# Fekete-Szegö Problem for Subclasses of Analytic Functions Defined by New Integral Operator 

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Abstract-The author introduced the integral operator, by using this operator a new subclasses of analytic functions are introduced. For these classes, several Fekete-Szeg" type coefficient inequalities are obtained.

Keywords-Integral operator, Fekete-Szeg" inequalities, Analytic functions.

## I. Introduction and definition

L ET $\mathcal{A}$ denote the class of functions of the form

$$
\begin{equation*}
f(z)=z+\sum_{n=2}^{\infty} a_{n} z^{n} \tag{1}
\end{equation*}
$$

which are analytic in the open unite disk $\mathbb{U}=\{z: z \in$ $\mathbb{C}$ and $|z|<1\}$.
Also let $\mathcal{S}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$.
In [2] Fekete and Szeg" proved a noticeable result that the estimate

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq 1+2 \exp \left(\frac{-2 \mu}{1-\mu}\right)
$$

holds for $f \in \mathcal{S}$ and for $0 \leq \mu \leq 1$. This inequality is sharp for each $\mu$. The coefficient functional

$$
\phi_{\mu}(f)=a_{3}-\mu a_{2}^{2}=\frac{1}{6}\left(f^{\prime \prime \prime}(0)-\frac{3 \mu}{2}\left(f^{\prime \prime}(0)\right)^{2}\right)
$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$
\phi_{\mu}\left(e^{-i \theta} f\left(e^{i \theta} z\right)=e^{2 i \theta} \phi_{\mu}(f), \quad(\theta \in \mathbb{R})\right.
$$

In fact, other than the simplest case when

$$
\phi_{0}(f)=a_{3}
$$

we have several important ones. For example,

$$
\phi_{1}(f)=a_{3}-a_{2}^{2}
$$

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represent $S_{f}(0) / 6$, where $S_{f}$ denotes the Schwarzian derivative

$$
S_{f}(z)=\left(\frac{f^{\prime \prime}(z)}{f^{\prime}(z)}\right)^{\prime}-\frac{1}{2}\left(\frac{f^{\prime \prime}(z)}{f^{\prime}(z)}\right)^{2}
$$

Moreover, the first two non-trivial coefficients of the $n$-th root transform

$$
\left(f\left(z^{n}\right)\right)^{\frac{1}{n}}==z+c_{n+1} z^{n+1}+c_{2 n+1} z^{2 n+1}+\ldots
$$

of $f$ with the power series (1), are written by

$$
c_{n+1}=\frac{a_{2}}{n}
$$

and

$$
c_{2 n+1}=\frac{a_{3}}{n}+\frac{(n-1) a_{2}^{2}}{2 n^{2}}
$$

so that

$$
a_{3}-\mu_{2}^{2}=n\left(c_{2 n+1}-\lambda c_{n+1}^{2}\right)
$$

where

$$
\lambda=\mu n+\frac{n-1}{2}
$$

Thus, it is quite natural to ask about inequalities for $\phi_{\mu}$ corresponding to subclasses of $\mathcal{S}$. This is called Fekete-Szeg" problem. Actually many authors have considered this problem for typical classes of univalent functions.

Recently, in [1] the author introduced a certain integral operator $\mathcal{I}-c^{\delta}$ defined by :

$$
\begin{equation*}
\mathcal{I}_{c}^{\delta} f(z)=\frac{(1+c)^{\delta}}{\Gamma(\delta)} \int_{0}^{1} t^{c-1}(\log 1 / t)^{\delta-1} f(t z) d t \tag{2}
\end{equation*}
$$

where $c>0, \delta>1$ and $z \in \mathbb{U}$.

