Explicit Feedback Linearization of Magnetic Levitation System

Bhawna Tandon, Shiv Narayan, Jagdish Kumar

Abstract—This study proposes the transformation of nonlinear Magnetic Levitation System into linear one, via state and feedback transformations using explicit algorithm. This algorithm allows computing explicitly the linearizing state coordinates and feedback for any nonlinear control system, which is feedback linearizable, without solving the Partial Differential Equations. The algorithm is performed using a maximum of N-1 steps where N being the dimension of the system.

Keywords—Explicit Algorithm, Feedback Linearization, Nonlinear control, Magnetic Levitation System.

I. INTRODUCTION

IN the field of control system, a lot of research effort has been focused on the control of a Magnetic Levitation System (MLS). It is widely used in various fields such as frictionless bearings, levitation of wind tunnel models and high-speed Maglev passenger trains etc. But the challenge is the inherent nonlinearities and unstable aspect of MLS. Several dynamic models of Magnetic Levitation System have been proposed over the past years and various control strategies with these models have been used to compare their performance. Both the linear and nonlinear techniques have been used. Linear system model only works well over a small region of operating point. Valer and Lia build a nonlinear model for magnetic levitation system and propose systems linearization principle based on the expansion in Fourier series and the preservation of the first order terms, in order to linearize the acquired nonlinear model [1]. Ying-Shing Shiao, employed system linearization and phase-lead compensation with virtual pole cancellation to design the controller of unstable nonlinear system to maintain better stability in a levitated ball [2]. Reference [3] presented a nonlinear model for the magnetic force of magnetic levitation device and a control technique for position control of a magnetically levitated permanent magnet was proposed. A Lyapunov based stability analysis was performed to prove the stability of the control technique. In [4] the author carried out a comparative study of linear and nonlinear controllers for Maglev system and stated that, feedback linearization controller has provided significantly better trajectory tracking.

This paper is devoted to design a nonlinear controller for a Magnetic Levitation System (MLS), by using the concept of

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the explicit feedback linearization [5]. Firstly, the dynamic model of Magnetic Levitation System (MLS) is described. Then the Feedback Linearization and summary of explicit algorithm is discussed. Finally, the implementation of algorithm to the Magnetic Levitation System (MLS) is explained in detail and the simulation is done using SIMULINK/MATLAB.

II. MAGNETIC LEVITATION SYSTEM

Magnetic levitation system considered in the current analysis is consisting of a ferromagnetic ball suspended in a voltage-controlled magnetic field. Fig. 1 shows the schematic diagram of magnetic levitation system.
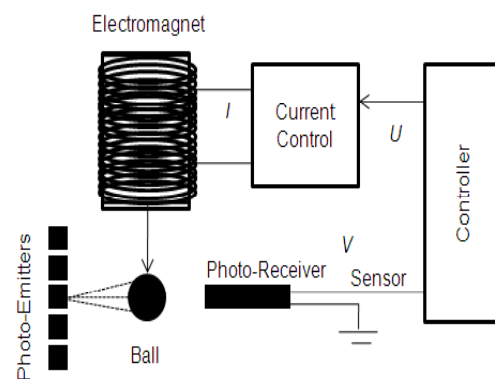

Fig. 1 Schematic diagram of magnetic levitation system

Coil acts as electromagnetic actuator, while an optoelectronic sensor determines the position of the ferromagnetic ball. By regulating the electric current in the circuit through a controller, the electromagnetic force can be adjusted to be equal to the weight of the steel ball, thus the ball will levitate in an equilibrium state. But it is a nonlinear, open loop and unstable system that demands a good dynamic model and a stabilized controller.

Dynamic behavior of magnetic levitation system can be modeled by the study of electromagnetic and mechanical sub systems.

A. Electromagnetic Dynamics Modeling

Electromagnetic force produced by current is given by the Kirchhoff's voltage law;

$$u(t) = V_R + V_L = iR + \frac{dL(x)i}{dt} \tag{1}$$

where u: applied voltage, i: current in the coil of electromagnet, R: coil's resistance and L: coil's inductance.

