# A Simplified Distribution for Nonlinear Seas

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**Abstract**—The exact theoretical expression describing the probability distribution of nonlinear sea-surface elevations derived from the second-order narrowband model has a cumbersome form that requires numerical computations, not well-disposed to theoretical or practical applications. Here, the same narrowband model is reexamined to develop a simpler closed-form approximation suitable for theoretical and practical applications. The salient features of the approximate form are explored, and its relative validity is verified with comparisons to other readily available approximations, and oceanic data.

*Keywords*—Ocean waves, probability distributions, second-order nonlinearities, skewness coefficient, wave steepness.

## I. INTRODUCTION

THE second-order narrowband model describing longcrested surface waves observed in time t at a fixed point in deep water is given by [1]

$$\eta = \eta_1 + \eta_2 = \xi \cos \chi + \frac{1}{2} \varepsilon \xi^2 \cos 2\chi.$$
 (1)

 $\eta_1$  is the linear Gaussian surface elevation,  $\xi$  amplitude or envelope of  $\eta_1$ , and  $\eta_2$  represents second-order nonlinear corrections, all scaled with the root-mean-square (*rms*)  $\sigma \sigma \eta_1$ . It is assumed that the spectral density of  $\eta_1$  is unimodal and narrowband over frequencies and directions. The *rms* of  $\eta_1$  is also  $\sigma$ , correct to  $O(\varepsilon)$ . The dimensionless parameter  $\varepsilon \ll 1$ and defined as  $\varepsilon = \sigma k_m$ , where  $k_m$  = spectral-mean wavenumber, corresponds to the *rms* gradient of  $\eta_1$ . In the context of narrowband approximation, it serves as an integral measure wave steepness. Further,  $\chi = \omega_m t + \phi(t) =$  total wave phase,  $\omega_m$  = spectral-mean frequency such that  $k_m = \omega_m^2 / g$ , g= gravitational acceleration, and  $\phi$ =wave phase. Finally,  $\xi$  is Rayleigh distributed while  $\chi$  and therefore  $\phi$  are uniformly random over an interval of  $2\pi$ .

Representation (1) has been used successfully to describe the distributions of nonlinear surface elevations and various other surface features, including wave crest and trough amplitudes, wave heights, envelopes and phases in nonlinear seas successfully in [4]-[13] and a variety other practical and theoretical studies, with extension and generalizations to large waves and shallower water depths in [14], [15]. Despite the apparent simplicity of  $\eta$ , its probability density function  $p_{\eta}$  cannot be obtained in an explicit closed form [1]. Therefore, its applications rely on either rather cumbersome numerical computations or require further approximations. One particular approximation that has been devised in [2] and often used in practical and theoretical studies has the form

$$p_{\eta}(z) \approx \frac{(1 - 7\varepsilon^2 / 8) \exp(-G^2 / 2\varepsilon^2)}{\sqrt{2\pi (1 + 3G + 2G^2)}},$$
 (2)

where  $G = \sqrt{1+2\varepsilon z} - 1$  and  $z > -3/8\varepsilon$ . A second and more conventional alternative that follows from expanding  $p_{\eta}$  in a Gram-Charlier [3] series is given, correct to  $\mathcal{O}(\varepsilon)$ , by

$$p_{\eta}(z) \approx \frac{\exp(-z^2/2)}{\sqrt{2\pi}} \left[ 1 + \frac{\lambda_3}{6} z (z^2 - 3) \right],$$
 (3)

where  $|z| < \infty$  and  $\lambda_3 = < \eta^3 >$  is the skewness coefficient. For (1),  $\lambda_3 = 3\varepsilon$  to  $O(\varepsilon^2)$  so that  $\varepsilon = \lambda_3/3$ . In oceanic wind seas, including those generated by extreme storms  $0 < \lambda_3 < 0.3$  typically. The preceding expression then suggests that  $0 < \varepsilon < 0.1$  as a typical range.

Since the first-order linear  $\eta_1$  is zero-mean Gaussian, its marginal probability density p and cumulative distribution P are given, respectively, by

$$p(x) = \frac{\exp(-x^2/2)}{\sqrt{2\pi}}; |x| < \infty,$$
 (4)

$$P(x) = \int_{-\infty}^{x} \frac{\exp(-u^2/2)}{\sqrt{2\pi}} \, du.$$
 (5)

Similarly, the conjugate  $\hat{\eta}_1 = \xi \sin \chi$  is also zero-mean Gaussian, with the marginal density and distribution identically described by (5) and (6), respectively. Finally, because  $\hat{\eta}_1$  and  $\eta_1$  are uncorrelated, their joint probability density reduces to a product of their marginal densities.

