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Circular Approximation by Trigonometric Bézier Curves

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***Abstract*—**We present a trigonometric scheme to approximate a circular arc with its two end points and two end tangents/unit tangents. A rational cubic trigonometric Bézier curve is constructed whose end control points are defined by the end points of the circular arc. Weight functions and the remaining control points of the cubic trigonometric Bézier curve are estimated by variational approach to reproduce a circular arc. The radius error is calculated and found less than the existing techniques.

***Keywords*—**Control points, rational trigonometric Bézier curves, radius error, shape measure, weight functions.

I. INTRODUCTION

CIRCLES are around us everywhere and inspired the mankind even before the beginning of recorded history. Circles are the main source of many wonderful inventions, such as circular ripples, wheels, circular gears which run the machines in urban factories and make our life much easier. Circular arcs are basic tools in engineering design, web design and mobile design. They are also used as user interface tool and in many other projects due to their exceptional properties such as constant curvature, constant distance from a fixed point etc.

The trigonometric polynomial curves were first introduced by Schoenberg [6]. The control point form of quadratic and cubic trigonometric polynomial curves were presented by Han in [3] and [4]. Wang, Chen and Zhou [7] defined trigonometric B-splines, known as algebraic-trigonometric B-splines (NUAT). These contributions only narrowed down the properties of trigonometric polynomial curves and did not focus on its applications. The authors in [7] used trigonometric polynomial curves for the shape preservation of data.

In this research paper, the circular approximation problem is considered. This problem is defined as: Given two end points and end tangents/unit tangents of a circular arc, construct a curve which interpolates the end points of circular arc and close to it between these endpoints.

Integral Bézier curves were in account for the approximation of circular arc by a number of researchers. Lee [5], Goldapp [2] and Fang [1] presented the G^0, G^1 and G^2 approximations of circular arc by quadratic, cubic and quintic Bézier curves respectively. The radius error between the circular arc and the approximating Bézier was minimized to obtain the best approximation.

In this research paper, rational cubic trigonometric Bézier curves are developed for approximating circular arc, particularly when its

1. Two end points and end tangents vectors are given.
2. Two end points and end unit tangents vectors are given.

II. RATIONAL TRIGONOMETRIC BÉZIER CURVE

In this section, rational quadratic and cubic trigonometric Bézier curves are introduced over the interval $\left[0, \frac{\pi}{4}\right]$.

The parametric form of newly developed rational quadratic trigonometric Bézier curve $p(t), t \in \left[0, \frac{\pi}{4}\right]$, is defined by the relation given in (1):

$$p(t) = \frac{\sum_{k=0}^{2} \boldsymbol{p}_k b_k^2(t) w_k}{\sum_{k=0}^{2} b_k^2(t) w_k}, \tag{1}$$

where $b_k^2(t)$ and w_k are the quadratic trigonometric basis functions and weight functions respectively. p_0, p_1, p_2 are the control points of rational quadratic trigonometric Bézier curve. The quadratic trigonometric basis functions $b_k^2(t)$ of (1), are defined as

$$b_0^2(t) = (1 - tan(t))^2,\ b_1^2(t) = 2(1 - tan(t))\, tan(t),\ b_2^2(t) = tan^2(t).$$

It can be easily verified that

$$\sum_{k=0}^{2} R_k^2(t) = 1 \text{ and } R_k^2(t) \geq 0 \text{ for } k = 0,1,2, \tag{2}$$

where

$$R_k^2(t) = \frac{b_k^2(t) w_k}{\sum_{k=0}^{2} b_k^2(t) w_k},$$

and

$$p(0) = \boldsymbol{p_0},\ p\left(\tfrac{\pi}{4}\right) = \boldsymbol{p_2}. \tag{3}$$

Thus from (2) and (3), the rational quadratic trigonometric Bézier curve (1), satisfies the convex-hull property and endpoints interpolation property.