B. Non Linear Model

On the basis of electro-mechanical modeling nonlinear model of magnetic levitation system can be described in terms of following set of differential equations

$$v = \frac{dx}{dt}$$

$$u = Ri + \frac{dL(x)}{dt}i$$

$$m\,\ddot{x} = mg - C\left(\frac{i}{x}\right)^2$$

(2)

Equation (2) indicates that $L(x)$ is a nonlinear function of balls position x. If we take the approximation that inductance varies with the inverse of ball's position, that is

$$L(x) = L + \frac{L_o x_o}{x}$$

(3)

where L is the constant inductance of the coil in the absence of ball, L_o is the additional inductance that is contributed by the presence of the ball, x_o is the equilibrium position. Substituting (3) into (2) results in

$$u(t) = iR + \frac{d}{dt}\left(L_c + \frac{L_o x_o}{x}\right)i$$

$$u(t) = iR + L\frac{di}{dt} - \left(\frac{L_o x_o i}{x^2}\right)\frac{dx}{dt}$$

(4)

By substituting

$$L_o x_o = 2C,$$

We get

$$u = iR + L\frac{di}{dt} - C\left(\frac{i}{x^2}\right)\frac{dx}{dt}$$

(5)

Taking $x = x_1, v = x_2, i = x_3$, (1) (2) and (4) can be expressed in vector format where position of ball is taken as output as under:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ g - \frac{C}{m}\left(\frac{x_3}{x_1}\right)^2 \\ -\frac{R}{L}x_3 + \frac{2C}{L}\left(\frac{x_2 x_3}{x_1^2}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$$

$$\dot{x} = f(x) + g(x)u$$

(6)

III. NON LINEAR SYSTEMS AND FEEDBACK LINEARIZATION

Consider a smooth control affine system:

$$\Pi: \dot{x} = f(x) + g(x)u = f(x) + \sum_{j=1}^{m} g_j(x)u_j, \quad x \in R^n$$

around an equilibrium (x_e, u_e).

Let us consider another smooth control affine system:

$$\tilde{\Pi}: \dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})v = \tilde{f}(\tilde{x}) + \sum_{j=1}^{m} \tilde{g}_j(\tilde{x})v_j, \quad \tilde{x} \in R^n$$

(7)

The systems Π and $\tilde{\Pi}$ are said to be feedback equivalent [5]-[7], if there exists a transformation

$$\Xi = \left\{ \begin{array}{c} \tilde{x} = \phi(x) \\ u = \alpha(x) + \beta(x)v \end{array} \right\}$$

that maps Π into $\tilde{\Pi}$, that is such that the following Partial Differential Equations hold:

$$\left\{ \begin{array}{c} d\phi(x).(f(x) + g(x)\alpha(x)) = \tilde{f}(\phi(x)) \\ d\phi(x).(g(x)\beta(x)) = \tilde{g}(\phi(x)) \end{array} \right\}$$

The following two problems were considered by Krener [8] and Brockett [9].

Problem 1: If there exists a local diffeomorphism $w = \phi(x)$ defining new coordinates $w = (w_1, w_2, \dots, w_n)$ in which the transformed system takes the linear form:
$\forall: \dot{w} = Fw + Gu = Fw + \sum_{j=1}^{m} G_i u_i, w \in R^n$ and $u \in R^m$ Then the system Π is said to be S- linearizable (state linearizable).

Problem 2: If there exists a local feedback transformation $\Xi = (\phi, \alpha, \beta)$ that takes Π into a linear system

$$\forall: \dot{w} = Aw + Bv = Aw + \sum_{j=1}^{m} b_i v_i, w \in R^n \text{ and } v \in R^m$$

Then the system Π is said to be F-linearizable (feedback linearizable).

