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Diameter of Zero Divisor Graphs of Finite Direct Product of Lattices

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Abstract—In this paper, we verify the diameter of zero divisor graphs with respect to direct product.

Keywords—Atomic lattice, complement of graph, diameter, direct product of lattices, 0-distributive lattice, girth, product of graphs, prime ideal, zero divisor graph.

I. Introduction

The study of zero divisor graphs was initiated by Istvan Beck [5] in 1988. He proposed a method for coloring a commutative ring by associating the ring to a simple graph, the vertices of which were defined to be the elements of the ring, with vertices x and y joined by an edge when $xy = 0$. In 1999, Anderson and Livingston [3] changed this definition, restricting the set of vertices to the non-zero zero divisors of the ring. Afterwards, the research work was taken up for non-commutative rings by Redmond [18], while DeMeyer, McKenzie, and Schneider [6] looked at the zero-divisor graphs of commutative semigroups with 0. Nimbhorkar, Wasadikar and DeMeyer [17] introduced the zero divisor graphs of meet semi-lattices with 0 and proved a form of Beck's Conjecture. They associated a zero divisor graph to a meet semi-lattice L with 0, whose vertices are the elements of L and two distinct elements $x, y \in L$ are adjacent if and only if $x \wedge y = 0$.

This work was further extended by Halaš and Jukl [8] to posets with 0 (see also,[9]). Halaš and Jukl [8] introduced the concept of zero divisor graph to posets with 0, where vertex set of the zero divisor graph $G(P)$ is the poset P and two vertices x and y are adjacent if and only if 0 is the only element below both x and y. There are many authors working in this area, see Alizadeh, et. al., [1], [2], Estaji [7], Joshi, et. al., [10], [11], [12], [13], [14], [15].

The zero divisor graph with respect to an ideal was first defined in the context of commutative rings by Redmond [18]. In [10], Joshi introduced a similar graph in the context of posets, which coincides with the definition of zero divisor graphs given by Lu and Wu [16].

The concept of a zero divisor graph of a poset P with respect to an ideal I is due to Joshi [10]. We consider this definition when P is a lattice.

Definition 1: Let I be an ideal of a lattice L with 0. We associate an undirected graph, called the *zero divisor graph of L with respect to the ideal* I, denoted by $G_I(L)$ in which the set of vertices is $V(G_I(L)) = \{x \notin I \mid x \wedge y \in I$ for some $y \notin I\} = Z_I(L)^*$ and two distinct vertices x, y are adjacent if and only if $x \wedge y \in I$. When $I = \{0\}$ then the corresponding zero divisor graph is denoted by $G_{\{0\}}(L)$.

We recall the following concepts from graph theory, see D. B. West [20].

Definition 2: Let G be a graph. Let x, y be distinct vertices in G. We denote by $d(x, y)$ the length of a shortest path from x to y (if it exists) and put $d(x, y) = \infty$ otherwise we write $d(x, x) = 0$ for $x \in V(G)$. The *diameter* of G is denoted by $diam(G)$, $diam(G) = sup\{d(x, y) \mid \ x, y \in V(G)$. A *cycle* in a graph G is a path that begins and ends at the same vertex. The *girth* of G, denoted $gr(G)$, is the length of a shortest cycle in G (and $gr(G) = \infty$ if G has no cycle).

In fact, in Section II, it is proved that the diameter and girth of the zero divisor graph of direct product of lattices with respect to different ideals is always 3. An immediate consequence of this result is diameter and girth of a Boolean lattice $\mathbf{2}^n$ (for $n \geq 3$) is 3. In Section III, we give a sufficient condition for connectedness of the complement of the zero divisor graph of a lattice.

II. Diameter of zero divisor graphs of finite direct product of lattices

The diameter of a zero divisor graph for finite direct product of commutative rings was studied by Atani and Kohan [4]. In this section we study the diameter of zero divisor graphs of finite direct product of lattices.

Throughout this paper, we assume that all lattices have the smallest element $\mathbf{0}$**.**

Definition 3: The *product of graphs* $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is showing by $G_1 \times G_2$ and is defined as following:
Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 is adjacent to $v_2]$ or $[u_2 = v_2$ and u_1 is adjacent to $v_1]$. The following Fig. 1, illustrates the product of two graphs.