The parametric form of the developed rational cubic trigonometric Bézier curve is given by the following relation.

$$x(t) = \frac{\sum_{i=0}^{3} r_i b_i^3(t) \mu_i}{\sum_{i=0}^{3} b_i^3(t) \mu_i},\ 0 \leq t \leq \tfrac{\pi}{4}, \tag{4}$$

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where $\boldsymbol{r_i}$, μ_i and $b_i^3(t)$are the control points, weight functions and basis functions respectively. These cubic trigonometric basis functions $b_i^3(t)$, are defined as

$$b_i^3(t) = \binom{3}{i}(1 - tan(t))^{3-i} tan^i(t), i = 0,1,2,3. \quad (5)$$

The cubic trigonometric Bézier curve (4) satisfies the following properties:

(i) End point interpolation property: $x(0) = \boldsymbol{r_0}$ and $x(\frac{\pi}{4}) = \boldsymbol{r_3}$,

(ii) Convex-hull property:

$$\sum_{i=0}^{3} \hat{R}_i^3(t) = 1, \hat{R}_i^3(t) \geq 0 \text{ for } i = 0,1,2,3, 0 \leq t \leq \frac{\pi}{4},$$

where

$$\hat{R}_i^3(t) = b_i^3(t)\mu_i / \sum_{i=0}^{3} b_i^3(t)\mu_i.$$

The weight functions $w_k, k = 0,1,2,$ and $\mu_i, i = 0,1,2,3,$ are positive real numbers.

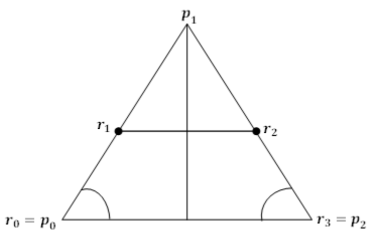

Fig. 1 The control polygon of rational quadratic and rational cubic Bézier curve

III. Representation of a Circle by Trigonometric Bézier Curves

In this section, the rational quadratic and rational cubic representations of a circular arc are established. If the control points p_0, p_1, p_2 form an isosceles triangle with base $p_0p_2$and the weight functions $w_k, k = 0,1,2$ taken as

$$w_0 = w_2 = 1, w_1 = \cos\theta, \quad (6)$$

then the rational quadratic trigonometric Bézier curve (1) represents circle. Here θ is the base angle of the triangle Δ $\boldsymbol{p_0 p_1 p_2}$ (See Fig. 1). Thus the rational quadratic trigonometric Bézier curve (1) is rewritten as

$$p(t) = \frac{\boldsymbol{p_0} b_0^2(t) + \boldsymbol{p_1}\omega_1 b_1^2(t) + \boldsymbol{p_2} b_2^2(t)}{b_0^2(t) + \omega_1 b_1^2(t) + b_2^2(t)}. \quad (7)$$

The control point $\boldsymbol{p_1}$ is the intersection of the lines given by the end points $\boldsymbol{p_0}$, $\boldsymbol{p_2}$ along with unit tangents $\boldsymbol{t_0}, \boldsymbol{t_1}$ at $\boldsymbol{p_0}$ and $\boldsymbol{p_2}$ respectively as shown in Fig. 1. Now by degree elevation of rational quadratic trigonometric Bézier curve (7), we get a rational cubic trigonometric representation of the same circular arc with control points$\boldsymbol{r_0}, \boldsymbol{r_1}, \boldsymbol{r_2}and\boldsymbol{r_3}$ and weights 1,μ_1,μ_2, 1. The relation between control points and weights of rational quadratic trigonometric Bézier curve (7) and rational cubic trigonometric Bézier curve (4) is given as follows

$$\boldsymbol{r_0} = \boldsymbol{p_0}, \boldsymbol{r_1} = \frac{2w_1\boldsymbol{p_1} + \boldsymbol{p_0}}{(1+2w_1)}, \boldsymbol{r_2} = \frac{2w_1\boldsymbol{p_1} + \boldsymbol{p_2}}{(1+2w_1)}, \boldsymbol{r_3} = \boldsymbol{p_2},$$
$$\mu_1 = \mu_2 = \frac{1}{3}(1 + 2w_1), \text{ where } w_1 = \cos\theta. \quad (8)$$

Thus the rational cubic trigonometric Bézier curve is rewritten as

$$x(t) = \frac{\boldsymbol{r_0} b_0^3(t) + \boldsymbol{r_1}\mu_1 b_1^3(t) + \boldsymbol{r_2}\mu_2 b_2^3(t) + \boldsymbol{r_3} b_3^3(t)}{b_0^3(t) + \mu_1 b_1^3(t) + \mu_2 b_2^3(t) + b_3^3(t)}. \quad (9)$$

where control points $\boldsymbol{r_i}, i = 0,1,2,3$ and weight functions μ_1,μ_2 are defined in (8).