We also note that the operator $\mathcal{I}_{c}^{\delta} f(z)$ defined by (1) can be expressed by the series expansion as following:

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$$
\begin{equation*}
\mathcal{I}_{c}^{\delta} f(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+c}{k+c}\right)^{\delta} a_{k} z^{k} \tag{3}
\end{equation*}
$$

Obviously, we have, for $(\delta, \lambda \geq 0)$

$$
\begin{equation*}
\mathcal{I}_{c}^{\delta}\left(I_{c}^{\lambda} f(z)\right)=I_{c}^{\delta+\lambda} f(z) . \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{I}_{c}^{\delta}\left(z f^{\prime}(z)\right)=z\left(I_{c}^{\delta} f(z)\right)^{\prime} \tag{5}
\end{equation*}
$$

Moreover, from (3), it follows that

$$
\begin{equation*}
z\left(\mathcal{I}_{c}^{\delta+1} f(z)\right)^{\prime}=(c+1) \mathcal{I}_{c}^{\delta} f(z)-c \mathcal{I}_{c}^{\delta+1} f(z) \tag{6}
\end{equation*}
$$

We note that :

- For $c=0$ and $\delta=n$ ( $n$ is any integer), the multiplier transformation $\mathcal{I}_{0}^{n} f(z)=I^{n} f(z)$ was studied by Flett [3] and Salagean [4];
- For $c=0$ and $\delta=-n\left(n \in \mathbb{N}_{0}=\{0,1,2,3 \ldots\}\right)$, the differential operator $\mathcal{I}_{0}^{-n} f(z)=D^{n} f(z)$ was studied by Salagean [4];
- For $c=1$ and $\delta=n(n$ is any integer), the operator $\mathcal{I}_{1}^{n} f(z)=\mathcal{I}^{n} f(z)$ was studied by Uralegaddi and Somanatha [5];
- For $c=1$, the multiplier transformation $\mathcal{I}_{1}^{\delta} f(z)=$ $\mathcal{I}^{\delta} f(z)$ was studied by Jung et al. [6];
- For $c=a-1(a>0)$, the integral operator $\mathcal{I}_{a-1}^{\delta} f(z)=$ $\mathcal{I}_{a-1}^{\delta} f(z)$ was studied by Komatu [7];

Using the operator $\mathcal{I}_{c}^{\delta}$, we now introduce the following classes:
Definition 1: we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{c, \delta}(b)$ if

$$
\begin{array}{r}
\Re\left\{1+\frac{1}{b}\left(\frac{z\left(\mathcal{I}_{c}^{\delta} f(z)\right)^{\prime}}{\mathcal{I}_{c}^{\delta} f(z)}-1\right)\right\}>0 \\
(c>0, \delta \geq 0, b \in \mathbb{C} \backslash\{0\}, z \in \mathbb{U}) \tag{7}
\end{array}
$$

Definition 2: we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{C}_{c, \delta}(b)$ if

$$
\begin{array}{r}
\Re\left\{1+\frac{1}{b} \frac{z\left(\mathcal{I}_{c}^{\delta} f(z)\right)^{\prime \prime}}{\mathcal{I}_{c}^{\delta} f(z)}\right\}>0, \\
(c>0, \delta \geq 0, b \in \mathbb{C} \backslash\{0\}, \quad z \in \mathbb{U}) . \tag{8}
\end{array}
$$

Note that

$$
\begin{equation*}
f \in \mathcal{C}_{c, \delta}(b) \Leftrightarrow z f^{\prime} \in \mathcal{S}_{c, \delta}(b) . \tag{9}
\end{equation*}
$$

In particular, we have starlike and convex function classes, $\mathcal{S}_{c, 0}(1)=\mathcal{S}^{*}$ and $\mathcal{C}_{c, 0}(1)=\mathcal{C}$, respectively.

We denote by $P$ a class of the analytic functions in $\mathbb{U}$ with

$$
p(0)=1 \text { and } \Re\{p(z)\}>0
$$

To prove our results, we need the following Lemmas considered by Duren [8] Ravichandran et al. [9] .

