

# Energy Budget Equation of Superfluid HVBK Model: LES Simulation

M. Bakhtaoui, L. Merahi

**Abstract**—The reliability of the filtered HVBK model is now investigated via some large eddy simulations (LES) of freely decaying isotropic superfluid turbulence. For homogeneous turbulence at very high Reynolds numbers, comparison of the terms in the spectral kinetic energy budget equation indicates, in the energy-containing range, that the production and energy transfer effects become significant except for dissipation. In the inertial range, where the two fluids are perfectly locked, the mutual friction may be neglected with respect to other terms. Also, the LES results for the other terms of the energy balance are presented.

**Keywords**—Superfluid turbulence, HVBK, Energy budget, Large Eddy Simulation.

## I. INTRODUCTION

At finite temperature, superfluid turbulence manifests itself as a tangle of vortex lines and can be generated in many ways. Turbulent thermal counter flow was the first turbulent flow which was studied in detail in a series of pioneering works of Vinen [1,2]. Liquid  $^4\text{He}$  is one of such quantum fluids. Helium II may be modeled using Landau's two-fluid theory, in which the fluid may be considered to be made up of two completely mixed fluids: a viscous normal fluid and an inviscid superfluid. Following the pioneering works of Landau [3], a macroscopic model for modeling  $^4\text{He}$  was derived by the Hall–Vinen–Bekharevich–Khalatnikov (HVBK) equations. In the previous flow simulation of a model described by the superfluid HVBK, It was concluded a similarity between the superfluid turbulence and the classical turbulence. In this respect, the HVBK model can be used to derive the Hall–Vinen–Bekharevich–Khalatnikov (HVBK) equations for large scales in isotropic superfluid turbulence. These equations were derived assuming flow configurations in which there are a large number of vortex lines aligned in roughly the same direction [4].

Our aim in this paper consists to use the superfluid energy budget equation in order to investigate the effect of different terms of the HVBK equations. Since the pioneering work from the LES of this model in superfluid  $^4\text{He}$  [5], the same method similar to those of Merahi et al. [5] is generated HVBK model numerically from large eddy simulation. It was predicted that this model is ideal to study the coupled dynamics of superfluid and normal fluid in the limit of intense turbulence at the lowest temperatures.

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## II. THE HVBK MODEL

Following the pioneering works of Landau [3], a macroscopic model for modeling  $^4\text{He}$  was derived by the HVBK equations. This model is also expected. These equations were derived assuming flow configurations in which there are a large number of vortex lines aligned in roughly the same direction. While in the original Landau model the two fluids were independent; a mutual friction force is introduced in the HVBK model. These equations are written as follows [6]:

$$\nabla \cdot \mathbf{V}_n = 0, \quad \nabla \cdot \mathbf{V}_s = 0 \quad (1-a)$$

$$\frac{\partial \mathbf{V}_n}{\partial t} + (\mathbf{V}_n \cdot \nabla) \mathbf{V}_n = -\frac{\nabla P}{\rho} - \frac{\rho_s}{\rho_n} S \nabla T + \nu_n \nabla^2 \mathbf{V}_n + \frac{\rho_s}{\rho} \mathbf{F}_{ns} \quad (1-b)$$

$$\frac{\partial \mathbf{V}_s}{\partial t} + (\mathbf{V}_s \cdot \nabla) \mathbf{V}_s = -\frac{\nabla P}{\rho} + S \nabla T - \mathbf{T} - \frac{\rho_n}{\rho} \mathbf{F}_{ns} \quad (1-c)$$

with:

$$\mathbf{F}_{ns} = \frac{1}{2} B \hat{\omega}_s \times (\omega_s \times (\mathbf{V}_n - (\mathbf{V}_s + \nu_s \nabla \times \hat{\omega}_s))) + \frac{1}{2} B' \omega_s (\mathbf{V}_n - \mathbf{V}_s - \nu_s \nabla \times \hat{\omega}_s) \quad (2)$$