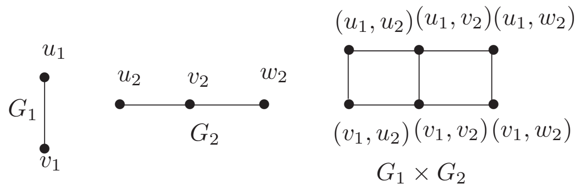

Fig. 1. The product of two graphs

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Definition 4: Let L and K be lattices. Define $\wedge$ and $\vee$ in $L \times K$ component-wise:

$$< a_0, b_0 > \wedge < a_1, b_1 > \ = \ < a_0 \wedge a_1, b_0 \wedge b_1 >$$
$$< a_0, b_0 > \vee < a_1, b_1 > \ = \ < a_0 \vee a_1, b_0 \vee b_1 >$$

This makes $L \times K$ into a lattice, called the *direct product* of L and K. As an example see the following Fig. 2.

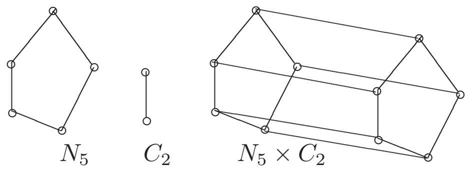

Fig. 2. The direct product $N_5 \times C_2$

Remark 1: Let L_1 and L_2 be two lattices. Let $G_{(0,0)}(L_1 \times L_2)$ be the zero divisor graph of product of lattices $L = L_1 \times L_2$ with respect to the ideal $I = (0, 0)$. Now we give the set of vertices (edges) of $G_{(0,0)}(L_1 \times L_2)$ in terms of vertex set (edge set) of $G_{\{0\}}(L_1)$ and $G_{\{0\}}(L_2)$ respectively. The set of vertices of $G_{(0,0)}(L_1 \times L_2)$ is $V(G_{(0,0)}(L_1 \times L_2)) = \Big\{(a, b) \neq (0, 0) \mid a \in V(G_{\{0\}}(L_1)) \cup \{0\}$ or $b \in V(G_{\{0\}}(L_2)) \cup \{0\}\Big\}$ and two distinct vertices (a, b) and (x, y) are adjacent ($e = ((a, b), (x, y)) \in E(G_{(0,0)}(L))$) if and only if one of the following conditions hold:

either $e \in G_{\{0\}}(L_1) \times G_{\{0\}}(L_2)$;
or $a = 0$, $x = 0$ and $(b, y) \in E(G_{\{0\}}(L_2))$;
or $b = 0$, $y = 0$ and $(a, x) \in E(G_{\{0\}}(L_1))$;
or $a = 0$, $x \neq 0$, $b \neq 0$, $y = 0$;
or $a \neq 0$, $x = 0$, $b = 0$, $y \neq 0$.

Definition 5: A non-empty subset I of a lattice L is an *ideal* of L if $a, b \in I$ and $c \in L$ with $c \leq a$ implies $c \in I$ and $a \vee b \in I$. An ideal $I \neq L$ is a *prime ideal* if $a \wedge b \in I$ implies either $a \in I$ or $b \in I$.

The following theorem is essentially due to Joshi [10].

Theorem 1: Let I be an ideal of a lattice L. Then $G_I(L)$ is a connected graph $diam(G_I(L)) \leq 3$.

Lemma 1: Let $L_1, L_2, ..., L_n$ be lattices with ideals $I_1, I_2, ..., I_n$, respectively. Then $I = I_1 \times I_2 \times ... \times I_n$ forms an ideal in $L = L_1 \times L_2 \times ... \times L_n$.

Proof: Easy to prove. ∎

Remark 2: Note that if one of I_j's, $j \in \{1, 2, ..., n\}$ is not prime, then $I = I_1 \times I_2 \times ... \times I_n$ is not prime.

Lemma 2: Let L_1 and L_2 be two lattices with I_2, a non-prime ideal, then $diam(G_{L_1 \times I_2}(L_1 \times L_2)) = diam(G_{I_2}(L_2))$.

Proof: Suppose $diam(G_{L_1 \times I_2}(L_1 \times L_2)) = n > diam(G_{I_2}(L_2))$. Then $n = 2$ or $n = 3$. Let $(a_0, x_0), (a_1, x_1), ..., (a_n, x_n) \in Z_{L_1 \times I_2}(L_1 \times L_2)^*$ be such that $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$ is a minimal path. This implies that $a_i \wedge a_{i+1} \in L_1$ and $x_i \wedge x_{i+1} \in I_1$ for $i \in \{0, 1, ..., n-1\}$. Hence we have a path $x_0 - x_1 - ... - x_n$ in $G_{I_2(L_2)}$. Since $n > diam(G_{I_2}(L_2))$, $x_0 - x_1 - ... - x_n$ is not a minimal path. This can happen in two ways.

