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Preconditioned Generalized Accelerated Overrelaxation Methods for Solving Certain Nonsingular Linear System

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Abstract**—**In this paper, we present preconditioned generalized accelerated overrelaxation (GAOR) methods for solving certain nonsingular linear system. We compare the spectral radii of the iteration matrices of the preconditioned and the original methods. The comparison results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent. Finally, we give two numerical examples to confirm our theoretical results.

Keywords**—**Preconditioned, GAOR method, linear system, convergence, comparison.

I Introduction

SOMETIMES, one has to solve a nonsingular linear system

$$Ay = f, \tag{1}$$

where

$$A = \begin{pmatrix} I-B & H \\ K & I-C \end{pmatrix}.$$

Here B and C are square nonsingular diagonal matrices of order n_1 and n_2, respectively, $H \in R^{n_1 \times n_2}$ and $K \in R^{n_2 \times n_1}$.

Yuan proposed a generalized SOR (GSOR) method to solve linear system (1) in [1]; afterwards, Yuan and Jin [2] established a generalized AOR (GAOR) method to solve linear system (1). In [3]-[5], authors studied the convergence of the GAOR method for solving the linear system (1). In [3], authors studied the convergence of the GAOR method when the coefficient matrices are consistently ordered matrices and gave the regions of convergence. In [4], authors studied the convergence of the GAOR method for diagonally dominant coefficient matrices and gave the regions of convergence. In [5], authors studied the convergence of GAOR method for strictly doubly diagonally dominant coefficient matrices and gave the regions of convergence.

In order to solve the linear system (1) using the GAOR method, we split H as

$$A = I - \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix} - \begin{pmatrix} B & -H \\ 0 & C \end{pmatrix}.$$

Then, for $\omega \neq 0$, one GAOR method can be defined by

$$y^{(k+1)} = L_{r,\omega} y^{(k)} + \omega g, k = 0,1,2,\cdots,$$

where

$$L_{r,\omega} = \begin{pmatrix} I & 0 \\ rK & I \end{pmatrix}^{-1} \left((1-\omega) I + (\omega - r) \begin{pmatrix} 0 & 0 \\ -K & 0 \end{pmatrix} + \omega \begin{pmatrix} B & -H \\ 0 & C \end{pmatrix} \right)$$
$$= \begin{pmatrix} (1-\omega) I + \omega B & -\omega H \\ \omega(r-1)K - \omega r K B & (1-\omega) I + \omega C + \omega r K H \end{pmatrix}$$

is the iteration matrix and

$$g = \begin{pmatrix} I & 0 \\ -rK & I \end{pmatrix} f.$$

In order to decrease the spectral radius of $L_{r,\omega}$, an effective method is to precondition the linear system (1), namely,

$$PA = \begin{pmatrix} I-B^* & H^* \\ K^* & I-C^* \end{pmatrix}$$

then the preconditioned GAOR method can be defined by

$$y^{(k+1)} = L^*_{r,\omega} y^{(k)} + \omega g^*, k = 0,1,2,\cdots, \tag{2}$$

where

$$L^*_{r,\omega} = \begin{pmatrix} (1-\omega) I + \omega B^* & -\omega H^* \\ \omega(1-r)K^* - \omega r K^* B^* & (1-\omega) I + \omega C^* + \omega r K^* H^* \end{pmatrix}$$

and

$$g^* = \begin{pmatrix} I & 0 \\ -rK^* & I \end{pmatrix} Pf.$$

This paper is organized as follows. In Section II, we propose new preconditioners and give the comparison theorems between the preconditioned and original methods. These results show that the preconditioned GAOR methods converge faster than the GAOR method whenever the GAOR method is convergent. In Section III, we give two examples to confirm our theoretical results.

We need the following definition:

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Definition 1: Let $A=(a_{ij})_{n\times n}$, and $B=(b_{ij})_{n\times n}$. We say $A>B$ if $a_{ij}>b_{ij}$ for all $i,j=1,2,\cdots,n.$ We say $A\geq B$ if $a_{ij}\geq b_{ij}$ for all $i,j=1,2,\cdots,n.$

In this paper, $\rho(\cdot)$ denotes the spectral radius of a square matrix.