With reference to Fig. 1, the following notations are introduced

$$m = \frac{\|\boldsymbol{r_3} - \boldsymbol{r_0}\|}{2}, m_0 = \|\boldsymbol{p_1} - \boldsymbol{p_0}\|, n_0 = \|\boldsymbol{r_1} - \boldsymbol{r_0}\|, \theta = m\angle \boldsymbol{p_1 p_0 p_2}. \quad (10)$$

It can be seen from Fig. 1 that $\cos\theta = \frac{m}{m_0}$, or $m_0 = \frac{m}{\cos\theta}$. Also, putting the values of $\boldsymbol{r_0}$ and $\boldsymbol{r_1}$ from (7) in (10) to get $n_0 = \frac{2w_1}{(1+2w_1)} m_0$. Since $w_1 = \cos\theta$, so n_0 can be written as a function of θ as

$$n_0 = \frac{2m}{(1 + 2\cos\theta)}. \quad (11)$$

By vector algebra, from Fig. 1, we have

$$\boldsymbol{r_1} - \boldsymbol{r_0} = n_0 \boldsymbol{t_0}. \quad (12)$$

From (10) and (11), we get

$$\boldsymbol{r_1} = \boldsymbol{r_0} + \frac{2m}{(1 + 2\cos\theta)} \boldsymbol{t_0}. \quad (13)$$

Similarly, $\boldsymbol{r_2} = \boldsymbol{r_3} - \frac{2m}{(1+2\cos\theta)} \boldsymbol{t_1}$.

IV. Rational Cubic Trigonometric Approximation of Circle when Two End Points and End Tangents are Given

In this section, the problem of approximation of a circle by rational cubic trigonometric Bézier curve (9) is discussed. If two end points and end tangents of a circular arc are given, then the end control points $\boldsymbol{r_0}$ and $\boldsymbol{r_3}$of the rational cubic trigonometric Bézier curve (9) are equal to the end points of the given circular arc. For solving this problem, we calculate the control points $\boldsymbol{r_1}$,$\boldsymbol{r_2}$ and the corresponding weights μ_1, μ_2. From Fig. 1, we have

$$\boldsymbol{r_1} = \boldsymbol{r_0} + k_0 \boldsymbol{t_0}, \boldsymbol{r_2} = \boldsymbol{r_3} - k_1 \boldsymbol{t_1}, \quad (14)$$

where $\boldsymbol{t_0}$and $\boldsymbol{t_1}$ are the end tangents, given as $\boldsymbol{t_0} = \frac{\dot{x}(0)}{\|\dot{x}(0)\|}$and $\boldsymbol{t_1} = \frac{\dot{x}(\frac{\pi}{4})}{\|\dot{x}(\frac{\pi}{4})\|}$.

The derivative vectors $\dot{x}(0)$ and $\dot{x}(\frac{\pi}{4})$ are computed as $\dot{x}(0) = 3\mu_1(\boldsymbol{r_1} - \boldsymbol{r_0})$ and $\dot{x}\left(\frac{\pi}{4}\right) = 3\mu_2(\boldsymbol{r_3} - \boldsymbol{r_2})$. Using (14), these derivative vectors are rewritten as

$$\dot{x}(0) = 3\mu_1 k_0 \boldsymbol{t_0} \text{and } \dot{x}\left(\frac{\pi}{4}\right) = 3\mu_2 k_1 \boldsymbol{t_1}.$$

This gives

$$k_0 = \frac{\|\dot{x}(0)\|}{3\mu_1} \text{ and } k_1 = \frac{\left\|\dot{x}\left(\frac{\pi}{4}\right)\right\|}{3\mu_2}, \tag{15}$$

where

$$\mu_1 = \frac{1}{3}(1 + 2\cos\theta_0) \text{ and } \mu_2 = \frac{1}{3}(1 + 2\cos\theta_1). \tag{16}$$

θ_0 is the angle between $\dot{x}(0)$ and $\boldsymbol{r_3} - \boldsymbol{r_0}$and $\theta_1$is the angle between $\dot{x}\left(\frac{\pi}{4}\right)$ and $\boldsymbol{r_3} - \boldsymbol{r_0}$. Now, when the input data is coming from a circle, then the rational cubic trigonometric Bézier curve, with control points and weights defined in (8), will reproduce the circle.