$$\omega_s = \text{curl} \mathbf{V}_s, \quad \hat{\omega}_s = \omega_s / |\omega_s| \quad (3)$$

$$\mathbf{T} = -\nu_s \omega_s \times (\nabla \times \hat{\omega}_s) \quad (4)$$

$$\nu_s = \frac{\kappa}{4\pi} \log(\Delta_{\text{tignes}}/a) \quad (5)$$

The HVBK equations have interesting limits. At  $T \rightarrow 0$  then  $\rho_n \approx 0$ , we have  $\nu_s = 0$ , the pure superflow equation (3) becomes the classical Euler equation. If  $T \rightarrow 2.17 \text{ K}$  corresponding to  $\rho_s \approx 0$ , the normal fluid equation (2) becomes the classical Navies–Stokes equation. This two equations model, which has been verified in the experiment for Reynolds numbers less than 400, proves valid also in the turbulent case through the numerical results found. Therefore, it is seen that the HVBK model is ideal to study the coupled dynamics of superfluid and normal fluid, and can be used to derive a LES model for large scales in isotropic superfluid turbulence.

In this work, we apply the same systematic approach based on the mathematical frame of the HVBK model and the convolution filter used in the work of Merahi et al. [5]. The filtered HVBK model is solved using a fully pseudo-spectral method, which is an extension of the classical Rogallo's method [7] to the two-fluid model. In this paper, we analyze the evolution of various terms in the momentum equations of the HVBK model via the energy budget equation of the function of spectrum energy.

### III. THE ENERGY-SPECTRUM BALANCE

An analysis of the evolution of various terms constituting the equations for momentum of HVBK model, in the overall energy balance, observed striking similarity between ordinary turbulence and quantum turbulence, that is in with the vortex are quantized. In this paper, we consider the balance equation for the energy-spectrum function  $E(k, t)$  [8], this equation can be written in the following form:

$$\frac{\delta}{\delta t} E(k, t) = \mathcal{P}_k(k, t) - \mathcal{T}_k(k, t) - \mathcal{D}(k, t) - F \quad (6)$$

The four terms on the right-hand side represent, respectively, production, spectral transfer, dissipation, and the friction force.

#### A. The Energy Transfer

It is the net rate at which energy is transferred from modes of lower wavenumber than  $k$  to those with wavenumbers higher than  $k$  up to the cutoff  $k_c$  ( $k_c$  is also assumed to be much larger than the smoothed quantized vortex spacing  $l \approx L^{-1/2}$ ). Some of its components have been measured directly. The nonlinear term of this spectral transfer is:

$$\mathcal{T}(k) = ik_j u_i u_j \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \cdot \hat{u}_i(k) \quad (7)$$

#### B. The dissipation spectrum

If the dissipating eddies are isotropic, can be expanded as follows:

$$\mathcal{D}(k, t) = 2\gamma k^2 E(k, t) \quad (8)$$

#### C. The mutual Friction Force

The final term in the balance equation (6) is the friction force taken as follows:

$$F = \tilde{F}_i \cdot \hat{u}_i(k) \quad (9-a)$$

$$\tilde{F}_i = \tilde{f}_1 + \tilde{f}_2 \quad (9-b)$$

with  $\tilde{f}_1$  and  $\tilde{f}_2$  representing, respectively, the reactive and dissipative parts of the friction force.

As well as in the rest of the energy-spectrum balance terms, the superfluid part for the term related to the tension  $\mathbf{T}$  is as follows:

$$\mathbf{T} = \hat{\mathbf{T}} \cdot \hat{u}_i(k) \quad (10)$$

Recent numerical studies of the energy balance were performed by Wacks and Barenghi [9] using a simple shell model simulation, and DNS of two fluids hydrodynamics of Roche et al. [10]. They found a build-up of energy develops at high wavenumbers suggesting the need for a further dissipative effect. In this paper, we compare the HVBK results with those results in the following.