Lemma 1 [6], [7]: Let $A\in R^{n\times n}$ be nonnegative and irreducible. Then

(i) A has a positive real eigenvalue equal to its spectral radius $\rho(A)$;

(ii) for $\rho(A)$, there corresponds an eigenvector $x>0$;

(iii) if $0\neq \alpha x\leq Ax\leq \beta x,\quad \alpha x\neq Ax,\quad Ax\neq \beta x$ for some nonnegative vector x, then $\alpha<\rho(A)<\beta$ and x is a positive vector.

II COMPARISON RESULTS

Let $A\in R^{n\times n}$. We denote by $A\geq 0$ a nonnegative matrix, $|A|$ the absolute value of matrix A, $\rho(A)$ the spectral radius of A, and $<A>$ the comparison matrix of A.

In [8], [9], authors presented several kinds of preconditioners for preconditioned GAOR method to solve systems of linear equations. They showed that the convergence rate of the preconditioned modified AOR methods is better than that of the original method, whenever the original method is convergent.

In this paper, we consider the preconditioned linear system

$$(I+\tilde{S})Ay=(I+\tilde{S})f, \tag{3}$$

with

$$\tilde{S}=\begin{pmatrix}0&0\\S&0\end{pmatrix},$$

S is a $n_2\times n_1$ matrix. We take S as follows:

$$S_1=\begin{pmatrix}0&0&\cdots&0&0\\0&0&\ddots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\ddots&0&0\\-\frac{K_{n_2,1}}{\alpha}&0&\cdots&0&0\end{pmatrix},\quad S_2=\begin{pmatrix}-K_{11}&0&\cdots&0\\-K_{21}&0&\cdots&0\\\vdots&\vdots&\ddots&0\\-K_{n_2,1}&0&\cdots&0\end{pmatrix}.$$

and let S_3 as

$$S_3=\begin{pmatrix}-K_{11}&0&\cdots&0\\0&-K_{22}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&-K_{n_2,n_2}\end{pmatrix},\ (n_1=n_2)$$

or

$$S_3=\begin{pmatrix}-K_{11}&0&\cdots&0&0&\cdots&0\\0&-K_{22}&\cdots&0&0&\cdots&0\\\vdots&\vdots&\ddots&0&\vdots&\ddots&\vdots\\0&0&\cdots&-K_{n_2,n_2}&0&\cdots&0\end{pmatrix},\ (n_1>n_2)$$

or

$$S_3=\begin{pmatrix}-K_{11}&0&\cdots&0\\0&-K_{22}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&-K_{n_1,n_1}\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix},(n_1<n_2).$$

Then the preconditioned GAOR methods for solving (3) are defined as follows:

$$y^{(k+1)}=L_{r,\omega}^{(i)}y^{(k)}+\omega\tilde{g},k=0,1,2,\cdots, \tag{4}$$

where for $i=1,2,3,$

$$L_{r,\omega}^{(i)}=\begin{pmatrix}I&0\\r(K+S_i(I-B))&I\end{pmatrix}^{-1}\left((1-\omega)I+(\omega-r)\begin{pmatrix}0&0\\-(K+S_i(I-B))&0\end{pmatrix}+\omega\begin{pmatrix}B&-H\\0&C-S_iH\end{pmatrix}\right)$$

are iteration matrices and

$$\tilde{g}=\begin{pmatrix}I&0\\-r(K+S_i(I-B))&I\end{pmatrix}\tilde{f}.$$

Now, we consider new preconditioners P_i^*

$$P_i^*=\begin{pmatrix}I&V_i\\S_i&I\end{pmatrix},i=1,2,3,$$

where S_i are defined as above, and V_i are defined as:

$$V_1=\begin{pmatrix}0&0&\cdots&0&0\\0&0&\ddots&0&0\\\vdots&\ddots&\ddots&\ddots&\vdots\\0&0&\ddots&0&0\\-\frac{H_{n_1,1}}{\beta}&0&\cdots&0&0\end{pmatrix},\quad V_2=\begin{pmatrix}-H_{11}&0&\cdots&0\\-H_{21}&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\-H_{n_1,1}&0&&0\end{pmatrix},$$