Lemma 1: [8] Let $p \in P$ with $p(z)=1+c_{1} z+c_{2} z^{2}+\ldots$. Then

$$
\left|c_{n}\right| \leq 2, \quad(n \geq 1)
$$

Lemma 2: [9] Let $p \in P$ with $p(z)=1+c_{1} z+c_{2} z^{2}+\ldots$. Then for any complex number $\gamma$

$$
\left|c_{2}-\gamma c_{1}^{2}\right| \leq 2 \max \{1,|2 \gamma-1|\},
$$

and the result is sharp for the functions given by

$$
p(z)=\frac{1+z^{2}}{1-z^{2}}, \quad p(z)=\frac{1+z}{1-z} .
$$

Lemma 3: [8] Let $p \in P$ with $p(z)=1+c_{1} z+c_{2} z^{2}+\ldots$.
Then

$$
\left|c_{2}-\frac{1}{2} \lambda c_{1}^{2}\right| \leq 2+\frac{1}{2}(|\lambda-1|-1)\left|c_{1}\right|^{2} .
$$

## II. Main results

Theorem 1: Let $c, \delta \geq 0 ; b \in \mathbb{C} \backslash\{0\}$. If $f \in \mathcal{S}_{c, \delta}(b)$, then

$$
\left|a_{2}\right| \leq 2|b|\left(\frac{c+2}{c+1}\right)^{\delta}
$$

$$
\left|a_{3}\right| \leq|b|\left(\frac{c+3}{c+1}\right)^{\delta} \max \{1,|1+2 b|\},
$$

and

$$
\left|a_{3}-\frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta} a_{2}^{2}\right| \leq|b|\left(\frac{c+3}{c+1}\right)^{\delta} .
$$

Proof. Denote

$$
\mathcal{I}_{c}^{\delta}=z+A_{2} z^{2}+A 3 z^{3}+\ldots .
$$

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Then by (3), we can write

$$
\begin{equation*}
A_{2}=\left(\frac{c+1}{c+2}\right)^{\delta} a_{2}, \quad A_{3}=\left(\frac{c+1}{c+3}\right)^{\delta} a_{3} \tag{10}
\end{equation*}
$$

by the definition of the class $\mathcal{S}_{c, \delta}(b)$, there exists $p \in P$ such that:

$$
\begin{aligned}
1+ & \frac{1}{b}\left(\frac{z\left(\mathcal{I}_{c}^{\delta} f(z)\right)^{\prime}}{\mathcal{I}_{c}^{\delta} f(z)}-1\right)=p(z) \\
& \frac{z\left(\mathcal{I}_{c}^{\delta} f(z)\right)^{\prime}}{\mathcal{I}_{c}^{\delta} f(z)}=1-b+b p(z)
\end{aligned}
$$

so that
$\frac{z\left(1+2 A_{2} z+3 A_{3} z^{2}+\ldots\right)}{z+A_{2} z^{2}+A_{3} z^{3}+\ldots}=1-b+b\left(1+c_{1} z+c_{2} z^{2}+\ldots\right)$, which implies the equality

$$
\begin{align*}
& z+2 A_{2} z^{2}+3 A_{3} z^{3}+\ldots  \tag{13}\\
& =z+\left(A_{2}+b c_{1}\right) z^{2}+\left(A_{3}+b c_{1} A_{2}+b c_{2}\right) z^{3}+\ldots
\end{align*}
$$

Moreover for each $\mu$, there is a function in $\mathcal{S}_{c, \delta}(b)$ such that equality holds.

