

# The Application of Hybrid Orthonormal Bernstein and Block-Pulse Functions in Finding Numerical Solution of Fredholm Fuzzy Integral Equations

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**Abstract**—In this paper, we have proposed a numerical method for solving fuzzy Fredholm integral equation of the second kind. In this method a combination of orthonormal Bernstein and Block-Pulse functions are used. In most cases, the proposed method leads to the exact solution. The advantages of this method are shown by an example and calculate the error analysis.

**Keywords**—Fuzzy Fredholm Integral Equation, Bernstein, Block-Pulse, Orthonormal.

## I. INTRODUCTION

**I**NTEGRAL equation is an equation which has integral sign and an unknown function. Naturally, in such an equation there can occur other terms as well. The topic of fuzzy integral equations (FIE) has been developed in recent years. In the first step often including applicable definitions of the fuzzy integrals was followed by introducing FIE and establishing sufficient conditions for the existence of unique solutions to these equations. Finally, numerical methods for calculation approximates to these solutions were designed. The concept of fuzzy sets which was originally introduced by Zadeh [1] led to the definition of the fuzzy number and implementation in fuzzy control [2] and approximate reasoning problems [1].

In recent years, many different basic functions have been used to estimate the solution of integral equations, such as Block-Pulse functions [3, 4], Triangular functions [5, 6], Haar functions [7], Hybrid Legendre and Block-Pulse functions [8, 9], Hybrid Chebyshev and Block-Pulse functions [10, 11], Hybrid Taylor, Block-Pulse functions [12], Hybrid Fourier and Block-Pulse functions [13]. In the first time, Block-Pulse functions were introduced to electrical engineering by Harmuth and several researchers discussed the Block-Pulse [14, 15]. Fuzzy integral equations arise in many applications such as physics, geographic, medical, biology, social sciences, etc. Many practical problems in science and engineering can be transformed into Fuzzy Fredholm integral equations of the second kind, thus their solution is one of the main goals in various areas of applied sciences and engineering.

In this paper, a hybrid of orthonormal Bernstein and Block-Pulse functions for numerical solution of fredholm integral equations are used. Leading to the exact solution is the advantage of proposed method.

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## II. PRELIMINARIES

In this section, we introduce Bernstein polynomials and their properties to get better approximation and we used orthonormal these polynomials and hybrid them with Block-pulse functions.

### A. Definition of Bernestein Polynomials

The Bernstein basis polynomial of degree  $n$  are defined by [16]

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \quad (1)$$

By using binomial expansion of  $(1-x)^{n-i}$ , we have

$$\binom{n}{i} x^i (1-x)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k} \quad (2)$$

Then,  $\{B_{0,n}, B_{1,n}, \dots, B_{n,n}\}$  in Hilbert space  $L^2[0, 1]$  is a complete basis. Therefore, any polynomial of degree  $n$  can be expanded in terms of linear combination of  $B_{i,n}(x)$  for  $i = 0, 1, 2, \dots, n$ . By using Gram-schmid algorithm, we obtain orthonormal polynomials to construct new basis, these new basis are  $OB_{i,n}(x)$ .

### B. Definition of Block-Pulse Functions and Their Properties

An  $M$ -set of Block-Pulse function is defined over the interval  $[0, T)$  as

$$b_i(x) = \begin{cases} 1 & \frac{iT}{M} \leq x < \frac{(i+1)T}{M} \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where,  $i = 0, 1, \dots, M-1$  with  $M$  as a positive integer. Also,  $h = \frac{T}{M}$  and  $b_i$  is the  $i$ th BPF. In this paper, it is assumed that  $T = 1$ , so BPFs are defined over  $[0, 1)$  and  $h = \frac{1}{M}$ . [17]

$$b_i(x)b_j(x) = \begin{cases} b_i(x) & i = j \\ 0 & i \neq j \end{cases} \quad (4)$$

where, other property is orthogonality. It is clear that [18]

$$\int_0^1 b_i(x)b_j(x)dx = h\delta_{i,j} \quad (5)$$

where,  $\delta_{i,j}$  is Kronecker delta. The third property is completeness. For every  $f \in L^2[0, 1]$  when  $M$  approaches to infinity, Parsevals identity holds [17]

$$\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2 \quad (6)$$

where,

$$f_i = \frac{1}{h} \int_0^1 f(x)b_i(x)dx \quad (7)$$

**Definition 1:** We define OHB on the interval [0,1] as follow:

$$OBH_{i,j}(x) = \begin{cases} B_{j,n}(Mx - i + 1) & \frac{i-1}{M} \leq x < \frac{i}{M} \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

where,  $i = 1, 2, \dots, M$  and  $j = 0, 1, \dots, n$ .

thus, our new basis is  $\{OBH_{1,0}, OBH_{1,1}, \dots, OBH_{M,n}\}$  and we can approximate function with the base.