It is noted that while both (2) and (3) are simple enough, (2) has a somewhat restricted domain. In contrast, the domain over which (3) is non-negative is restricted to  $\eta \ge -(\alpha^+ + \alpha^-)$ , where  $\alpha^{\pm} = \{ [1 \pm (1 - \varepsilon^2)^{1/2}]/\varepsilon \}^{1/3}$ .

Evidently, the latter expression can also be expressed in terms of  $\lambda_3$  since  $\varepsilon = \lambda_3/3$ . For instance, when  $\lambda_3 = 0.3$ , (3) is non-negative provided that  $\eta \ge -3.08$  whereas (2) is valid

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for  $\eta > -3/8\varepsilon = -3.75$ , a somewhat larger domain than that of (3). In addition, both (2) and (3) are not entirely zero-mean or normalized to unity over their restricted domains.

This study represents an attempt to develop a new simpler approximation for the density and cumulative distribution of (1), and explores its salient features relative to (2) and (3). Eventually, the new approximation is compared with (2), (3) and oceanic measurements to investigate its relative validity.

## II. APPROXIMATE PROBABILITY STRUCTURE

For simplicity in notation, (1) is rewritten as

$$\eta = z(x, y) = x + \frac{1}{2}\varepsilon (x^2 - y^2), \tag{6}$$

where  $x = \xi \cos \chi$  and  $y = \xi \sin \chi$ . It is recalled that the joint probability density of x and y is a product of two marginal standard Gaussian densities, each of the form (4). Consequently, the conditional density of  $\eta$ , given y, follows from a simple change of variables as

$$p_{\eta|y}(z \mid y) = p(x) \left| \frac{\partial x}{\partial z} \right|_{x=z^{-1}(x|y)},$$
(7)

where

$$x = z^{-1}(x \mid y) = (-1 + \sqrt{1 + 2\varepsilon z + \varepsilon^2 y^2}) / \varepsilon,$$
(8)

and

$$\partial x / \partial z |_{x=z^{-1}(x|y)} = (1 + \varepsilon x)^{-1} |_{x=z^{-1}(x|y)}.$$
 (9)

The approximation

$$x = z^{-1}(x \mid y) \approx (-1 + \sqrt{1 + 2\varepsilon z}) / \varepsilon = G / \varepsilon,$$
(10)

can now be used to rewrite (9) as

$$\left. \partial x \, / \, \partial \eta \right|_{x=z^{-1}(x|y)} \approx 1 \, / \, \sqrt{1 + 2\varepsilon \, z} \,. \tag{11}$$

Finally, the substitutions of (10) and (11) in (7) leads to an expression independent of y. As a result, (7) reduces to the approximation sought in the form

$$p_{\eta}(z) \approx \exp[-G^2/2\varepsilon^2)/(1+G)\sqrt{2\pi},$$
 (12)

The corresponding cumulative distribution is given by

$$P_{\eta}(z) \approx P(G/\varepsilon) - P(-1/\varepsilon). \tag{13}$$

Both expressions are defined for  $z > -1/2\varepsilon$ , a wider domain than either (2) or (3) in particular. The normalization of (12) and (13) requires that

$$\lim_{z \to \infty} P_{\eta}(z) = 1 - P(-1/\varepsilon) = P(1/\varepsilon) = 1$$
(14)

For  $1/\varepsilon >>1$ , the asymptotic expansion of  $P(1/\varepsilon)$  gives [16]

$$P(1/\varepsilon) = 1 - \varepsilon \exp(-1/2\varepsilon^2)(1 - \varepsilon^2 + \dots)/\sqrt{2\pi}.$$
 (15)

Therefore,

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$$P(1/\varepsilon) = 1 - O(\varepsilon e^{-1/2\varepsilon^2}).$$
(16)

The variation of  $\beta = 1/P(1/\varepsilon) = 1/P(3/\lambda_3)$  with  $\lambda_3 = 3\varepsilon$  is shown in Fig. 1. It is seen that that  $\beta = 1$  for  $0 < \lambda_3 < 0.6$ , a much wider range than the typical range  $0 < \lambda_3 < 0.3$ observed under oceanic conditions. For instance, if  $\lambda_3 = 0.3$ , then  $\beta = 1 + O(10^{-16})$ . For  $\lambda_3 = 0.6$ ,  $\beta = 1 + O(10^{-7})$ . Therefore, there is essentially no need for introducing any normalization in (12) or (13) for oceanic applications.