### IV. RESULT

The spectral kinetic energy budgets for various height ranges are computed as a function of total wavenumber. The contribution of all the terms to the balance equation  $E(k, t)$  for the superfluid helium as shown in Fig. 1, positive values indicate gain and negative values indicate energy loss. The production and spectral energy transfer dominate the turbulent energy budget. The blue line of Fig. 1 shows the function of the energy transfer as given by (7). Some of the directly measured components of the energy transfer are negative at large scales and positive over the rest of the spectrum, according to [11], [12] in the classical turbulence. This is consistent with a predominantly downscale nonlinear cascade of kinetic energy. This figure clearly demonstrates the non-existence of an inertial subrange. The energy spectrum becomes isotropic at such wavenumbers; because the energy transfer spectrum only becomes isotropic some time after the production goes to zero. In the energy-containing range, the production and energy-transfer effects become significant and constitute a source or a sink at each wavenumber except for dissipation. The representation of the dissipation spectrum by  $2\gamma k^2 E(k, t)$  gives an estimate of the separation between the production and dissipation ranges. In the inertial range, starting at  $k \sim 5$ , there is not much dissipation in the energy-producing range. At other scales, which are equivalent to the small scales, this dissipation term works as a viscosity.

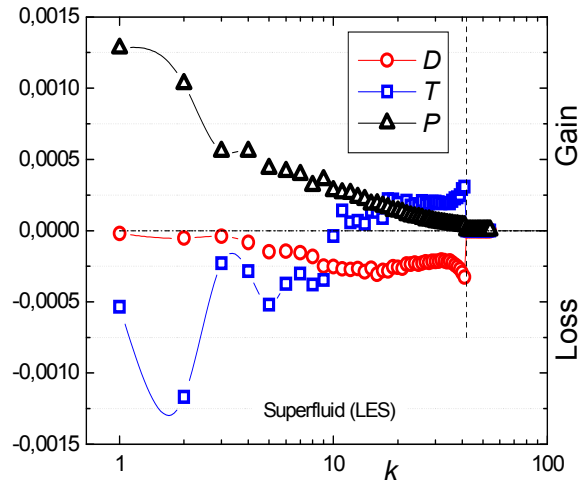


Fig. 1 Log-linear plots of terms in the global spectral energy budget  $E(k, t)$  versus wavenumber  $k$ , at 1.6 K. Superfluid: evolution of the various terms production  $P(k)$ , spectral transfer  $T(k)$  and the dissipation  $D(k)$ , (6)–(8). The vertical dashed line marks  $k_c$

In contrast, the contribution of the tension  $\mathbf{T}$  and the friction force  $\mathbf{F}$  in the kinetic energy-balance terms are also taken into account. Figs. 2 and 3 show the importance of the friction force  $\mathbf{F}$  and the tension  $\mathbf{T}$ , respectively. We observe that their order of magnitude is very small. So, the influence of the mutual friction term, especially, can be neglected in the range  $10^{-10}$  for superfluid and in the range  $10^{-11}$  for normal fluid; the reason is that two fluids have the same velocity field

( $Vn \approx Vs$ ) at large scales, obviously the same energy spectrum. At these lowest temperatures, the relevant length scales are significantly larger than the expected spacing of quantized vortex lines [13]. So Fig. 2 shows that for large scales the mutual friction is sufficient to keep the two fluids locked together on these length scales.