### C. Function Approximation by Using OHB Functions

A function  $u(x)$ , square integrable in  $[0, 1]$ , may be expressed in terms of the OBH basis as follow [18]

$$u(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \cdot OBH_{i,j}(x) \quad (9)$$

If we truncate the infinite series in (8), then we have

$$u(x) \simeq \sum_{i=1}^M \sum_{j=0}^n c_{i,j} \cdot OBH_{i,j}(x) = C^T OBH(x), \quad (10)$$

where,

$$OBH(x) = [OBH_{1,0}, OBH_{1,1}, \dots, OBH_{M,n}]^T \quad (11)$$

and

$$C = [c_{1,0}, c_{1,1}, \dots, c_{M,n}]^T$$

Therefore, we have

$$C^T < OBH(x), OBH(x) > = < u(x), OBH(x) >$$

then

$$C = D^{-1} < u(x), OBH(x) >$$

where,

$$\begin{aligned} D &= < OBH(x), OBH(x) > \\ &= \int_0^1 OBH(x) \cdot OBH^T(x) dx \\ &= \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_M \end{pmatrix} \end{aligned} \quad (12)$$

then, by using (7),  $D_i (i = 0, 1, 2, \dots, M)$  is defined as follow:

$$(D_n)_{i+1,j+1} = \int_{\frac{i-1}{M}}^{\frac{i}{M}} B_{i,n}(Mx - i + 1) B_{j,n}(Mx - j + 1) dx$$

$$\begin{aligned} (D_n)_{i+1,j+1} &= \frac{1}{M} \int_0^1 B_{i,n}(x) B_{j,n}(x) dx \\ &= \frac{\binom{n}{i} \binom{n}{j}}{M(2n+1) \binom{2n}{i+j}} \end{aligned} \quad (13)$$

We can also approximate the function  $k(x, t) \in L^2[0, 1]$  as follow:

$$k(x, t) \simeq OBH^T(x) K OBH(t) \quad (14)$$

where,  $K$  is an  $M(n+1)$ -matrix that we can obtain as follows:

$$K = D^{-1} < OBH(x) < k(x, t), OBH(t) > > D^{-1} \quad (15)$$

### III. BASIC DEFINITIONS FUZZY

**Definition 2** [19]: A fuzzy number is a set  $v : R^1 \rightarrow I = [0, 1]$  which satisfies:

- $v$  is upper semi continuous,
- $v(x) = 0$  outside some interval  $[c, d]$ ,
- There are real numbers  $a, b : c \leq a \leq b \leq d$  which
  - $v(x)$  is monotonic increasing on  $[c, d]$ ,
  - $v(x)$  is monotonic decreasing on  $[b, d]$ ,
  - $v(x) = 1, a \leq x \leq b$ .

The set of all such fuzzy number is denoted by  $R_F$ .

**Definition 3** [19]: Let  $V$  be a fuzzy set on  $R$ .  $V$  is called a fuzzy interval if:

- $V$  is normal: there exists  $x_0 \in R$  such that  $V(x_0) = 1$ .
- $V$  is convex: for all  $x, t \in R$  and  $0 \leq \lambda \leq 1$ , it holds that  $V(\lambda x + (1 - \lambda)t) \geq \min\{V(x), V(t)\}$ ,
- $V$  is upper semi-continuous: for any  $x_0 \in R$ , it holds that  $V(x_0) \geq \lim_{x \rightarrow x_0^+} V(x)$ ,
- $[V]^0 = Cl\{x \in R | V(x) > 0\}$  is a compact subset of  $R$ . The  $\alpha$ -cut of a fuzzy interval  $V$ , with  $0 < \alpha \leq 1$  is the crisp set,  $[V]^\alpha = \{x \in R | V(x) > \alpha\}$ . For a fuzzy interval  $V$ , its  $\alpha$ -cuts are closed intervals in  $R$ . Let denote them by  $[V]^\alpha = [\underline{V}(\alpha), \overline{V}(\alpha)]$ .

An alternative definition or parametric form of a fuzzy number which yields the same  $E^1$  is given by Kaleva [19] as follows:

**Definition 4** [20]: An arbitrary fuzzy number  $u$  in the parametric form is represented by an ordered pair of functions  $(\underline{u}(r), \overline{u}(r))$  which satisfy the following requirements:

- $\underline{u}(r)$  is a bounded left-continuous non-decreasing function over  $[0, 1]$ ,
- $\overline{u}(r)$  is a bounded right-continuous non-increasing function over  $[0, 1]$ ,
- $\underline{u}(r) \leq \overline{u}(r)$ , for all  $0 \leq r \leq 1$ .

