The Study of Relative Efficiency in Growth Curve Model

Nan Chen, Baoguang Tian

***Abstract*—**In this paper, some relative efficiency have been discussed, including the LSE estimate with respect to BLUE in curve model. Four new kinds of relative efficiency have defined, and their upper bounds have been discussed.

***Keywords*—**Relative efficiency, LSE estimate, BLUE estimate, Upper bound, Curve model.

I. Introduction

CONSIDERING the general growth model

$$\begin{cases} Y = XBZ + e \\ E(e) = 0 \\ Cov(\vec{e}) = V \otimes \Sigma \end{cases} \quad (1)$$

Among them, Y is the $n \times q$ random observation matrix; e is the $n \times q$ random error matrix, X_{nm}, Z_{kq} is the design matrix that we are known, and $r(X) = m$, $r(Z) = k$, B is the $p \times k$ regression parameter matrix that we are unknown, V and Σ are positive definite matrices with the size $q \times q$ and $n \times n$ -order respectively. $\otimes$ expresses product of matrix Kronecker. $\vec{e}$ expresses a column vector of the $\vec{e}$ straightened by column.

In the model of (1), there are two more common kinds in the B estimator class [1], [2]: One is the best linear unbiased estimator (BLUE), recorded as:

$$B^* = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}YV^{-1}Z'\left(ZV^{-1}Z'\right)^{-1}$$

and

$$Cov\left(B^*\right) = \left(ZV^{-1}Z'\right)^{-1} \otimes \left(X'\Sigma^{-1}X\right)^{-1}.$$

Another is the least squares estimation (LSE), recorded as:

$$\hat{B} = \left(X'X\right)^{-1} X'YZ'\left(ZV^{-1}Z'\right)^{-1}$$

and

$$Cov\left(\hat{B}\right) = \left(ZZ'\right)^{-1} ZVZ'\left(ZZ'\right)^{-1} \otimes \left(X'X\right)^{-1} X'\Sigma^{-1}X\left(X'X\right)^{-1}.$$

Refer to Gauss-Markov theorem: $Cov\left(B^*\right) \le Cov\left(\hat{B}\right)$.

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It is often used $\hat{B}$ instead of B^* when n is too large or V, Σ is not clear, thus creating a difference. To measure the size of difference, some relative efficiency has been studied of the LSE estimate with respect to BLUE [3]. This paper has defined four new relative efficiencies, and has given their upper bound:

$$e_1\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k} \lambda_i^p \left[Cov\left(\hat{B}\right) - Cov\left(B^*\right)\right]$$

$$e_2\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k} \lambda_i^p \left[P_w \Lambda^2 P_w - \left(P_w \Lambda^2 P_w\right)^2\right]$$

$$e_3\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k} \lambda_i^p \left[Cov\left(\hat{B}\right) Cov^{-1}\left(B^*\right)\right]$$

$$e_4\left(\hat{B}/B^*\right) = \left[tr\left[Cov\left(\hat{B}\right) Cov^{-1}\left(B^*\right)\right]^p\right]^{\frac{1}{p}}$$

There $W = Z'X$, $\Lambda = V \otimes \Sigma$, $P_w = W\left(W'W\right)^{-1}W'$, $\lambda_i(A)$ is the i-th order characteristic roots for matrix A from small to large.

II. Upper Bound of $e_1\left(\hat{B}/B^*\right)$

Mark:

$a_i = \lambda_i(V)$, $a_1 \ge \cdots \ge a_n \ge 0$;

$b_i = \lambda_i(\Sigma)$, $b_1 \ge \cdots \ge b_n \ge 0$;

$r_i = \lambda_i(X'X)$, $r_1 \ge \cdots \ge r_m \ge 0$;

$s_i = \lambda_i(X'X)$, $s_1 \ge \cdots \ge s_k \ge 0$.