V. Rational Trigonometric Cubic Approximation of Circle when Two End Points and End Unit Tangents are Given

In this section, the problem of approximation of a circle is discussed when its two end points and end unit tangents, $\boldsymbol{t_0}$ and $\boldsymbol{t_1}$ are given. Again the end points of the rational cubic trigonometric Bézier curve are identical to the end points of the given circular arc. Thus we have to find the Bézier points $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ and the weights μ_1 and μ_2. The unit tangents $\boldsymbol{t_0}$ and $\boldsymbol{t_1}$make the angles $\theta_0$and $\theta_1$with the base $\boldsymbol{r_0 r_3}$ respectively. Using (13), we can find the Bézier points $\boldsymbol{r_1}$and $\boldsymbol{r_2}$ i.e.

$$\boldsymbol{r_1} = \boldsymbol{r_0} + \frac{2m}{(1+2\cos\theta_0)}\boldsymbol{t_0}, \boldsymbol{r_2} = \boldsymbol{r_3} + \frac{2m}{(1+2\cos\theta_1)}\boldsymbol{t_1}. \tag{17}$$

The corresponding weight functions μ_1 and μ_2 are calculated by (16). The other weights are unity. The angles θ_0 and $\theta_1$are approximated by minimizing the shape measure quantities $s_1(x)$ and $s_2(x)$ defined in (18) and (19) respectively.

$$s_1(x) = \int_0^{\frac{\pi}{4}} [k'(t)]^2\, dt. \tag{18}$$

$$s_2(x) = |approximated\ radius - exact\ radius| \tag{19}$$

Equation (18) shows that the change in curvature is more significant than its magnitude. The discontinuous curvature is acceptable as long as the slope $k'(t)$ is continuous. The quality of the developed rational cubic trigonometric circular approximation scheme is measured by $q(x)$, defined by

$$q(x) = \int_a^b \int_c^d s(\theta_0, \theta_1) d\theta_0 d\theta_1. \tag{20}$$

Here,a, b, c, d are the limitations of admissible values for $\theta_0$and θ_1. Here $a = -90^o, b = 90^o$ and $c = 90^o$, $d = 270^o$ i.e. all acceptable $\boldsymbol{t_0}$ have a positive x-component, and all admissible $\boldsymbol{t_1}$have a negative x-component.

VI. Numerical Examples

In this section, a circular arc is estimated by the schemes developed in Sections IV and V.

(a). Minimization of $s_1(x)$: In this case the values of θ_0 and θ_1 are obtained by minimizing the integral given in (18) for rational cubic trigonometric Bézier curve (9). The end points of the circle are taken as $\boldsymbol{r_0}(-1,0)$and $\boldsymbol{r_3}(1,0)$. The optimized values of $\theta_0 =$ 1.6435and θ_1=1.4732, and corresponding to these values we have $\boldsymbol{r_1}(-1,1.6896)$, $\boldsymbol{r_2}(1,2.0350)$, $\mu_1 = 0.3946$ and $\mu_2 = 0.3276$. The resulting graph is close to circle with radius error between 0 and 0.0836. The approximated circle is shown in Fig. 2.

(b). Minimization of $s_2(x)$: In this case the values of θ_0 and θ_1 are obtained by minimizing the integral given in (19) for (9). The end points of the circle are taken as $\boldsymbol{r_0}(-1,0)$ and $\boldsymbol{r_3}(1,0)$. The optimized values of θ_0 =1.5708 and θ_1=1.5708, and corresponding to these values we have $\boldsymbol{r_1}(-1,2)$, $\boldsymbol{r_2}(1,2)$, $\mu_1 = 0.333$ and $\mu_2 = 0.3333$. The resulting graph is close to circle with radius error between 0 and 9.9880$\times 10^{-4}$. The approximated circle is shown in Fig. 3.