If there exist i, j such that $0 \leq i < j \leq n$, $j \neq i + 1$ and $x_i - x_j$, then $(a_i, x_i) - (a_j, x_j)$, a contradiction to $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n)$ is a minimal path. So $diam(G_{L_1 \times I_2}(L_1 \times L_2)) \leq diam(G_{I_2}(L_2))$. Suppose $diam(G_{L_1 \times I_2}(L_1 \times L_2)) = n < diam(G_{I_2}(L_2))$ such that $1 \leq n \lneqq 3$. Then there exist $x_0, x_1, .., x_n \in Z_{I_2}(L_2)^*$ such that $x_0 - x_1 - ... - x_{n+1}$ is a minimal path. Since $L_1 = I_1$, $\forall a_0, a_1, ..., a_{n+1} \in L_1$, $(a_0, x_0) - (a_1, x_1) - ... - (a_n, x_n) - (a_{n+1}, x_{n+1})$ is a minimal path of length $n+1$, a contradiction.

Thus $diam(G_{L_1 \times I_2}(L_1 \times L_2)) = diam(G_{I_2}(L_2))$. ∎

Definition 6: Let I be an ideal of a lattice L. We define the set $Z_I(L)^* = \{r \notin I \mid r \wedge a \in I$ for some $a \notin I\}$. Clearly, $Z_I(L) = Z_I(L)^* \cup I$.

Lemma 3: Let $L_1, L_2, ..., L_{n-1}$ and L_n be lattices with ideals $I_1, I_2, ..., I_n$ respectively such that $Z_{I_i}(L_i)^* \neq \emptyset$ for $\forall i$ and let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ $(n \geq 2)$ and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ $(n \geq 2)$. Then $diam(G_{I_1 \times I_2 \times ... \times I_n}(L_1 \times L_2 \times ... \times L_n)) > 1$.

Proof: Let $x_1 \in Z_{I_1}(L_1)^*$ and $y_1 \in Z_{I_2}(L_2)^*$. So there exist $x_2 \in L_1 \backslash I_1$ and $y_2 \in L_2 \backslash I_2$ such that $x_1 \wedge x_2 \in I_1$ and $y_1 \wedge y_2 \in I_2$. Consider, $(x_1, y_1, 0, ..., 0), (0, y_1, 0, ..., 0) \in L_1 \times L_2 \times ... \times L_n$. It is easy to see that $(x_1, y_1, 0, ..., 0), (0, y_1, 0, ..., 0) \in V(G_{I_1 \times I_2 \times .... \times I_n}(L_1 \times L_2 \times ... \times L_n))$. Since $(x_1, y_1, 0, ..., 0), (0, y_1, 0, ..., 0)$ are not adjacent, $diam(G_{I_1 \times I_2 \times ... \times I_n}(L_1 \times L_2 \times ... \times L_n)) > 1$. ∎

Theorem 2: Let $L_1, L_2, ..., L_{n-1}$ and L_n be lattices with ideals $I_1, I_2, ..., I_n$ respectively, such that at least two of them are non-prime. Let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ $(n \geq 2)$ and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ $(n \geq 2)$. If $diam(G_I(L)) = 2$ then $L_i - Z_{I_i}(L_i) = \emptyset$ for some $i \in \{1, 2, ..., n\}$.