When $q \ge 2k$, $a_{1i} = \left(\sqrt{a_i} - \sqrt{a_{q-i+1}}\right)^2$, $i = 1, 2, \cdots, k$

When $q < 2k$, $a_{1i} = \begin{cases} \left(\sqrt{a_i} - \sqrt{a_{q-i+1}}\right)^2 & i = 1, 2, \cdots, q-k \\ 0 & i = q-k+1, \cdots, k \end{cases}$

When $n \ge 2m$, $a_{2i} = \left(\sqrt{b_i} - \sqrt{b_{q-i+1}}\right)^2$, $i = 1, 2, \cdots, q-m$

When $n < 2m$, $a_{2i} = \begin{cases} \left(\sqrt{b_i} - \sqrt{b_{q-i+1}}\right)^2 & i = 1, 2, \cdots, q-m \\ 0 & i = q-m+1, \cdots, m \end{cases}$.

Lemma 1: Assume $\Delta = diag\left(\delta_1, \cdots, \delta_p\right)$, $\delta_1 > \cdots > \delta_p > 0$, $A_{nn} > 0$, where U is an $n \times p$ -column matrix. Then:

$$\min_{U'U=\Delta} tr\left(U'AU\right) = \sum_{i=1}^{p} \lambda_i(A)\delta_i;$$

$$\min_{U'U=\Delta} tr\left(U'AU\right) = \sum_{i=1}^{p} \lambda_{n-i+1}(A)\delta_i;$$

$$\max_{U'U=\Delta} tr\left(U'AU\right)^{-1} = \sum_{i=1}^{p} \lambda_{n-i+1}(A)\delta_{p-i+1}^{-1};$$

$$\max_{U'U=\Delta} tr\left(U'AU\right)^{-1} = \sum_{i=1}^{p} \lambda_{p-i+1}^{-1}\left(A\right)\delta_i^{-1} . [4]$$

Lemma 2: Assume $\mu_i = \lambda_i\left[U'\Lambda U - \left(U'\Lambda U\right)^{-1}\right]$, there $U_X = X\left(X'X\right)^{-1}X'$, $\Lambda = diag\left(b_1,\cdots,b_n\right)$, then $\sum_{i=1}^{t}\mu_i \le \sum_{i=1}^{t} r_i$, $1 \le t \le k$.[5]

Lemma 3: Assume A, B is the $n\times n$ -order non-negative definite matrix, then $\sum_{i=1}^{k}\lambda_i\left(AB\right) \le \sum_{i=1}^{k}\lambda_i\left(A\right)\sum_{i=1}^{k}\lambda_i\left(B\right)$.[6]

Lemma 4: Assume $X = \left(x_1, x_2, \cdots, x_m\right)$, $Y = \left(y_1, y_2, \cdots, y_m\right)$ are two vectors in the space of R^m. And $x_1 \ge \cdots \ge x_m$, $y_1 \ge \cdots \ge y_m$. I is a interval in R, $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_m \in I$, and $\sum_{i=1}^{t} x_i \le \sum_{i=1}^{t} y_i, 1 \le t \le m$ .If $\phi(x)$ is the monotony but not drop convex function, then $\sum_{i=1}^{m}\phi\left(x_i\right) \le \sum_{i=1}^{m}\phi\left(y_i\right)$. This is Advantages Theorem. [6]

Theorem 1:

$$e_1\left(\hat{B}/B^*\right) \le \sum_{i=1}^{m,k}\left[\left(\frac{a_1 b_1}{s_{k-i+1} r_{m-i+1}}\right)^p - \left(\frac{a_{q-k+1} b_{n-m+1}}{s_i r_i}\right)^p\right].$$

Proof: Assume $X = MAN$, $M'M = I_m$, $N'N = NN' = I_m$, where $A = diag\left(\sqrt{r_1}, \cdots, \sqrt{r_m}\right)$, obtained by Lemma 1:

$$\begin{aligned} &\sum_{i=1}^{m}\lambda_i\left(X'\Sigma^{-1}X\right)^{-1} = \sum_{i=1}^{m}\lambda_i\left(N'AM'\Sigma^{-1}MAN\right)^{-1} \\ &= \sum_{i=1}^{m}\lambda_i\left(AM'\Sigma^{-1}MA\right)^{-1} \ge \sum_{i=1}^{m}\lambda_{M-I+1}^{-1}\left(M'\Sigma^{-1}M\right)^{-1} r_i^{-1} \end{aligned} \quad (2)$$