and let V_3 as

$$V_3=\begin{pmatrix}-H_{11}&0&\cdots&0\\0&-H_{22}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&-H_{n_1n_1}\end{pmatrix},(n_1=n_2)$$

or

$$V_3=\begin{pmatrix} -H_{11} & 0 & \cdots & 0 \\ 0 & -H_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -H_{n_2,n_2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix},(n_1>n_2)$$

or

$$V_3=\begin{pmatrix} -H_{11} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -H_{22} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -H_{n_1,n_1} & 0 & \cdots & 0 \end{pmatrix},(n_1<n_2).$$

Then

$$\widetilde{A}_i^*=P_i^*A=\begin{pmatrix} I-(B-V_iK) & H+V_i(I-C) \\ K+S_i(I-B) & I-(C-S_iH) \end{pmatrix},$$

where

$$K+S_1(I-B)=\begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1,n_1} \\ K_{21} & K_{22} & \cdots & K_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n_2,1}(1-\frac{1-B_{11}}{\alpha}) & K_{n_2,2} & \cdots & K_{n_2,n_1} \end{pmatrix},$$

$$K+S_2(I-B)=\begin{pmatrix} K_{11}B_{11} & K_{12} & \cdots & K_{1,n_1} \\ K_{21}B_{11} & K_{22} & \cdots & K_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n_2,1}B_{11} & K_{n_2,2} & \cdots & K_{n_2,n_1} \end{pmatrix},$$

$$H+V_1(I-C)=\begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1,n_2} \\ H_{21} & H_{22} & \cdots & H_{2,n2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n_1,1}(1-\frac{1-C_{11}}{\beta}) & H_{n_1,2} & \cdots & H_{n_1,n_2} \end{pmatrix},$$

$$H+V_2(I-C)=\begin{pmatrix} H_{11}C_{11} & H_{12} & \cdots & H_{1,n_2} \\ H_{21}C_{11} & H_{22} & \cdots & H_{2,n2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n_1,1}C_{11} & H_{n1,2} & \cdots & H_{n1,n_2} \end{pmatrix}.$$

Let $K+S_3(I-B)$ and $H+V_3(I-C)$ as

$$K+S_3(I-B)=\begin{pmatrix} K_{11}B_{11} & K_{12} & \cdots & K_{1,n_1} \\ K_{21} & K_{22}B_{22} & \cdots & K_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n_2,1} & K_{n_2,2} & \cdots & K_{n_2,n_2}B_{n_1,n1} \end{pmatrix},(n_1=n_2)$$

$$H+V_3(I-C)=\begin{pmatrix} H_{11}C_{11} & H_{12} & \cdots & H_{1,n_2} \\ H_{21} & H_{22}C_{22} & \cdots & H_{2,n_2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n_1,1} & H_{n_1,2} & \cdots & H_{n_1,n1}C_{n_2,n_2} \end{pmatrix},(n_1=n_2)$$

or

$$K+S_3(I-B)=\begin{pmatrix} K_{11}B_{11} & K_{12} & \cdots & K_{1,n_2} & \cdots & K_{1,n1} \\ K_{21} & K_{22}B_{22} & \cdots & K_{2,n_2} & \cdots & K_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ K_{n_2,1} & K_{n_2,2} & \cdots & K_{n_2,n_2}B_{n_2n_2} & \cdots & K_{n_2,n_1} \end{pmatrix},(n_1>n_2)$$

$$H+V_3(I-C)=\begin{pmatrix} H_{11}C_{11} & H_{12} & \cdots & H_{1,n_2} \\ H_{21} & H_{22}C_{22} & \cdots & H_{2,n_2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n_2,1} & H_{n_2,2} & \cdots & H_{n_2,n_2}C_{n_2,n_2} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n_1,1} & H_{n_1,2} & \cdots & H_{n_1,n_2} \end{pmatrix},(n_1>n_2)$$