Proof: Since at least two of the ideals $I_1, I_2, ..., I_n$ are non-prime, we have I is non-prime. This gives $Z_I(L)^* \neq \emptyset$. Assume that $diam(G_I(L)) = 2$. We claim that $L_i - Z_{I_i}(L_i) = \emptyset$ for some $i \in \{1, 2, ..., n\}$. Suppose on the contrary that $L_i - Z_{I_i}(L_i) \neq \emptyset, \forall i$. Then there must exist $x_i \in L_i - Z_{I_i}(L_i)$ for each $i \in \{1, 2, ..., n\}$. Without loss of generality, let I_1 and I_2 be two non-prime ideals. Then $z_j \in Z_{I_j}(L_j)^*$ for $j \in \{1, 2\}$. So there is an element z'_j of $Z_{I_j}(L_j)^*$ such that $z_j \wedge z'_j \in I_j$ for $j \in \{1, 2\}$. If $a = (z_1, x_2, x_3, ..., x_n)$ and $b = (x_1, z_2, x_3, ..., x_n)$ then $a \wedge a' \in I$ and $b \wedge b' \in I$ where $a' = (z'_1, 0, ..., 0)$ and $b' = (0, z'_2, 0, ..., 0)$. So $a, b \in Z_I(L)^*$. Clearly, $a \wedge b \notin I$. Since $diam(G_I(L)) = 2$, there must be some $c = (c_1, c_2, ..., c_n) \in Z_I(L)^*$ such that $a \wedge c, b \wedge c \in I$. But $a \wedge c = (z_1 \wedge c_1, x_2 \wedge c_2, ..., x_n \wedge c_n) \in I$, *i.e,* $z_1 \wedge c_1 \in I_1$ and $x_i \wedge c_i \in I_i$ for $i \in \{2, 3, ..., n\}$ but $x_i \in L_i - Z_{I_i}(L_i)$. Hence $x_i \notin I_i$. This together with $x_i \wedge c_i \in I_i$ gives $c_i \in I_i$ for $i \in \{2, 3, ..., n\}$. (1)

Similarly, $b \wedge c \in I$, but $b \wedge c = (x_1 \wedge c_1, z_2 \wedge c_2, ..., x_n \wedge c_n) \in I$, *i.e,* $z_2 \wedge c_2 \in I_2$ and $x_i \wedge c_i \in I_i$ for $i \in \{1, 3, ..., n\}$ but $x_i \in L_i - Z_{I_i}(L_i)$. Therefore we must have $c_i \in I_i$ for $i \in \{1, 3, ..., n\}$. (2)

From (1) and (2) we get $c = (c_1, c_2, ..., c_n) \in I$, a contradiction to the fact that $c \notin I$. Thus $L_i = Z_{I_i}(L_i)$ for some $i \in \{1, 2, ..., n\}$. ∎

Remark 3: We provide an example of a lattice L such that $L = Z_I(L)$ for an ideal I of L. Consider the lattice of all proper subsets of $\mathbb{N}$, the set of all natural numbers under set inclusion. Then it is easy to observe that $L = Z_{\{\emptyset\}}(L)$.

In view of Theorem 2, it is clear that

$diam(G_{\{0\}}(L_1 \times L_2 \times ... \times L_n)) = 3$ whenever L_i's are finite for every i.

Corollary 1: If $L_i - Z_{I_i}(L_i) \neq \emptyset$ for every $i \in \{1, 2, ..., n\}$, then $diam(G_I(L)) = 3$. In particular $diam(G_{\{0\}}(L)) = 3$ for $L = \mathbf{2}^n$, a Boolean lattice, for $n \geq 3$.

Proof: It is easy to observe that diameter of the zero divisor graph of $L = \mathbf{2}^3$ is 3. Hence the result follows from Theorem 1, Lemma 3 and Theorem 2. ∎

Theorem 4: Let $L_1, L_2, ..., L_{n-1}$ and L_n be lattices with ideals $I_1, I_2, ..., I_n$ respectively, such that at least two of them are non-prime. Let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ $(n \geq 2)$ and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ $(n \geq 2)$. If $diam(G_{I_1}(L_1)) = diam(G_{I_2}(L_2)) = ... = diam(G_{I_n}(L_n)) = 3$ Then $diam(G_I(L)) = 3$.

Proof: Since for each $i \in \{1, 2, .., n\}$, $diam(G_{I_i}(L_i)) = 3$, there exist non adjacent vertices $x_i, y_i \in Z_{I_i}(L_i)^*$ such that there is no $z_i \in Z_{I_i}(L_i)^*$ with $x_i \wedge z_i, y_i \wedge z_i \in I_i$. Consider $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$. For each $i \in \{1, 2, ..., n\}$, there are elements $x_i', y_i' \in Z_{I_i}(L_i)^*$ such that $x_i \wedge x_i' \in I_i$ and $y_i \wedge y_i' \in I_i$, so $x, y \in Z_I(L)^*$. As $x \wedge y \notin I$ and $diam(G_I(L)) \neq 1$, therefore $diam(G_I(L)) = 2$ or 3. If $diam(G_I(L)) = 2$, then there exist an element $a = (a_1, a_2, ..., a_n) \in Z_I(L)^*$ such that we have a path $x - a - y$ in $G_I(L)$. Therefore, we have $x_i \wedge a_i, y_i \wedge a_i \in I_i$. Hence $d(x_i, y_i) = 2$, which is a contradiction to the fact that $diam(G_{I_i}(L_i)) = 3$. So $diam(G_I(L)) = 3$. ∎