Theorem 1: A control system

$$\Pi: \dot{x} = f(x) + g(x)u$$

$$\varphi_j(x) = x_j + \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} L_{\sigma_k \eta}^{s-1}(\sigma_k \eta_j)(x)$$

$$\varphi_k(x) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_k^s}{s!} L_{\sigma_k \eta}^{s-1}(\sigma_k)(x)$$

(8)

is locally equivalent to a linear controllable system $\forall : \dot{w} = Aw + Bv$ via change of coordinates $w = \phi(x)$ and feedback $v = \alpha(x) + \beta(x)u,$ if and only if

(1) Dim $\Psi^n(x) = n$

(2) Ψ^{n-1} is involutive

where,

$$\Psi^k = \left\{ ad_f^{\; r} g_j, \quad 0 \leq r \leq k-1, \quad 1 \leq j \leq m \right\}, k = 1, \ldots\ldots\ldots n$$

with

$$ad_f^{\; 0} g_j = g_j \text{ and } ad_f^{\; t} g_j = \left[f, ad_f^{\; t-1} g_j \right]$$

For all $t \geq 1.$

An algorithm proposed by Issa Amadou Tall [5]-[7] gives a complete solution to problem 2 without solving the Partial Differential Equations. This algorithm allows constructing explicitly feedback linearizing coordinates. Consider $\prod : \dot{x} = f(x) + g(x)u$ and let $1 \leq q \leq n-1.$ System $\prod$ is in form $(FB)_q$ denoted by $\prod_q^{FB}$ if in some coordinates $x_q = (x_{q1}, \ldots\ldots\ldots, x_{qn}),$ it takes the form

$$\prod_q^{FB} : \begin{cases} \dot{x}_{qi} = F_{qi}(x_{q1}, \ldots\ldots\ldots, x_{qk+1}), \text{ if } 1 \leq i \leq q \\ \dot{x}_{qk+1} = F_{qk+1}(x_{q1}, \ldots\ldots\ldots, x_{qk+2}) \\ \ldots\ldots\ldots\ldots \\ \dot{x}_{qn-1} = F_{qn-1}(x_{q1}, \ldots\ldots\ldots, x_{qn}) \\ \dot{x}_{qn} = F_{qn}(x_{q1}, \ldots\ldots\ldots, x_{qn}) + u \end{cases}$$

where $q = k.$

Theorem 2: Consider a linearly controllable system

$$\prod : \dot{x} = f(x) + g(x)u, \quad x \in R^n, \quad u \in R$$

Assume it is Feedback Linearizable; there exists a sequence of explicit coordinates changes

$$\phi_n(x_n), \phi_{n-1}(x_{n-1}), \ldots\ldots\ldots\ldots \phi_2(x_2)$$

That gives rise to a sequence of $(FB)_q$ forms

$$\prod_{n-1}^{FB}, \prod_{n-2}^{FB}, \ldots\ldots\ldots\ldots, \prod_1^{FB}$$

Such that for any $2 \leq q \leq n$ we get

$$\prod_{q-1}^{FB} = (\phi_q) \bullet \prod_q^{FB}$$

moreover, in the coordinates $z = \phi_2(x_2)$ the system $\prod$ takes the feedback form (FB).

The proof of this theorem is based on the following theorem no. 3:

Theorem 3: Let η be a smooth vector field on $R^n,$ any integer $1 \leq k \leq n$ such that $\eta_k(0) \neq 0$ and $\sigma_k(x) = 1/\eta_k.$

Define $z = \phi(x)$ by its components as following:

For any, $1 \leq j \leq n, \quad j \neq k$ the diffeomorphism $z = \phi(x)$ satisfies $\phi * (\eta) = \partial_{zk}.$

The diffeomorphism $x = \lambda(z)$ given by its components

$$\lambda_j(z) = z_j + \sum_{s=1}^{\infty} \frac{z_k^{\; s}}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^{\; i} \partial_{zk}^{\; i} . L_\eta^{\; s-i-1} (\eta_j)(z) \right)$$

$$\lambda_k(z) = \sum_{s=1}^{\infty} \frac{z_k^{\; s}}{s!} \left(\sum_{i=0}^{s-1} (-1)^i C_s^{\; i} \partial_{zk}^{\; i} . L_\eta^{\; s-i-1} (\eta_k)(z) \right)$$

For any, $1 \leq j \leq n, \quad j \neq k$ is the inverse of $z = \phi(x),$ such that $\dfrac{\partial \lambda(z)}{\partial z_k} = \eta(\lambda(z)).$

A. Summary of Algorithm

Start with the system

$$\prod : \dot{x} = f(x) + g(x)u, \quad x \in R^n, \quad u \in R$$

Step 0: Normalize the vector field $g \leftrightarrow g = (0,0,\ldots\ldots0,1)^T$ and apply a linear feedback to put the linearization in Brunovsky form.