(c). Geometric Approach: When we used the geometric approach, the circle is reproduced with $\theta_0 = \frac{\pi}{2}$ and θ_1=$\frac{\pi}{2}$ (See Fig. 4). Corresponding to these values, we have $\boldsymbol{r_1}(-1,2)$, $\boldsymbol{r_2}(1,2)$, $\mu_1 = \frac{1}{3}$ and $\mu_2 = \frac{1}{3}$. In this case, the rational trigonometric curve (9) will reproduce the circle (See. Fig. 5) which is the same as obtained by the minimization of $s_2(x)$. In Figs. 6 and 7 the curvature derivative plots of (9) are shown for the above mentioned cases.

VII. Conclusion

In this research paper, we have discussed the rational cubic trigonometric techniques to approximate the circle. We have used approaches, the variational and the geometric one. The quality of the rational cubic trigonometric problem is calculated using (20), which is found to be $q(x) = 17.6099$. It is 375.3 for polynomial Hermite interpolant it is 375.3. This shows that our rational cubic trigonometric schemes did better than integral and rational polynomial Hermite interpolant. Radius error is found less than many existing schemes [1].

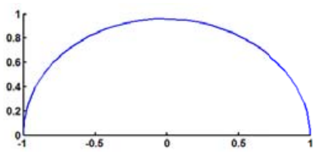

Fig. 2 A rational cubic trigonometric approximation of circle using $s_1(x)$

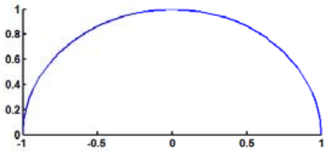

Fig. 3 A rational cubic trigonometric approximation of circle using $s_2(x)$

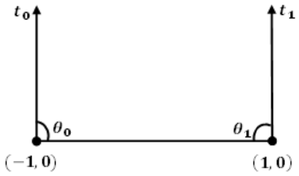

Fig. 4 Tangents direction

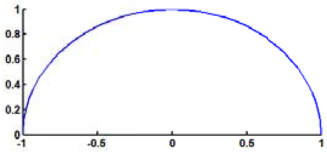

Fig. 5 The graphical representation of circle using geometric approach

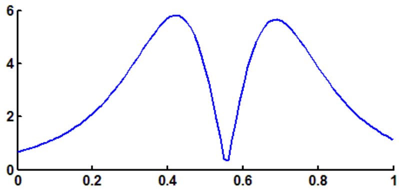

Fig. 6 Curvature derivative plot of cubic trigonometric Bézier using geometric approach

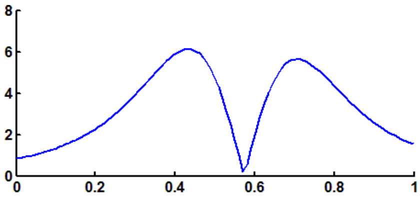

Fig. 7 Curvature derivative plot of cubic trigonometric Bézier using variational approach

References

[1] L. Fang, "Circular arc approximation by quintic polynomial curves," Computer Aided Geometric Design, vol. 15, 1998, pp. 843-861.
[2] M. Goldapp, "Approximation of circular arcs by cubic polynomials," Computer Aided Geometric Design,vol. 8, 1991, pp. 227-238.
[3] X. Han, "Quadratic trigonometric polynomial curves with a shape parameter," Computer Aided Geometric Design, vol. 19, 2002, pp. 503-512.
[4] X. Han, "Cubic trigonometric polynomial curves with a shape parameter," Computer Aided Geometric Design, vol. 21, 2004, pp. 535-548.
[5] I. K. Lee, M. S. Kim and G. Elber, "Planer curve offset based on circular approximation," Computer Aided Design,vol.28, no. 8, 1996, pp. 617-630.
[6] I. J. Schoenberg, "On trigonometric spline interpolation,"Journal of Mathematics and Mechanics,vol. 13, no. 5, 1964, pp. 795-825.
[7] G. Wang, Q. Chenand M. Zhou, "NUAT B-spline curves,"Computer Aided Geometric Design,vol. 21, 2004, pp. 193-205.