Theorem 5: Let $L_1, L_2, ..., L_{n-1}$ and L_n be lattices with ideals $I_1, I_2, ..., I_n$ respectively, such that at least two of them are non-prime. Let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ $(n \geq 2)$ and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ $(n \geq 2)$. Then $G_I(L)$ has a cycle of length 3. Hence $gr(G_I(L)) = 3$.

Proof: Take non-zero elements $a = (a_1, 0, ..., 0)$, $b = (0, b_2, 0, ..., 0)$ and $c = (0, 0, c_3, 0, ..., 0)$ of a lattice L. Clearly, $a, b, c \in V(G_I(L))$ and $a \wedge b, a \wedge c, b \wedge c \in I$. Therefore, we get a cycle $a - b - c - a$, hence the girth is 3. ∎

Lemma 4: Let $L_1, L_2, ..., L_{n-1}$ and L_n be lattices with ideals $I_1, I_2, ..., I_n$ respectively, such that at least two of them are non-prime. Let $L = L_1 \times L_2 \times L_3 \times ... \times L_n$ $(n \geq 2)$ and $I = I_1 \times I_2 \times I_3 \times ... \times I_n$ $(n \geq 2)$. If a is a cut vertex of $G_I(L)$, then there exists some $a_i \neq 0$; $(1 \leq i \leq n)$ such that $a = (0, 0, ..., a_i, ...0)$.

Proof: Let a be a cut vertex of $G_I(L)$, with $a = (a_1, a_2, ..., a_i, ...a_n)$ where $a_i \in L_i$. Since a is a cut vertex, for any two arbitrary elements $b, c \in V(G_I(L))$, the path between b and c goes through of a. Consider the element $d = (0, 0, ..., a_i, 0, ..., 0)$. Then we get a path $b - d - c$. Since a is a cut vertex, we have $a = d$. Then $a = (0, 0, ..., a_i, ...0)$. ∎

III. Complement of Zero Divisor Graphs of Direct Product of Lattices

The complement of the zero divisor graph of a lattice was studied by Joshi and Khiste [11].

In this section, we study the connectivity of the complement of zero divisor graphs of direct product of lattices.

Definition 7: Let $G = (V, E)$ be a simple graph. The complement of G, denoted by G^c, is defined by setting $V(G^c) = V(G) = V$ and two distinct vertices $u, v \in V$are joined by an edge in G^c if and only if there exists no edge in G joining u and v.

We give examples of two lattices L_1 and L_2 such that $(G_{\{0\}}(L_i))^c$, the complement of the zero divisor graph of a lattice L_i $(i = 1, 2)$ is disconnected and connected respectively.

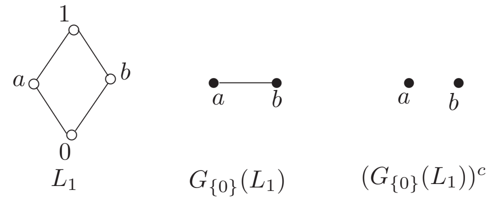

Fig. 3. Connected zero divisor graph whose complement is disconnected

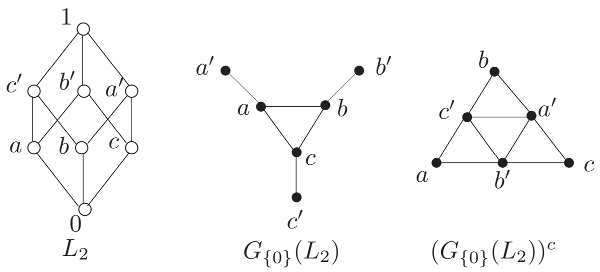

Fig. 4. Zero divisor graph and its complement both are connected

From Fig. 3, it is clear that $G_{\{0\}}(L_1)$ is connected but not $(G_{\{0\}}(L_1))^c$ whereas in Fig. 4, $G_{\{0\}}(L_2)$ and $(G_{\{0\}}(L_2))^c$ both are connected. Hence it is natural to ask the following question.