Step $n-k$: If the condition

$$\frac{\partial^2 f_j}{\partial x_{k+1}^{\; 2}} = \gamma_{n-k}(x) \frac{\partial f_j}{\partial x_{k+1}}, \quad 1 \leq j \leq k \tag{9}$$

fails, then the system is not feedback linearizable and algorithm stops.

If the above condition is satisfied, then first k components

$$f_1, f_2, \ldots\ldots\ldots, f_k$$

can be decomposed as follows:

$$f_j(x_1, \ldots\ldots\ldots, x_{k+1}) = F_j(x) + \eta_j(x)\zeta(x) \quad 1 \leq j \leq k$$

to construct a change of coordinates $z = \phi(x),$ theorem in [6] is applied to rectify the non-singular vector field.

$$\eta(x) = \eta_1(x)\partial x_1 + \ldots\ldots + \eta_k(x)\partial x_k + 0.\partial x_{k+1} + \ldots\ldots + 0.\partial x_n$$

That is such that, $\phi * (\eta)(z) = \partial_{zk}$.

Repeat step $n-k$ for $k = n-1,\ldots,2$. End if the algorithm fail or the system is in FB form.

IV. DESIGN EXAMPLE

Consider the Magnetic Levitation System modeled by (6):

$$f(x) = \begin{bmatrix} x_2 \\ g - \dfrac{C}{m}\left(\dfrac{x_3}{x_1}\right)^2 \\ -\dfrac{R}{L}x_3 + \dfrac{2C}{L}\left(\dfrac{x_2 x_3}{x_1^2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} x_2 \\ 9.8 - \dfrac{0.0001}{0.05}\left(\dfrac{x_3}{x_1}\right)^2 \\ -\dfrac{1}{0.01}x_3 + \dfrac{2(0.0001)}{0.001}\left(\dfrac{x_2 x_3}{x_1^2}\right) \end{bmatrix}$$

$$f(x) = \begin{bmatrix} x_2 \\ 9.8 - 0.002\left(\dfrac{x_3}{x_1}\right)^2 \\ -100 x_3 + 0.02\left(\dfrac{x_2 x_3}{x_1^2}\right) \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix}$$

Denote $\eta(x) = g(x)$ and apply Theorem 2 with $n = 3$ and $\sigma_3(x) = 1/100$, thus

$$\sigma_3 \eta = 0.01 \begin{bmatrix} 0 \\ 0 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

$$L_{\sigma_3 \eta}{}^0(\sigma_3 \eta_1) = \sigma_3 \eta_1 = 0.01(0) = 0$$

It follows that

$$z_1 = \phi_1(x) = x_1 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_{\sigma_k \eta}{}^{s-1}(\sigma_k \eta_j)(x)$$

$$= x_1 + \begin{bmatrix} (-1)x_3(L_{\sigma_3 \eta}{}^0(\sigma_3 \eta_1)(x)) + \\ (-1)^2 \dfrac{x_3^2}{2!}(L_{\sigma_3 \eta}(\sigma_3 \eta_1)(x)) + \ldots\ldots\ldots\ldots \end{bmatrix}$$

$$= x_1$$

and

$$z_2 = \varphi_2(x) = x_2 + \sum_{s=1}^{\infty} \frac{(-1)^s x_3^s}{s!} L_{\sigma_k \eta}{}^{s-1}(\sigma_k \eta_j)(x) = x_2$$