Obtained by Poincare Theorem [7], [8]:

$$\lambda_{m-i+1}\left(M'\Sigma^{-1}M\right) \le \lambda_{m-i+1}\left(\Sigma^{-1}\right) = b_{n-m+i}^{-1} \quad (3)$$

Insert (3) into (2):

$$\sum_{i=1}^{m}\lambda_i\left(X'\Sigma^{-1}X\right) \ge \sum_{i=1}^{m} b_{n-m+1} r_i^{-1} \quad (4)$$

Then

$$\begin{aligned} &\sum_{i=1}^{m}\lambda_i\left[\left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1}\right] \\ &= \sum_{i=1}^{m}\lambda_i\left[\left(N'AM'MAN\right)^{-1}N'AM'\Sigma MAN\left(N'AM'MAN\right)^{-1}\right] \\ &= \sum_{i=1}^{m}\lambda_i\left(N'A^{-1}M'\Sigma MA^{-1}N\right) \\ &= \sum_{i=1}^{m}\lambda_i\left(A^{-1}M'\Sigma MA^{-1}\right) \end{aligned}$$

Obtained by Lemma 3:

$$\begin{aligned} &\sum_{i=1}^{m}\lambda_i\left(A^{-1}M'\Sigma MA^{-1}\right) \\ &\le \sum_{i=1}^{m}\lambda_i\left(A^{-2}\right)\lambda_1\left(M'\Sigma M\right) \\ &\le \sum_{i=1}^{m}\lambda_i\left(A^{-2}\right)\lambda_1\left(\Sigma\right) = \sum_{i=1}^{m} b_1 r_{m-i+1}^{-1} \end{aligned} \quad (5)$$

Obtained by Lemma 4 [9]: Assume $\phi(x) = x^p, p \ge 1$, then

$$\sum_{i=1}^{m}\lambda_i^p\left(X'\Sigma^{-1}X\right) \ge \sum_{i=1}^{m} b_{n-m+i}^p r_i^{-p} \quad (6)$$

$$\sum_{i=1}^{m}\lambda_i^p\left[\left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1}\right] \ge \sum_{i=1}^{m} b_1^p r_{m-i+1}^{-p} \quad (7)$$

By the same token:

$$\sum_{i=1}^{k}\lambda_i^p\left(Z'V^{-1}Z\right) \ge \sum_{i=1}^{m} a_{q-k+i}^p s_i^{-p} \quad (8)$$

$$\sum_{i=1}^{k}\lambda_i^p\left[\left(Z'Z\right)^{-1}Z\Sigma Z'\left(Z'Z\right)^{-1}\right] \ge \sum_{i=1}^{m} a_1^p s_{k-i+1}^{-p} \quad (9)$$

Insert (6)-(9) into:

$$\begin{aligned} &e_1\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k}\lambda_i^p\left[Cov\left(\hat{B}\right) - Cov\left(B^*\right)\right] \\ &= \sum_{i=1}^{m,k}\lambda_i^p\left[\left(ZZ'\right)^{-1}ZVZ'\left(ZZ'\right)^{-1}\otimes\left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1} \right. \\ &\left. -\left(ZV^{-1}Z'\right)\otimes\left(X'\Sigma X\right)^{-1}\right] \\ &= \sum_{i=1}^{k}\lambda_i^p\left[\left(ZZ'\right)^{-1}ZVZ'\left(ZZ'\right)^{-1}\right]\sum_{i=1}^{m}\lambda_i^p\left[\left(X'X\right)^{-1}X'\Sigma X\left(X'X\right)^{-1}\right] \\ &-\sum_{i=1}^{k}\lambda_i^p\left[\left(ZV^{-1}Z'\right)\right]\sum_{i=1}^{m}\lambda_i^p\left[\left(X'\Sigma X\right)^{-1}\right] \\ &\le \sum_{i=1}^{m,k}\left[\left(\frac{a_1 b_1}{s_{k-i+1} r_{m-i+1}}\right)^p - \left(\frac{a_{q-k+1} b_{n-m+1}}{s_i r_i}\right)^p\right] \end{aligned}$$

The theorem 1 has been proved.