Fig. 1 Variation of normalization parameter  $\beta$  with  $\lambda 3 = 3 \varepsilon$ 

The mean of  $\eta$  implied by (13) is given by

$$<\eta>=\varepsilon/2+O(e^{-1/2\varepsilon^2})\approx\varepsilon/2.$$
 (17)

The approximate density (13) can now be centralized to zero mean by replacing z with  $z + \varepsilon/2$  to obtain

$$p_n(z) \approx \exp[-H^2/2\varepsilon^2) / (1+H)\sqrt{2\pi},$$
 (18)

where  $z > -(1 + \varepsilon^2)/2\varepsilon$ , and

$$H = \sqrt{1 + 2\varepsilon \left(z + \varepsilon/2\right)} - 1. \tag{19}$$

The corresponding cumulative distribution is obtained by

replacing G in (14) with H as

$$P_{\eta}(z) \approx P(H/\varepsilon) - P(-1/\varepsilon) \sim P(H/\varepsilon).$$
(20)

The mean-square of  $\eta$  follows after some algebra as

$$<\eta^2>=1+3rac{\varepsilon^2}{4}+O(e^{-1/2\varepsilon^2)}/\varepsilon)\approx 1+O(\varepsilon^2).$$
 (21)

So, the corresponding variance and *rms* are given by, respectively,

$$\operatorname{var}(\eta) = 1 + \varepsilon^2 / 2 = 1 + O(\varepsilon^2),$$
 (22)

and

$$\sigma_n = 1 + \varepsilon^2 / 4 = 1 + O(\varepsilon^2). \tag{23}$$

Finally, the skewness coefficient associated with the final form of the approximate density (19) is given by

$$<\eta^3>=3\varepsilon+O(e^{-1/2\varepsilon^2}/\varepsilon^2)\approx\lambda_3,$$
 (25)

as it is to be expected.

## III. COMPARISONS

The comparisons between the final form of present approximation (18) with (2), (3) and (4) are shown in Fig. 2, where  $\lambda_3 = 0.3$ . Numerical comparisons indicate that the present approximation is by and large the same as the exact density such that the differences between the two cannot be differentiated in this figure. It is evident that the present approximation (18) is also the same as (2) except over large negative values at and beyond  $z = -3/8\varepsilon = -3.75$ , where (2) has a singularity. The differences between (2) and/or (18) and Gram-Charlier series (3) are more noticeable, particularly over the range of relatively large negative as well as positive values. Further, (3) becomes negative for z < -3.08. In contrast with (2) and/or (3), (18) is well-behaved and valid up to z = -5.05, where it also has a singularity. In oceanic applications where  $0 < \lambda_3 < 0.3$ , it has already been shown above that the domain  $-(1 + \varepsilon^2)/2\varepsilon < z < \infty$  of (18) contains practically the whole probability mass over it.

The comparisons with oceanic data will consider two sets of wave measurements. These datasets represent oceanic waves measured during severe storms, one set comprising surface measurements collected by a wave staff from an offshore platform in the Gulf of Mexico during the passage of hurricane Camille in August, 1969, and the other measurements gathered by a Marex radar from the Tern platform in the northern North Sea in January, 1993. These data will be denoted as Camille and Tern93, respectively.

Both Tern93 and Camille are characterized with nonlinear distortions and rather large waves with heights nearly as large

as 25 m in Tern93 and 27 m in Camille. Tern93 is suggestive of a relatively steady sea state whereas Camille is conspicuously not so, judging from the temporal variation of  $\sigma$ from about 1.85 m almost linearly to about 3.5 m as the hurricane approached the measurement site.

There is no rigorous analytical proposition of practical value for analyzing the statistical structure of non-stationary random functions based on a single realization. To compensate for the general temporal variability of  $\sigma$ , surface displacements from the mean-zero level are estimated in both cases from 30minute consecutive segments and scaled with the corresponding segmental estimates of  $\sigma$ . The parameters in Table I represent overall averages derived from the segmental estimates. The variations of segmental estimates of  $\sigma$  and  $\lambda_3$ are displayed in Fig. 3.

The comparisons of the density estimates observed in Tern93 and Camille to the Gaussian form of (4) and the present approximation (18) are shown in Figs. 4 and 5, respectively. It appears that the present approximation does reasonably well in describing the observed data.