Question: *When $(G_I(L))^c$ is connected ?*

We answer this question in the Theorem 5. To prove this theorem, we need the following results in sequel and the proof of Theorem 5 is mentioned at the end of this section.

We use the notation, $\mathbf{0} = (0, 0, ..., 0)$.

Lemma 5: Let $L = L_1 \times L_2 \times ... \times L_n$. If $(G_{\{\mathbf{0}\}}(L))^c$ is connected, then $diam(G_{\{\mathbf{0}\}}(L))^c \geq 2$.

Proof: Let $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in Z_{\{\mathbf{0}\}}(L)^*$ be two distinct elements. By Theorem 1, $G_{\{\mathbf{0}\}}(L)$

is connected; hence there exists $c = (c_1, c_2, ..., c_n) \in Z_{\{\mathbf{0}\}}(L)^*$ such that $c \wedge a = \mathbf{0}$. Hence, if $(G_{\{\mathbf{0}\}}(L))^c$ is connected, then $d(a,c) \geq 2$ in $(G_{\{\mathbf{0}\}}(L))^c$ and so $diam(G_{\{\mathbf{0}\}}(L))^c \geq 2$. ■

Definition 8: A lattice L with 0 is said to be 0-*distributive* if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$ for $a, b, c \in L$, see Varlet [19].

A lattice L with 1 is said to be 1-*distributive* if $a \vee b = 1$ and $a \vee c = 1$ imply $a \vee (b \wedge c) = 1$ for $a, b, c \in L$.

A bounded lattice which is both 0-distributive and 1-distributive is called *0-1-distributive* lattice.

Lemma 6: Let L_1, L_2 be 1-distributive lattices. Then direct product of L_1 and L_2 is also a 1-distributive lattice.

Proof: Let L_1 and L_2 be 1-distributive lattices. To show that $L = L_1 \times L_2$ is 1-distributive lattice, it is enough to show that if $(x_1, y_1) \vee (x_2, y_2) = (1, 1)$ and $(x_1, y_1) \vee (x_3, y_3) = (1, 1)$ then $(x_1, y_1) \vee ((x_2, y_2) \wedge (x_3, y_3)) = (1, 1)$ for any $x_i \in L_1$ and $y_i \in L_2$ where $i \in \{1, 2, 3\}$. From the hypothesis we can conclude that $(x_1 \vee x_2, y_1 \vee y_2) = (1, 1) = (x_1 \vee x_3, y_1 \vee y_3)$, *i.e,* $x_1 \vee x_2 = x_1 \vee x_3 = 1$ and $y_1 \vee y_2 = y_1 \vee y_3 = 1$. Since L_1 and L_2 are 1-distributive lattices, we have $x_1 \vee (x_2 \wedge x_3) = 1$ and $y_1 \vee (y_2 \wedge y_3) = 1$.

Therefore $(x_1, y_1) \vee ((x_2, y_2) \wedge (x_3, y_3)) = (x_1, y_1) \vee (x_2 \wedge x_3, y_2 \wedge y_3) = (x_1 \vee (x_2 \wedge x_3), y_1 \vee (y_2 \wedge y_3)) = (1, 1)$. ■

Lemma 7: Let $L_1, L_2, ..., L_n$ be 1-distributive lattices. Then $L = L_1 \times L_2 \times ... \times L_n$ is also a 1-distributive lattice.

Proof: Follows by using mathematical induction. ■

Corollary 2: Let $L_1, L_2, ..., L_n$ be 0-distributive lattices. Then $L = L_1 \times L_2 \times ... \times L_n$ is also 0-distributive lattice.

Definition 9: A bounded lattice L is *complemented* if, for each element x, there exists at least one element y such that $x \wedge y = 0$ and $x \vee y = 1$. In a lattice L with 0, an element y is called a *semi-complement* of x if $x \wedge y = 0$; and L is said to be *semi-complemented(SC)* if each $x \in L$ (with $x \neq 1$, if 1 exists in L)admits at least one non zero semi-complement.

Definition 10: A lattice L is called *atomic* if L has 0 and, for every $(\neq 0)a \in L$, there is an atom $p \leq a$. A lattice L is called *co-atomic* if L has 1 and, for every $(\neq 1)a \in L$, there is a co-atom $q \geq a$.

Lemma 8: Let $L_1, L_2, ..., L_n$ be semi-complemented lattices. Then $L = L_1 \times L_2 \times ... \times L_n$ is also semi-complemented lattice.