III. Upper Bound of $e_2\left(\hat{B}/B^*\right)$

Mark: $a_i = \lambda_i\left(V\right), a_1 \ge \cdots \ge a_n \ge 0$;
$b_i = \lambda_i\left(\Sigma\right), b_1 \ge \cdots \ge b_n \ge 0$;

Lemma 5: $P_{A\otimes B} = P_A \otimes P_B$. [10]

Lemma 6: Assume $P_X = X\left(X'X\right)^{-1}X'$, $\Sigma = diag\left(b_1,\cdots,b_n\right)$, $\omega_i = \lambda_i\left[P_X\Sigma^2P_X - \left(P_X\Sigma P_X\right)^2\right]$, then $\sum_{i=1}^{m}\omega_i^p \le \sum_{i=1}^{\min(m,n-m)}\beta_i^{2p} \quad (p \ge 1)$, $\beta_i = \frac{1}{2}\left(b_i - b_{n-k+1}\right), i = 1, 2, \cdots, \min\left(m, n-m\right)$, there X is an $n\times m$ -column matrix of full rank . [11]

Theorem 2: $e_2\left(\hat{B}/B^*\right) \le \sum_{i=1}^{\min(m,n-m)}\beta_i^{2p}a_i^{2p}\left(p \ge 1\right)$.

Proof: Because $W = Z' \otimes X$. So [12]:

$$e_2\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k} \lambda_i^p \left[P_W \Lambda^2 P_W - \left(P_W \Lambda P_W\right)^2 \right]$$
$$= \sum_{i=1}^{m,k} \lambda_i^p \left[P_Z V^2 P_Z \otimes P_X \Sigma^2 P_X - P_Z V P_Z V P_Z \left(P_Z \Sigma P_X\right)^2 \right]$$
$$= \sum_{i=1}^{m,k} \lambda_i^p \left[P_Z V^2 \otimes \left(P_X \Sigma^2 P_X - \left(P_Z \Sigma P_X\right)^2 \right) \right]$$

Obtained by Lemma 5:

$$e_2\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k} \lambda_i^p P_Z V^2 \otimes \sum_{i=1}^{m,k} \lambda_i^p \left[P_X \Sigma^2 P_X - \left(P_Z \Sigma P_X\right)^2 \right]$$

Obtained by Lemma 6:

$$e_2\left(\hat{B}/B^*\right) \le \sum_{i=1}^{\min(m,n-m)} \beta_i^{2p} a_i^{2p} \left(p \ge 1\right).$$

IV. Upper Bound of $e_3\left(\hat{B}/B^*\right)$

Mark: $a_i = \lambda_i(V)$, $a_1 \ge \cdots \ge a_n \ge 0$;
$b_i = \lambda_i(\Sigma)$, $b_1 \ge \cdots \ge b_n \ge 0$;

When $q \ge 2k$, $a_{1i} = \left(a_i - a_{q-i+1}\right)^2 / 4a_i a_{q-i+1}$, $i = 1,2,\cdots,k$

When $q < 2k$, $a_{1i} = \begin{cases} \left(a_i - a_{q-i+1}\right)^2 / 4a_i a_{q-i+1} & i = 1,2,\cdots,k \\ 1 & i = q-k+1,\cdots,k \end{cases}$

When $n \ge 2m$, $a_{2i} = \left(b_i - b_{n-i+1}\right)^2 / 4b_i b_{n-i+1}$, $i = 1,2,\cdots,m$

When $n < 2m$, $a_{2i} = \begin{cases} \left(b_i - b_{n-i+1}\right)^2 / 4b_i b_{n-i+1} & i = 1,2,\cdots,m \\ 1 & i = n-m+1,\cdots,m \end{cases}$