or

$$K+S_3(I-B)=\begin{pmatrix} K_{11}B_{11} & K_{12} & \cdots & K_{1,n1} \\ K_{21} & K_{22}B_{22} & \cdots & K_{2,n_1} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1,1} & K_{n1,2} & \cdots & K_{n_1,n1}B_{n_2n_2} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n_2,1} & K_{n_2,2} & \cdots & K_{n_2,n_1} \end{pmatrix},(n_1<n_2)$$

$$H+V_3(I-C)=\begin{pmatrix} H_{11}C_{11} & H_{12} & \cdots & H_{1,n_1} & \cdots & H_{1,n_2} \\ H_{21} & H_{22}C_{22} & \cdots & H_{2,n_1} & \cdots & H_{2,n_2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ H_{n_1,1} & H_{n_1,2} & \cdots & H_{n_1,n_1}C_{n_1,n_1} & \cdots & H_{n_1,n_2} \end{pmatrix},(n_1<n_2).$$

Then the preconditioned GAOR methods for solving $P_i^*Ay=P_i^*f$ are defined as follows

$$y^{(k+1)}=L_{r,\omega}^{*(i)}y^{(k)}+\omega\tilde{g}^*,k=0,1,2,\cdots,$$

where for $i=1,2,3$,

$$L_{r,\omega}^{*(i)}=\begin{pmatrix} I & 0 \\ r(K+S_i(I-B)) & I \end{pmatrix}^{-1}[(1-\omega)I+(\omega-r)\begin{pmatrix} 0 & 0 \\ -(K+S_i(I-B)) & 0 \end{pmatrix}$$
$$+\omega\begin{pmatrix} B-V_iK & -(H+V_i(I-C)) \\ 0 & C-S_iH \end{pmatrix}]$$

and

$$\widetilde{g}_i^*=\begin{pmatrix} I & 0 \\ -r(K+S_i(I-B)) & I \end{pmatrix}\tilde{f}.$$

Theorem 1: Let $L_{r,\omega},L_{r,\omega}^{*(1)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods,

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respectively. If the matrix A is irreducible with $K \geq 0$, $H \geq 0, I-B \geq 0, I-C \geq 0, 0<\omega \leq 1, 0 \leq r<1$, $\alpha>1-B_{11}, \beta>1-C_{11}$, then either

$$\rho\left(L_{r,\omega}^{*(1)}\right)<\rho\left(L_{r,\omega}\right)<1$$

or

$$\rho\left(L_{r,\omega}^{*(1)}\right)>\rho\left(L_{r,\omega}\right)>1.$$

Proof: By assumptions, it is easy to prove that both $L_{r,\omega}^{*(1)}$ and $L_{r,\omega}$ are irreducible and non-negative. By Lemma 1, there is a positive vector x such that

$$L_{r,\omega}x=\lambda x,$$

where $\lambda=\rho(L_{r,\omega})$.

Then

$$\left((1-\omega)I+(\omega-r)\begin{pmatrix}0&0\\-K&0\end{pmatrix}+\omega\begin{pmatrix}B&-H\\0&C\end{pmatrix}\right)x=\lambda\begin{pmatrix}I&0\\rK&I\end{pmatrix}x.$$

or

$$\left((1-\omega)I\right)x=\lambda\begin{pmatrix}I&0\\rK&I\end{pmatrix}x-(\omega-r)\begin{pmatrix}0&0\\-K&0\end{pmatrix}x-\omega\begin{pmatrix}B&-H\\0&C\end{pmatrix}x.$$