Now,

$$\phi_k(x) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_k^s}{s!} L_{\sigma_k \eta}{}^{s-1}(\sigma_k)(x)$$

This implies

$$z_3 = \begin{bmatrix} \dfrac{(-1)^2 x_3}{1!} L_{\sigma_3 \eta}{}^0(\sigma_3)(x) + \\ (-1)^2 \dfrac{x_3^2}{2!}(L_{\sigma_3 \eta}(\sigma_3)(x)) + \ldots\ldots\ldots\ldots \end{bmatrix}$$

$$= x_3(0.01) = 0.01 x_3$$

We apply the change of coordinates:

$$z_1 = x_1,\ z_2 = x_2,\ z_3 = 0.01 x_3$$

To transform the original system into

$$\dot{z} = \tilde{f}(z) + \tilde{g}(z)u \cong \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = 9.8 - 20\dfrac{z_3^2}{z_1^2} \\ \dot{z}_3 = -100 z_3 + 0.02\dfrac{z_2 z_3}{z_1^2} + u \end{cases}$$

The system is in (FB) form and can be put into the linear Brunovsky form [6] via

$$w_1 = \hat{h}(z) = z_1$$

$$w_2 = L_{\hat{f}}\hat{h}(z) = \frac{\partial \hat{h}(z)}{\partial z}(\hat{f}) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_2 \\ 9.8 - 20\dfrac{z_3^2}{z_1^2} \\ -100 z_3 + 0.02\dfrac{z_2 z_3}{z_1^2} \end{bmatrix} = z_2$$

$$w_3 = L_{\hat{f}}{}^2\left(\hat{h}(z)\right) = L_{\hat{f}}\left(L_{\hat{f}}\hat{h}(z)\right) = L_{\hat{f}}(z_2) = 9.8 - 20\frac{z_3^2}{z_1^2}$$

and $v = L_{\hat{f}}{}^3\hat{h}(z) + L_{\hat{g}}L_{\hat{f}}{}^2\hat{h}(z)u.$

The composition of the two-step changes of coordinates and feedback gives linearizing coordinates for the original system

$$w_1 = z_1 = x_1$$

$$w_2 = z_2 = x_2$$

$$w_3 = 9.8 - 20\frac{z_3^2}{z_1^2} = 9.8 - 20(0.01)^2\frac{x_3^2}{x_1^2} = 9.8 - 0.002\frac{x_3^2}{x_1^2}$$

Brings system $\prod$ into Brunovsky form [10]

$$\dot{w}_1 = \dot{x}_1 = w_2$$

$$\dot{w}_2 = \dot{x}_2 = w_3$$

$$\dot{w}_3 = 0.004\frac{x_3^2}{x_1^3}\dot{x}_1 - 0.004\frac{x_3}{x_1^2}\dot{x}_3 = v \tag{10}$$

where,

$$v = 0.004\frac{x_3^2 x_2}{x_1^3} + 0.4\frac{x_3^2}{x_1^2} - 8\times10^{-5}\frac{x_3^2 x_2}{x_1^4} - 0.4\frac{x_3}{x_1^2}u \tag{11}$$
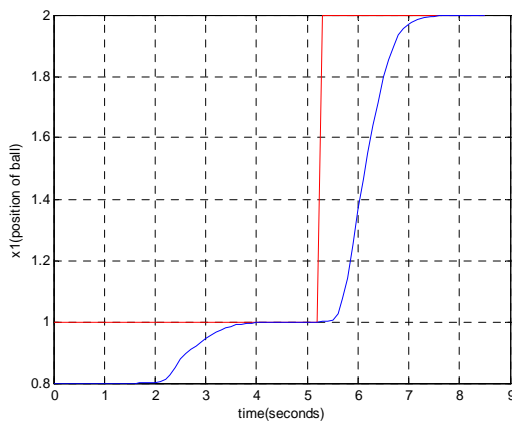

Fig. 2 Unit step response of x1(Position of ball)

On simulating the above linear system (10) with nonlinear feedback (11) using Nelinsys/SIMULINK/MATLAB, unit step response of x1 is shown in the Fig. 2.

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