Lemma 7: If $Y_{n1} = X_{nm}\beta_{m1} + \varepsilon$, $Cov(\varepsilon) = \sigma^2\Sigma > 0$, $r(X) = m \le n$, $B^* = \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}Y$, $\hat{B} = \left(X'X\right)^{-1} X'Y$, $E(\varepsilon) = 0$, then $\sum_{i=1}^{m} \lambda_i^p \left[\left(X'X\right)^{-1} X'\Sigma X \left(X'X\right)^{-1} \left(X'\Sigma^{-1}X\right)^{-1} \right]$ $\le \sum_{i=1}^{m} a_{2i}^p$. [13]

Lemma 8: Assume $A_i, i = 1,2$ is the m rank negative definite square matrix, $D_i, i = 1,2$ is the n rank negative definite square matrix, then $\lambda_i\left(A_1 \otimes D_1\right) = \lambda_i\left(A_1\right) \otimes \lambda_i\left(D_1\right)$, $\lambda_i\left(A_1 A_2\right) \le \lambda_i\left(A_1\right)\lambda_i\left(A_2\right)$, $i = 1,2,\cdots,m$. [14]

Theorem 3: $e_3\left(\hat{B}/B^*\right) \le \sum_{i=1}^{m,k} a_{1i}^p a_{2i}^p$

Proof: $e_3\left(\hat{B}/B^*\right) = \sum_{i=1}^{m,k} \lambda_i^p \left[Cov\left(\hat{B}\right) Cov^{-1}\left(B^*\right) \right]$

$$= \sum_{i=1}^{m,k} \lambda_i^p \left[\left(\left(ZZ'\right)^{-1} ZVZ' \left(ZZ'\right)^{-1} \otimes \left(XX'\right)^{-1} X'\Sigma X \left(X'X\right)^{-1} \right) \right.$$
$$\left. \left(\left(ZV^{-1}Z'\right)^{-1} \otimes \left(X'\Sigma^{-1}X\right)^{-1} \right)^{-1} \right]$$
$$= \sum_{i=1}^{m,k} \lambda_i^p \left[\left(\left(ZZ'\right)^{-1} ZVZ' \left(ZZ'\right)^{-1} ZV^{-1}Z' \right) \otimes \right.$$
$$\left. \left(\left(XX'\right)^{-1} X'\Sigma X \left(X'X\right)^{-1} X'\Sigma^{-1}X \right) \right]$$
$$= \sum_{i=1}^{k} \lambda_i^p \left[\left(ZZ'\right)^{-1} ZVZ' \left(ZZ'\right)^{-1} ZV^{-1}Z' \right] \otimes$$
$$\sum_{i=1}^{m} \lambda_i^p \left[\left(XX'\right)^{-1} X'\Sigma X \left(X'X\right)^{-1} X'\Sigma^{-1}X \right]$$

Obtained by Lemma 7:

$$\sum_{i=1}^{k} \lambda_i^p \left[\left(ZZ'\right)^{-1} ZVZ' \left(ZZ'\right)^{-1} ZV^{-1}Z' \right] \le \sum_{i=1}^{k} a_{1i}^p$$
$$\sum_{i=1}^{m} \lambda_i^p \left[\left(XX'\right)^{-1} X'\Sigma X \left(X'X\right)^{-1} X'\Sigma^{-1}X \right] \le \sum_{i=1}^{m} a_{2i}^p$$

To sum up: $e_3\left(\hat{B}/B^*\right) \le \sum_{i=1}^{m,k} a_{1i}^p a_{2i}^p$.

V. Upper Bound of $e_4\left(\hat{B}/B^*\right)$

Mark: $a_i = \lambda_i(V)$, $a_1 \ge \cdots \ge a_n \ge 0$;
$b_i = \lambda_i(\Sigma)$, $b_1 \ge \cdots \ge b_n \ge 0$;

When $q \ge 2k$, $a_{1i} = \left(a_i - a_{q-i+1}\right)^2 / 4a_i a_{q-i+1}$, $i = 1,2,\cdots,k$.