By substituting the above equation into the following formula

$$\begin{aligned}L_{r,\omega}^{*(1)}x-\lambda x&=\begin{pmatrix}I&0\\r(K+S_1(I-B))&I\end{pmatrix}^{-1}\left[(1-\omega)I+(\omega-r)\begin{pmatrix}0&0\\-(K+S_1(I-B))&0\end{pmatrix}\right.\\&\quad\left.+\omega\begin{pmatrix}B-V_1K&-(H+V_1(I-C))\\0&C-S_1H\end{pmatrix}\right]x-\lambda x\\&=\begin{pmatrix}I&0\\r(K+S_1(I-B))&I\end{pmatrix}^{-1}\left[(1-\omega)I+(\omega-r)\begin{pmatrix}0&0\\-(K+S_1(I-B))&0\end{pmatrix}\right.\\&\quad\left.+\omega\begin{pmatrix}B-V_1K&-(H+V_1(I-C))\\0&C-S_1H\end{pmatrix}-\lambda\begin{pmatrix}I&0\\r(K+S_1(I-B))&I\end{pmatrix}\right]x\\&=\begin{pmatrix}I&0\\r(K+S_1(I-B))&I\end{pmatrix}^{-1}\left[\begin{pmatrix}-\omega V_1K&-\omega V_1(I-C)\\-\omega S_1(I-B)&-\omega S_1H\end{pmatrix}x\right.\\&\quad\left.+(\lambda-1)\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}x\right],\end{aligned}$$

where

$$\begin{aligned}\begin{pmatrix}-\omega V_1K&-\omega V_1(I-C)\\-\omega S_1(I-B)&-\omega S_1H\end{pmatrix}x&=\begin{pmatrix}0&V_1\\S_1&0\end{pmatrix}\begin{pmatrix}-\omega(I-B)&-\omega H\\-\omega K&-\omega(I-C)\end{pmatrix}x\\&=\begin{pmatrix}0&V_1\\S_1&0\end{pmatrix}\left(\begin{pmatrix}(1-\omega)I+\omega B&-\omega H\\-(\omega-r)K&(1-\omega)I+\omega C\end{pmatrix}x-\begin{pmatrix}I&0\\rK&I\end{pmatrix}x\right)\\&=\begin{pmatrix}0&V_1\\S_1&0\end{pmatrix}\left(\lambda\begin{pmatrix}I&0\\rK&I\end{pmatrix}x-\begin{pmatrix}I&0\\rK&I\end{pmatrix}x\right)=(\lambda-1)\begin{pmatrix}0&V_1\\S_1&0\end{pmatrix}\begin{pmatrix}I&0\\rK&I\end{pmatrix}x.\end{aligned}$$

So we have

$$\begin{aligned}L_{r,\omega}^{*(1)}x-\lambda x&=\begin{pmatrix}I&0\\r(K+S_1(I-B))&I\end{pmatrix}^{-1}\\&\quad\left((\lambda-1)\begin{pmatrix}0&V_1\\S_1&0\end{pmatrix}\begin{pmatrix}I&0\\rK&I\end{pmatrix}+(\lambda-1)\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x\\&=(\lambda-1)\begin{pmatrix}I&0\\r(K+S_1(I-B))&I\end{pmatrix}^{-1}\left(\begin{pmatrix}0&V_1\\S_1&0\end{pmatrix}\begin{pmatrix}I&0\\rK&I\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x\\&=(\lambda-1)\begin{pmatrix}I&0\\-r(K+S_1(I-B))&I\end{pmatrix}\left(\begin{pmatrix}rV_1K&V_1\\S_1&0\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x.\end{aligned}$$

By assumptions, we know that

$$\begin{aligned}&-r(K+S_1(I-B))<0, rV_1K<0,\\&V_1<0, S_1<0, -rS_1(I-B)<0.\end{aligned}$$

So

$$\begin{pmatrix}I&0\\-r(K+S_1(I-B))&I\end{pmatrix}\left(\begin{pmatrix}rV_1K&V_1\\S_1&0\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)>0.$$

If $\lambda<1$, then $L_{r,\omega}^{*(1)}x-\lambda x<0$. By Lemma 1, we know that $\rho(L_{r,\omega}^{*(1)})<\rho(L_{r,\omega})<1$.

If $\lambda>1$, then $L_{r,\omega}^{*(1)}x-\lambda x>0$. By Lemma 1, we know that $\rho(L_{r,\omega}^{*(1)})>\rho(L_{r,\omega})>1$.

Corollary 1: Let $L_{r,\omega}, L_{r,\omega}^{(1)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix A is irreducible with $I-B \geq 0, I-C \geq 0, K \geq 0, H \geq 0, 0<\omega \leq 1, 0 \leq r<1$, $\alpha>1-B_{11}$, then either

$$\rho\left(L_{r,\omega}^{(1)}\right)<\rho\left(L_{r,\omega}\right)<1 \text{ or } \rho\left(L_{r,\omega}^{(1)}\right)>\rho\left(L_{r,\omega}\right)>1.$$

By the analogous proof of Theorem 1, we can prove the following two theorems.