Proof. Taking into account (12) we have

$$
\begin{aligned}
a_{3}-\mu a_{2}^{2} & =\frac{b}{2}\left(\frac{c+2}{c+1}\right)^{\delta}\left(c_{2}+b c_{1}^{2}\right)-\mu b^{2} c_{1}^{2}\left(\frac{c+2}{c+1}\right)^{2 \delta} \\
& =\frac{b}{2}\left(\frac{c+2}{c+1}\right)^{\delta}\left(c_{2}+\beta c_{1}^{2}\right)
\end{aligned}
$$

where

$$
\beta=-b+2 \mu b\left(\frac{\left.(c+2)^{2}\right)}{(c+1)(c+3)}\right)^{\delta}
$$

Then, with the aid of Lemma 2, we obtain

$$
\begin{align*}
& \left|a_{3}-\mu a_{2}^{2}\right|  \tag{14}\\
& \leq|b|\left(\frac{c+3}{c+1}\right)^{\delta} \max \left\{1,\left|1+2 b-4 \mu b\left(\frac{\left.(c+2)^{2}\right)}{(c+1)(c+3)}\right)^{\delta}\right|\right\} .
\end{align*}
$$

as asserted. An examination of the proof shows that equality is attained for the first case when $c_{1}=0$ and $c_{2}=2$ and the corresponding $f \in \mathcal{S}_{c, \delta}(b)$ is given by

$$
\begin{equation*}
\frac{z\left(\mathcal{I}_{c}^{\delta} f(z)\right)^{\prime}}{\mathcal{I}_{c}^{\delta} f(z)}=\frac{1+(2 b-1) z^{2}}{1-z^{2}} \tag{15}
\end{equation*}
$$

and likewise for the second case when $c_{1}=c_{2}=2$ the corresponding $f \in \mathcal{S}_{c, \delta}(b)$ is given by

$$
\begin{equation*}
\frac{z\left(\mathcal{I}_{c}^{\delta} f(z)\right)^{\prime}}{\mathcal{I}_{c}^{\delta} f(z)}=\frac{1+(2 b-1) z}{1-z} \tag{16}
\end{equation*}
$$

respectively.

Taking $\delta=0$ and $b=1$ in Theorem 2, we have :
Corollary 1: [10] If $f \in \mathcal{S}^{*}$, then for $\mu \in \mathbb{C}$ we have

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \max \{1,|4 \mu-3|\} .
$$

Moreover for each $\mu$, there is a function in $\mathcal{S}^{*}$ such that equality holds.
We next consider the case when $\mu$ and $b$ are real. Then we have
Theorem 3: Let $c, \delta \geq 0 ; b>0$. If $f \in \mathcal{S}_{c, \delta}(b)$, then for $\mu \in \mathbb{R}$, we have

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq\left\{\begin{array}{l}
b\left(\frac{c+3}{c+1}\right)^{\delta}\left[1+2 b-4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)^{\delta}\right] \\
\text { if } \mu \leq \frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta} \\
b\left(\frac{c+3}{c+1}\right)^{\delta} \\
\text { if } \frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta} \leq \mu \leq \frac{1+b}{2 b}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta} \\
b\left(\frac{c+3}{c+1}\right)^{\delta}\left[-1-2 b+4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)^{\delta}\right] \\
\text { if } \mu \geq \frac{1+b}{2 b}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta}
\end{array}\right.
$$

Moreover for each $\mu$, there is a function in $\mathcal{S}_{c, \delta}(b)$ such that equality holds.

Proof. By (14), we obtain

$$
\begin{align*}
& a_{3}-\mu a_{2}^{2}=\frac{b}{2}\left(\frac{c+3}{c+1}\right)^{\delta}  \tag{17}\\
& {\left[c_{2}-\frac{c_{1}^{2}}{2}+\frac{c_{1}^{2}}{2}\left(1+2 b-4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)\right)^{\delta}\right]}
\end{align*}
$$

First, let $\mu \leq \frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta}$. in this case, by (17), Lemma 1 and Lemma 3 give

$$
\begin{aligned}
& \left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{b}{2}\left(\frac{c+3}{c+1}\right)^{\delta} \\
& {\left[2-\frac{\left|c_{1}\right|^{2}}{2}+\frac{\left|c_{1}\right|^{2}}{2}\left(1+2 b-4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)\right)^{\delta}\right]} \\
& \leq b\left(\frac{c+3}{c+1}\right)^{\delta}\left[1+2 b-4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)^{\delta}\right]
\end{aligned}
$$