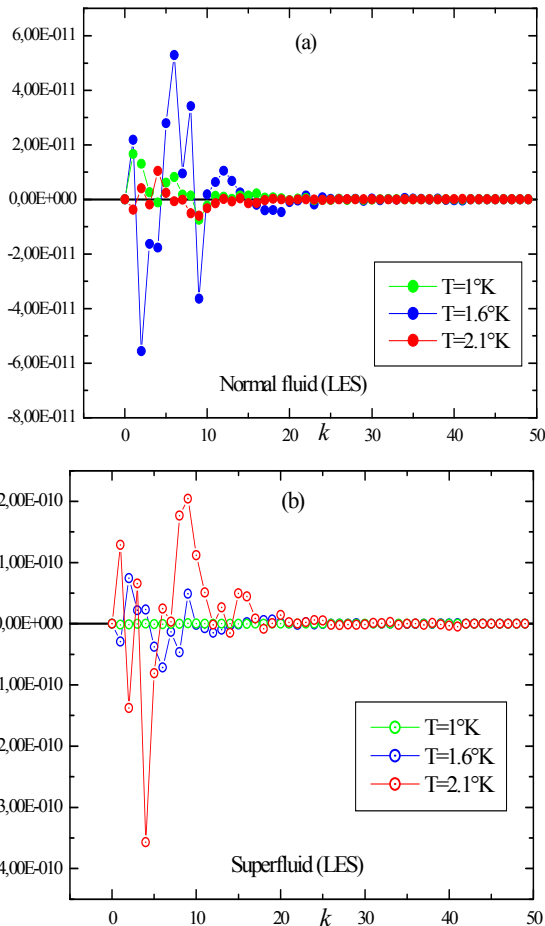


Fig. 2 (a) Normal fluid and (b) superfluid: contribution of the mutual friction force for both fluid components, (9) at  $T = 1, 1.6$ , and  $2.1$  K

Even for the tension  $T$  of the superfluid part, in Fig. 3, at temperature as low as  $T=1.6$  K, the tension  $T$  would be insignificant in the range  $10^{-5}$ . The locked fluids behave as a single turbulent fluid in which there is negligible dissipation. At  $T \rightarrow 2.1$  K corresponding to  $\rho_s/\rho \ll 1$ , where the superfluid component can be neglected, the statistics of turbulent superfluid  $^4\text{He}$  become similar to those of classical fluids. In this respect, the contribution of the mutual friction of the pure superflow increases slightly at large scales as shown in Fig. 2 (b).

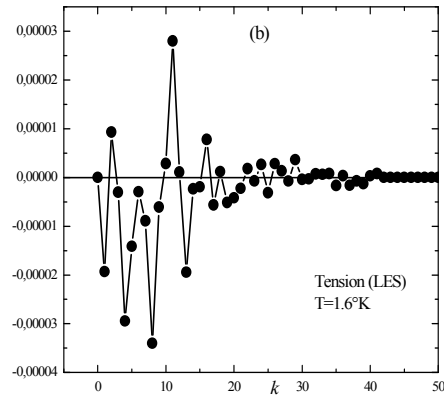


Fig. 3 Contribution of tension  $T(10)$ , at low temperature  $1.6$  K

It is not the same for the normal fluid part which shows its maximum value at  $T = 1.6$  K (see Fig. 2 (a)). It follows that dissipation must be due to a combination of viscous dissipation in the normal fluid and mutual friction, as it is discussed by Vinen [13]. Therefore, a Kolmogorov spectrum forms, as in a classical fluid, provided of course that energy is dissipated by some means at or beyond the Kolmogorov wavenumber.

## V. CONCLUSION

In this paper, we investigate the energy spectrum of superfluid turbulence generated by LES of the HVBK equations. Unknown subgrid terms are formally closed using an approximate deconvolution-type method, supplemented with a spectral viscosity for convection terms. The energy-balance spectrum calculated from the LES simulation is compared, similarity between the superfluid turbulence and classical turbulence for temperature as low as  $1.6$  K. The production term and energy transfer through the spectrum play a vital role for the energy exchange between the mean and the fluctuating motion. Of all the other terms, the production seems to be decaying faster with distance downstream. The LES simulations for a wide temperature range ( $1 \leq T \leq 2.1$  K) confirm that both components of  $^4\text{He}$  are locked together with a single velocity field at large scales, the locking mechanism is the mutual friction force between the two fluids. Therefore, in the inertial range, where the dissipation effects are absent, the contribution of the mutual friction term can be neglected.

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