Theorem 2 Let $L_{r,\omega}, L_{r,\omega}^{*(2)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix A is irreducible with $I-B \geq 0, I-C \geq 0, K \geq 0, H \geq 0, 0<\omega \leq 1, 0 \leq r<1$, then either

$$\rho\left(L_{r,\omega}^{*(2)}\right)<\rho\left(L_{r,\omega}\right)<1 \text{ or } \rho\left(L_{r,\omega}^{*(2)}\right)>\rho\left(L_{r,\omega}\right)>1.$$

Corollary 2: Let $L_{r,\omega}$, $L_{r,\omega}^{(2)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix A is irreducible with $I-B \geq 0, I-C \geq 0, K \geq 0, H \geq 0, 0<\omega \leq 1, 0 \leq r<1$, then either

$$\rho\left(L_{r,\omega}^{(2)}\right) < \rho\left(L_{r,\omega}\right) < 1 \text{ or } \rho\left(L_{r,\omega}^{(2)}\right) > \rho\left(L_{r,\omega}\right) > 1.$$

Theorem 3: Let $L_{r,\omega}$, $L_{r,\omega}^{*(3)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix A is irreducible with $I-B \ge 0, I-C \ge 0, K \ge 0, H \ge 0, 0 < \omega \le 1, 0 \le r < 1,$ then either

$$\rho\left(L_{r,\omega}^{*(3)}\right) < \rho\left(L_{r,\omega}\right) < 1 \text{ or } \rho\left(L_{r,\omega}^{*(3)}\right) > \rho\left(L_{r,\omega}\right) > 1.$$

Corollary 3 Let $L_{r,\omega}$, $L_{r,\omega}^{(3)}$ be the iteration matrices associated of the GAOR and preconditioned GAOR methods, respectively. If the matrix A is irreducible with $I-B \ge 0, I-C \ge 0, K \ge 0, H \ge 0, 0 < \omega \le 1, 0 \le r < 1,$ then either

$$\rho\left(L_{r,\omega}^{(3)}\right) < \rho\left(L_{r,\omega}\right) < 1 \text{ or } \rho\left(L_{r,\omega}^{(3)}\right) > \rho\left(L_{r,\omega}\right) > 1.$$

Theorem 4: Under the assumptions of Theorem 1, then either

$$\rho\left(L_{r,\omega}^{*(1)}\right) < \rho\left(L_{r,\omega}^{(1)}\right) < 1, \text{ if } \rho\left(L_{r,\omega}^{(1)}\right) < 1 \text{ or}$$
$$\rho\left(L_{r,\omega}^{*(1)}\right) > \rho\left(L_{r,\omega}^{(1)}\right) > 1, \text{ if } \rho\left(L_{r,\omega}^{(1)}\right) > 1.$$

Proof: By the proof of Theorem 1, we know that

$$L_{r,\omega}^{(1)}x-\lambda x=\begin{pmatrix} I & 0 \\ r(K+S_1(I-B)) & I\end{pmatrix}^{-1}\left((\lambda-1)\begin{pmatrix}0&0\\S_1&0\end{pmatrix}\begin{pmatrix}I&0\\rK&I\end{pmatrix}+(\lambda-1)\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x$$
$$=(\lambda-1)\begin{pmatrix} I & 0 \\ r(K+S_1(I-B)) & I\end{pmatrix}^{-1}\left(\begin{pmatrix}0&0\\S_1&0\end{pmatrix}\begin{pmatrix}I&0\\rK&I\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x$$
$$=(\lambda-1)\begin{pmatrix} I & 0 \\ -r(K+S_1(I-B)) & I\end{pmatrix}\left(\begin{pmatrix}0&0\\S_1&0\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x,$$