When $q < 2k$, $a_{1i} = \begin{cases} \left(a_i - a_{q-i+1}\right)^2 / 4a_i a_{q-i+1} & i = 1,2,\cdots,k \\ 1 & i = q-k+1,\cdots,k \end{cases}$

When $n \ge 2m$, $a_{2i} = \left(b_i - b_{n-i+1}\right)^2 / 4b_i b_{n-i+1}$, $i = 1,2,\cdots,m$

When $n < 2m$, $a_{2i} = \begin{cases} \left(b_i - b_{n-i+1}\right)^2 / 4b_i b_{n-i+1} & i = 1,2,\cdots,m \\ 1 & i = n-m+1,\cdots,m \end{cases}$

Lemma 9: Assume $\mu_i = \lambda_i\left[U'\Lambda UU'\Lambda^{-1}U\right]$, where U is an $n \times p$ -column matrix, and $U'U = I_m$, $\Lambda = diag\left(b_1,\cdots,b_n\right)$, $b_1 \ge \cdots \ge b_n > 0$, then $\sum_{i=1}^{t} \mu_i \le \sum_{i=1}^{t} a_{1t}$, $t = 1,\cdots,k$. [15]

Theorem 4: $e_4\left(\hat{B}/B^*\right) \le k^{\frac{1}{p}} \left(\sum_{i=1}^{m} a_{2i}^p \right)^{\frac{1}{p}}$.

Proof:

$$e_4\left(\hat{B}/B^*\right)=\left[tr\left[Cov\left(\hat{B}\right)Cov^{-1}\left(B^*\right)\right]^p\right]^{\frac{1}{p}}$$
$$=\left(tr\left[\left((ZZ')^{-1}ZVZ'(ZZ')^{-1}\otimes(X'X)^{-1}\right)\cdot\right.\right.$$
$$\left.\left.\left((ZV^{-1}Z')\otimes(X'X)\right)\right]^p\right)^{\frac{1}{p}} \qquad (10)$$
$$=\left(tr\left[(ZZ')^{-1}ZVZ'(ZZ')^{-1}\left(ZV^{-1}Z'\right)\otimes I_m\right]^p\right)^{\frac{1}{p}}$$
$$=k^{\frac{1}{p}}\left[tr\left[(ZZ')^{-1}ZVZ'(ZZ')^{-1}\left(ZV^{-1}Z'\right)\right]^p\right]^{\frac{1}{p}}$$

By thinking the following:

$$tr\left[(ZZ')^{-1}ZVZ'(ZZ')^{-1}\left(ZV^{-1}Z'\right)\right]^p=$$
$$\sum_{i=1}^{m}\lambda_i^p\left[(ZZ')^{-1}ZVZ'(ZZ')^{-1}\left(ZV^{-1}Z'\right)\right]$$

Because $V>0$, there is N which is satisfied that $N'N=NN'=I$, $V=N\Lambda N'$, $\Lambda=diag\left(b_1,\cdots,b_n\right)$, and let $M=ZN$, $U=M'(MM')^{-\frac{1}{2}}$, $U'U=I_m$, then it is believed that:

$$\lambda_i\left[(ZZ')^{-1}ZVZ'(ZZ')^{-1}\left(ZV^{-1}Z'\right)\right]$$
$$=\lambda_i\left[(ZNN'Z')^{-1}ZN\Lambda N'Z'(ZNN'Z')^{-1}\left(ZN\Lambda^{-1}Z'\right)\right]$$
$$=\lambda_i\left[(MM')^{-1}MVM'(MM')^{-1}\left(M\Lambda^{-1}M'\right)\right]$$
$$=\lambda_i\left(U'\Lambda UU'\Lambda^{-1}U\right)=\mu_i$$

Obtained by the Lemma 9: $\sum_{i=1}^{t}\mu_i\le\sum_{i=1}^{t}a_{2i}$, $1\le t\le m$. Assume $\varphi(x)=x^p$, $p\ge 1$, there

$$\sum_{i=1}^{m}\mu_i^p\le\sum_{i=1}^{m}a_{i2}^p \qquad (11)$$

Insert (11) into (10): $e_4\left(\hat{B}/B^*\right)\le k^{\frac{1}{p}}\left(\sum_{i=1}^{m}a_{2i}^p\right)^{\frac{1}{p}}$. The Theorem 4 has been proved.

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