\overset{d_0\times id_A}{\longleftarrow}\dots\overset{d_{n-1}\times id_A}{\longleftarrow}(A\otimes\wedge^n V\otimes A)\otimes A$ where the terms are isomorphic to the sequence

$$
\begin{gathered}
HH_n(A)=\wedge^n V\otimes A\cong A\otimes\wedge^n V. \\
A\overset{0}{\leftarrow} V\otimes A\overset{0}{\leftarrow}\dots\overset{0}{\leftarrow}\wedge^n V\otimes A
\end{gathered}
$$

Thus, $HH_0(A)=A, HH_1(A)=V\otimes A\cong A\otimes V$ and the nth-Hochschild homology group is

$$
HH_n(A)=\wedge^n V\otimes A\cong A\otimes\wedge^n V
$$

where the higher Hochschild homology groups for $m\ge n+1$, by definition 3 and example 1, we get

$$
HH_1(A)=\Omega_A^1, HH_2(A)=\Omega_A^2,\dots,HH_n(A)=\Omega_A^n.
$$

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