where $\lambda=\rho(L_{r,\omega})$. So

$$L_{r,w}^{*(1)}x-L_{r,w}^{(1)}x=(L_{r,w}^{*(1)}x-\lambda x)-(L_{r,w}^{(1)}x-\lambda x)=(\lambda-1)\begin{pmatrix} I & 0 \\ -r(K+S_1(I-B)) & I\end{pmatrix}$$
$$\left(\begin{pmatrix}rV_1K&V_1\\S_1&0\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x-(\lambda-1)\begin{pmatrix} I & 0 \\ -r(K+S_1(I-B)) & I\end{pmatrix}$$
$$\left(\begin{pmatrix}0&0\\S_1&0\end{pmatrix}+\begin{pmatrix}0&0\\-rS_1(I-B)&0\end{pmatrix}\right)x=(\lambda-1)\begin{pmatrix} I & 0 \\ -r(K+S_1(I-B)) & I\end{pmatrix}\begin{pmatrix}rV_1K&V_1\\0&0\end{pmatrix}x.$$

From the assumptions of Theorem 1, we know that

$$-r(K+S_1(I-B))<0, rV_1K<0, V_1<0.$$

So

$$\begin{pmatrix} I & 0 \\ -r(K+S_1(I-B)) & I\end{pmatrix}\begin{pmatrix}rV_1K&V_1\\0&0\end{pmatrix}>0.$$

If $\lambda<1$, then $L_{r,\omega}^{*(1)}x-L_{r,\omega}^{(1)}x<0$, that is $\rho(L_{r,\omega}^{*(1)})<\rho(L_{r,\omega}^{(1)})<1$.

If $\lambda>1$, then $L_{r,\omega}^{*(1)}x-L_{r,\omega}^{(1)}x>0$, that is $\rho(L_{r,\omega}^{*(1)})>\rho(L_{r,\omega}^{(1)})>1$.

By the analogous proof of Theorem 4, we can prove the following two theorems.

Theorem 5: Under the assumptions of Theorem 2, then either

$$\rho\left(L_{r,\omega}^{*(2)}\right) < \rho\left(L_{r\omega}^{(2)}\right) < 1 \text{ or } \rho\left(L_{r,\omega}^{*(2)}\right) > \rho\left(L_{r,\omega}^{(2)}\right) > 1.$$

Theorem 6: Under the assumptions of Theorem 3, then either

$$\rho\left(L_{r,\omega}^{*(3)}\right) < \rho\left(L_{r,\omega}^{(3)}\right) < 1 \text{ or } \rho\left(L_{r,\omega}^{*(3)}\right) > \rho\left(L_{r,\omega}^{(3)}\right) > 1.$$

III Examples

Example 1: The coefficient matrix A in (1) is given by

$$A=\begin{pmatrix} I-B & H \\ K & I-C\end{pmatrix},$$

where

$$B=\left(b_{ij}\right)_{p\times p}, C=\left(c_{ij}\right)_{(n-p)\times(n-p)},$$
$$K=\left(k_{ij}\right)_{(n-p)\times p}, \text{ and } H=\left(h_{ij}\right)_{p\times(n-p)}$$

with

$$b_{ii}=\frac{1}{10\times(i+1)},\ i=1,2,\cdots,p,$$
$$b_{ij}=\frac{1}{30}-\frac{1}{30\times j+i},\ i<j,\ i=1,2,\cdots,p-1,\ j=2,\cdots,p,$$
$$b_{ij}=\frac{1}{30}-\frac{1}{30\times(i-j+1)+i},\ i>j,\ i=2,\cdots,p,$$
$$j=1,2,\cdots,p-1,$$
$$c_{ii}=\frac{1}{10\times(p+i+1)},\ i=1,2,\cdots,n-p,$$
$$c_{ij}=\frac{1}{30}-\frac{1}{30\times(p+j)+p+i},\ i<j,\ i=1,2,\cdots,n-p+1,$$
$$j=2,\cdots,n-p,$$
$$c_{ij}=\frac{1}{30}-\frac{1}{30\times(i-j+1)+p+i},\ i>j,\ i=,2,\cdots,n-p,$$
$$j=1,2,\cdots,n-p-1,$$
$$k_{ij}=\frac{1}{30}-\frac{1}{30\times(p+i-j+1)+p+i},\ i=1,2,\cdots,n-p,$$
$$j=1,2,\cdots,p,$$
$$h_{ij}=\frac{1}{30}-\frac{1}{30\times(p+j)+i},\ i=1,2,\cdots,p,\ j=1,2,\cdots,n-p.$$