Now let $\frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta} \leq \mu \leq \frac{1+b}{2 b}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta}$. Then, using the above calculations, we get

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq b\left(\frac{c+3}{c+1}\right)^{\delta}
$$

Finally, if $\mu \geq \frac{1+b}{2 b}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta}$, then we obtain

$$
\begin{aligned}
& \left|a_{3}-\mu a_{2}^{2}\right| \leq \frac{b}{2}\left(\frac{c+3}{c+1}\right)^{\delta} \\
& {\left[2-\frac{\left|c_{1}\right|^{2}}{2}+\frac{\left|c_{1}\right|^{2}}{2}\left(-1-2 b+4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)\right)^{\delta}\right]} \\
& \leq \frac{b}{2}\left(\frac{c+3}{c+1}\right)^{\delta} \\
& {\left[2+\frac{\left|c_{1}\right|^{2}}{2}\left(-2-2 b+4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)\right)^{\delta}\right]} \\
& \leq b\left(\frac{c+3}{c+1}\right)^{\delta}\left[-1-2 b+4 \mu b\left(\frac{(c+2)^{2}}{(c+1)(c+3)}\right)^{\delta}\right] .
\end{aligned}
$$

Equality is attained for the second case on choosing $c_{1}=0, c_{2}=2$ in (15) and in (16) $c_{1}=c_{2}=2 ; c_{1}=2 i, c_{2}=-2$ for the firs and third case, respectively. Thus the proof is complete.

Using the relation (9), we easily obtain bounds of coefficients and a solution of the Fekete-Szegö problem in $\mathcal{C}_{c, \delta}$.
Theorem 4: Let $c, \delta \geq 0 ; b \in \mathbb{C} \backslash\{0\}$. If $f \in \mathcal{C}_{c, \delta}(b)$, then

$$
\left|a_{2}\right| \leq|b|\left(\frac{c+2}{c+1}\right)^{\delta}
$$

$$
\left|a_{3}\right| \leq \frac{|b|}{3}\left(\frac{c+3}{c+1}\right)^{\delta} \max \{1,|1+2 b|\}
$$

and

$$
\left|a_{3}-\frac{2}{3}\left(\frac{(c+1)(c+3)}{(c+2)^{2}}\right)^{\delta} a_{2}^{2}\right| \leq \frac{|b|}{3}\left(\frac{c+3}{c+1}\right)^{\delta} .
$$

Reasoning in the same line as in proof of Theorem 2 obtain
Theorem 5: Let $c, \delta \geq 0 ; b \in \mathbb{C} \backslash\{0\}$. If $f \in \mathcal{C}_{c, \delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$
\begin{aligned}
\left|a_{3}-\mu a_{2}^{2}\right| \leq & \frac{|b|}{3}\left(\frac{c+3}{c+1}\right)^{\delta} \\
& \max \left\{1,\left|1+2 b-3 \mu b\left(\frac{\left.(c+2)^{2}\right)}{(c+1)(c+3)}\right)^{\delta}\right|\right\}
\end{aligned}
$$

Moreover for each $\mu$, there is a function in $\mathcal{C}_{c, \delta}(b)$ such that equality holds.
By taking $\delta=0$ and $b=1$ in Theorem 5, we have
Corollary 2: [10] If $f \in \mathcal{C}^{*}$, then for $\mu \in \mathbb{C}$ we have

$$
\left|a_{3}-\mu a_{2}^{2}\right| \leq \max \left\{\frac{1}{3},|\mu-1|\right\}
$$

Moreover for each $\mu$, there is a function in $\mathcal{C}^{*}$ such that equality holds.

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