Proof: By mathematical induction. ■

The following lemma is essentially due to Joshi and Mundlik [12].

Lemma 9: Let L be a co-atomic lattice with the greatest element 1. Then the following are equivalent.

(a) L is a 1-distributive lattice.

(b) $(q]$ is a prime ideal of L for every co-atom $q \in L$.

Lemma 10: Let $L_1, L_2, L_3, ..., L_n$ $(n \geq 3)$ be co-atomic, 1-distributive lattices. Then $L = L_1 \times L_2 \times L_3 ... \times L_n$ has at least three prime ideals.

Proof: By applying Lemma 7, the finite direct product of 1-distributive lattices is again a 1-distributive lattice. We consider the elements $(q_1, 1, ..., 1), (1, q_2, 1, ..., 1), (1, 1, q_3, 1, ..., 1)$ in $L = L_1 \times L_2 \times L_3 ... \times L_n$, where q_i are co-atoms of L_i. It is easy to see that $(q_1, 1, ..., 1), (1, q_2, 1, 1, ..., 1), (1, 1, q_3, 1..., 1)$ are co-atoms of L. By applying Lemma 9, we get at least three prime ideals in L. ■

Now, we close this section by proving Theorem 5.

Theorem 5: Let $L_1, L_2, L_3, ..., L_n$ $(n \geq 3)$ be co-atomic, 1-distributive semi-complemented lattices and $L = L_1 \times L_2 \times L_3 \times ... \times L_n$. Then $(G_{\{\mathbf{0}\}}(L))^c$ is connected.

Proof: We claim that there exist $x, y \in V((G_{\{\mathbf{0}\}}(L))^c)$ such that $x \wedge y = 0$, where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$. If $x \wedge y \neq 0$ for any $x, y \in V((G_{\{\mathbf{0}\}}(L))^c)$, then $diam((G_{\{\mathbf{0}\}}(L))^c) = 1$, a contradiction to $diam((G_{\{\mathbf{0}\}}(L))^c) \geq 2$, by Lemma 5. Thus x and y are not adjacent in $(G_{\{\mathbf{0}\}}(L))^c$. By Lemma 10, at least three prime ideals, say $(q_1], (q_2], (q_3]$ do exist, where q_i are co-atoms of L of the form $q_1 = (d_1, 1, 1, ..., 1)$, $q_2 = (1, d_2, 1, ..., 1)$ and $q_3 = (1, 1, d_3, 1, ..., 1)$ where d_i are co-atoms of L_i.
Let x and y be two non-adjacent vertices. We have the following cases:

(Case I) If $x, y \in (q_1]$, then $x \wedge q_1 = x \neq 0$ and $y \wedge q_1 = y \neq 0$. Since L_i's are semi-complemented, it is easy to observe that L is also semi-complemented. Then every non zero element is in $Z_{\{\mathbf{0}\}}(L)^*$. Hence $q_1 \in V(G_{\{\mathbf{0}\}}(L))^c$. Hence there is a path $x - q_1 - y$ in $(G_{\{\mathbf{0}\}}(L))^c$.

(Case II) If $x \in (q_1]$ and $y \in (q_2]$. Since $x \wedge y = 0 \in (q_3]$ and $(q_3]$ is a prime ideal, at least one of x or $y \in (q_3]$. Without loss of generality, we assume that $y \in (q_3]$. Therefore $y \wedge q_2 = y \neq 0$ and $y \wedge q_3 = y \neq 0$. We claim that $(q_1] \cap (q_2] \neq \{\mathbf{0}\}$ or $(q_1] \cap (q_3] \neq \{\mathbf{0}\}$. For otherwise, assume that $(q_1] \cap (q_2] = (q_1 \wedge q_2] = \{\mathbf{0}\}$ and $(q_1] \cap (q_3] = (q_1 \wedge q_3] = \{\mathbf{0}\}$, *i.e,* $q_1 \wedge q_2 = \mathbf{0}$ and $q_1 \wedge q_3 = \mathbf{0}$. But this gives $q_1 \wedge q_2 \in (q_3]$. By primeness of $(q_3]$ and q_i's are dual atoms, we have either $q_1 = q_3$ or $q_2 = q_3$, a contradiction to the fact that q_i are distinct. Hence without loss of generality, we assume that $q_1 \wedge q_2 \neq \mathbf{0}$. Then we get a path $x - q_1 - q_2 - y$ in $(G_{\{\mathbf{0}\}}(L))^c$.

■

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