Fig. 2 Comparisons of probability densities predicted from (2), (3), present approximation (19) and Gaussian density (5) for  $\lambda_3 = 0.30$ 

TABLE I PARTICULARS OF OCEANIC MEASUREMENTS		
description	Tern93	Camille
duration (h)	9.00	6.30
Sampling rate (Hz)	5.12	10.0
$\sigma_{\rm avg}({ m m})$	3.02	2.61
$\lambda_{3 avg}$	0.17	0.15
$\varepsilon \sim \lambda_{3 \text{ avg}} / 3$	0.06	0.05

## **IV. CONCLUSIONS**

The second-order narrowband model describing longcrested deep-water waves was used to develop a simple and yet fairly accurate approximation to the probability structure of sea-surface elevations.

Comparisons with the two sets of measurements representative of nonlinear storm seas indicate that the approximation developed works fairly well in representing the observed data. The comparisons of the approximation with two presently popular approximations also suggest that it has advantages over these in terms of its relative simplicity and, in particular, its validity over a wider range of surface elevations.



Fig. 3 Segmental variations of (a)  $\sigma$  and (b)  $\lambda_3$  in Tern93 and Camille



Fig. 4 Tern93: the density estimates observed (points) compared to Gaussian density (5) and present approximation (19)



Fig. 5 Same as Fig 4 except for Camille

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#### REFERENCES

- M. A. Tayfun, "Narrow-band nonlinear sea waves," J. Geophys. Res., vol. 85, pp. 1548–1552, 1980.
- [2] H. Socquet-Juglard, K. Dysthe, K. Trulsen, H. Krogstad, and J. Liu, "Probability distributions of surface gravity waves during spectral changes," J. Fluid Mech., vol. 542, 195–216, 2005.
- [3] M. S. Longuet-Higgins, "The effects of nonlinearities on statistical distributions in the theory of sea waves," J. Fluid Mech. vol. 17, pp. 459–480, 1963.
- [4] M. A. Tayfun, and J-M. Lo, "Envelope, phase, and narrowband models of sea waves," J. Waterw. Port, Coast. Ocean Eng., vol. 115, pp. 594-613, 1990.
- [5] M. A. Tayfun, "Statistics of nonlinear wave crests and groups," *Ocean Eng.*, vol. 33, 1589–1622, 2006.
- [6] M. A. Tayfun, and F. Fedele, "Wave-height distributions and nonlinear effects," *Ocean Eng.*, vol. 34, 1631 – 1649, 2007.
- [7] M. A. Tayfun, and F. Fedele, "Expected shape of extreme waves in storm seas," in *Proc. 26th Inter. Conf. on Offshore Mech. & Arctic Eng.*, San Diego, paper no. OMAE2007-29073, pp. 1-8, 2007.
- [8] M. A. Tayfun, "Distributions of envelope and phase in wind waves," J. Phys. Oceanogr., vol. 38, pp. 2784–2800, 2008.
- [9] A. Toffoli, E. Bitner-Gregersen, M. Onorato, A. R. Osborne, and A. V. Babanin, "Surface gravity waves from direct numerical simulations of the Euler equations: A comparison with second-order theory," *Ocean Eng.*, vol. 35, 367–379, 2008.
- [10] Z. Cherneva, M. A. Tayfun, and C. Guedes-Soares, "Statistics of nonlinear waves generated in an offshore wave basin", *J. Geophys. Res.*, vol. 114, C08005, 2009.
- [11] F. Fedele, and M. A. Tayfun, "On nonlinear wave groups and crest statistics," J. Fluid Mech., vol. 620, 221–239, 2009.
- [12] F. Fedele, Z. Cherneva, M. A. Tayfun, and C. Guedes-Soares, "Nonlinear Schrödinger invariants and wave statistics," *Phys. Fluids*, vol. 22, 2010.
- [13] F. Arena, and F. Fedele, "A family of narrow-band nonlinear stochastic processes for the mechanics of sea waves," *Eur. J. Mech. B/Fluids*, vol. 21, pp. 125–137, 2005.
- [14] A. K. Jha, and S. R. Winterstein,"Non-linear random ocean waves: prediction and comparison with data," in *Proc. ETCE/OMAE2000 Joint Conference Energy for the New Millenium*, vol. 1, paper OMAE2000-6125, 2000, ISBN: 0791819949, 9780791819944.
- [15] M. Prevosto, H. E. Krogstad, and A. Robin, "Probability distributions for maximum wave and crest heights," *Coastal Eng.*, vol. 40, 329–360, 2000.
- [16] M. Abramowitz, and I. A. Stegun, Handbook of Mathematical Functions, Dover Publications, New York, 1968, p. 932.