Table I displays the spectral radii of the corresponding iteration matrices with some randomly chosen parameters $r,\ \omega,\ n,\ p$. The randomly chosen parameters α, β satisfy the conditions in Theorems 1-6. All numerical experiments have been carried out using Matlab 7.0.

TABLE I
THE SPECTRAL RADII OF THE GAOR AND PRECONDITIONED GAOR ITERATION MATRICES

n	10	15	20	40	50
p	5	5	10	25	30
ω	0.95	0.9	0.9	0.7	0.6
r	0.9	0.8	0.8	0.75	0.5
α	0.98	1	1	2	2
β	1	1	1	1	3
ρ	0.2325	0.4158	0.5472	1.1875	1.3931
ρ_1	0.2311	0.4152	0.5467	1.1876	1.3933
ρ_2	0.2260	0.4096	0.5422	1.1897	1.3984
ρ_3	0.2272	0.4133	0.5427	1.1896	1.3983
ρ_1^*	0.2301	0.4145	0.5462	1.1877	1.3934
ρ_2^*	0.2216	0.4065	0.5379	1.1922	1.4048
ρ_3^*	0.2229	0.4100	0.5386	1.1922	1.4024

where $\rho = \rho\left(L_{r,\omega}\right), \rho_1 = \rho\left(L_{r,\omega}^{(1)}\right), \rho_2 = \rho\left(L_{r,\omega}^{(2)}\right),$ $\rho_3 = \rho\left(L_{r,\omega}^{(3)}\right), \rho_1^* = (L_{r,\omega}^{*(1)}),\ \ \rho_2^* = (L_{r,\omega}^{*(2)}), \rho_3^* = (L_{r,\omega}^{*(3)})$.

From Table I, we see that these numerical results accordance with Theorems 1-6.

Example 2: We consider the following linear system $Ax = b$, where

$$A = \begin{pmatrix} I-B & H \\ K & I-C \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & \frac{1}{8} & 0 & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & 0 & 0 & 0 & 0 & \frac{1}{8} \end{pmatrix},$$

$$H = \begin{pmatrix} 0 & \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} \\ 0 & \frac{1}{8} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8} & \frac{1}{8} & 0 \\ \frac{1}{8} & 0 & 0 & \frac{1}{8} & \frac{1}{8} \end{pmatrix}.$$

Table II displays the spectral radii of the corresponding iteration matrices with some chosen parameters $r,\ \omega$.

TABLE II
THE SPECTRAL RADII OF THE GAOR AND PRECONDITIONED GAOR ITERATION MATRICES

ω	0.95	0.95	0.65	0.65
r	0.70	0.70	0.55	0.55
α	0.55	0.95	2	5
β	0.55	0.85	3	5
ρ	0.8553	0.8553	0.9042	0.9042
ρ_1	0.8536	0.8543	0.9039	0.9041
ρ_2	0.85	0.85	0.9005	0.9005
ρ_3	0.8531	0.8531	0.9027	0.9027
ρ_1^*	0.8523	0.8535	0.9037	0.9040
ρ_2^*	0.8467	0.8467	0.8982	0.8982
ρ_3^*	0.8517	0.8517	0.9017	0.9017

where $\rho = \rho\left(L_{r,\omega}\right), \rho_1 = \rho\left(L_{r,\omega}^{(1)}\right), \rho_2 = \rho\left(L_{r,\omega}^{(2)}\right),$ $\rho_3 = \rho\left(L_{r,\omega}^{(3)}\right), \rho_1^* = (L_{r,\omega}^{*(1)}),\ \ \rho_2^* = (L_{r,\omega}^{*(2)}), \rho_3^* = (L_{r,\omega}^{*(3)})$.

From Table II, we see that these numerical results accordance with Theorems 1-6.

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