For arbitrary fuzzy numbers  $v = (\underline{v}(r), \overline{v}(r))$ ,  $w = (\underline{w}(r), \overline{w}(r))$  and real number  $\lambda$ , one may define the addition

and the scalar multiplication of the fuzzy numbers by using the extension principle as follows:

- $v = w$  if and only if  $\underline{v}(r) = \underline{w}(r)$  and  $\bar{v}(r) = \bar{w}(r)$
- $v \oplus w = (\underline{v}(r) + \underline{w}(r), \bar{v}(r) + \bar{w}(r))$
- $(\lambda \otimes v) = \begin{cases} \lambda \underline{v}(r), \lambda \bar{v}(r) & \lambda \geq 0 \\ \lambda \bar{v}(r), \lambda \underline{v}(r) & \lambda < 0 \end{cases}$

**Definition 5** [21]: For arbitrary numbers  $v = (\underline{v}(r), \bar{v}(r))$  and  $w = (\underline{w}(r), \bar{w}(r))$

$$D(v, w) = \max\left\{\sup_{0 \leq r \leq 1} |\bar{v}(r) - \bar{w}(r)|, \sup_{0 \leq r \leq 1} |\underline{v}(r) - \underline{w}(r)|\right\}$$

in the distance between  $v$  and  $w$ . It is proved that  $(R_F, D)$  is a complete metric space with the properties [21]

- $D(u \oplus w, v \oplus w) = D(u, v) \quad ; \quad \forall u, v, w \in R_F,$
- $D(k \otimes u, k \otimes v) = |k| D(u, v) \quad ; \quad \forall u, v \in R_F \quad \forall k \in R,$
- $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e) \quad ; \quad \forall u, v, w, e \in R_F.$

**Definition 6:** [22] Let  $f, g : [a, b] \rightarrow R_F$ , be fuzzy real number valued functions. The uniform distance between  $f, g$  is defined by:

$$D_U(f, g) = \sup\{D(f(x), g(x)) | x \in [a, b]\} \quad (16)$$

In [23], the authors proved that if the fuzzy function,  $f(t)$ , is continuous in the metric  $D$ , its definite integral exists and also,

$$\frac{\int_a^b f(t, r) dt}{\int_a^b f(t, r) dt} = \frac{\int_a^b \underline{f}(t, r) dt}{\int_a^b \bar{f}(t, r) dt}$$

**Definition 7** [24]: A fuzzy real number valued function  $f : [a, b] \rightarrow R_F$  is said to be continuous in  $x_0 \in [a, b]$ , if for each  $\epsilon > 0$  there is  $\delta > 0$  such that  $D(f(x), f(x_0)) < \epsilon$ , whenever  $x \in [a, b]$  and  $|x - x_0| < \delta$ . We say that  $f$  is fuzzy continuous on  $[a, b]$  if  $f$  is continuous at each  $x_0 \in [a, b]$  and denote the space of all such functions by  $C_F([a, b])$ .

**Definition 8** [22]: Let  $f : [a, b] \rightarrow R_F$  be a bounded function, then function  $\omega_{[a, b]}(f, \cdot) : R_+ \cup \{0\} \rightarrow R_+$

$$\omega_{[a, b]}(f, \delta) = \sup\{D(f(x), f(y)) | x, y \in [a, b], |x - y| \leq \delta\} \quad (17)$$

where,  $R_+$  is the set of positive real numbers, is called the modulus of continuity of  $f$  on  $[a, b]$ .

**Definition 9** [22]: Let  $f : [a, b] \rightarrow R_F$ ,  $f$  is fuzzy-Riemann integrable to  $I \in R_F$  if for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any division  $P = \{[u, v] ; \xi\}$  of  $[a, b]$  with the norms  $\Delta(p) < \delta$ , we have:

$$D\left(\sum_p^* (v - u) \otimes f(\xi), I\right) < \epsilon \quad (18)$$

where,  $\sum^*$  denotes the fuzzy summation. In this case it is denoted by  $I = (FR) \int_a^b f(x) dx$ .

**Lemma 1** [22] If  $f, g : [a, b] \subseteq R \rightarrow R_F$  are fuzzy continuous functions, then the function  $F : [a, b] \rightarrow R_+$  by  $F(x) = D(f(x), g(x))$  is continuous on  $[a, b]$  and by

$$D\left((FR) \int_a^b f(x) dx, (FR) \int_a^b g(x) dx\right) \leq \int_a^b D(f(x), g(x)) dx$$

#### IV. SOLVING FUZZY FREDHOLM INTEGRAL EQUATION VIA OBH FUNCTIONS

In this section, first the fuzzy integral equations of the second kind are introduced then we solving it's via OBH function. The Fredholm fuzzy integral equation of the second kind is [25]

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^b k(x, t) \tilde{u}(t) dt \quad (19)$$

Where,  $\lambda > 0$ ,  $k(x, t)$  is an arbitrary kernel function over the square  $a \leq x, t \leq b$  and  $\tilde{u}(x)$ ,  $\tilde{f}(x)$  are fuzzy functions such that  $k(x, t), \tilde{f}(x) \in L^2[0, 1]$ . If  $f(x)$  is a crisp function then the solutions of (19) is crisp. However, if  $\tilde{f}(x)$  is a fuzzy function then this equation may only possess fuzzy solutions [25]. Now, if we introduce (19) by definition 4 then we have,

$$\underline{u}(x, r) = \underline{f}(x, r) + \lambda \int_a^b V_1(k(x, t) \underline{u}(t, r)) dt \quad (20)$$

$$\bar{u}(x, r) = \bar{f}(x, r) + \lambda \int_a^b V_2(k(x, t) \bar{u}(t, r)) dt \quad (21)$$

Where,

$$V_1(k(x, t) \underline{u}(t, r)) = \begin{cases} k(x, t) \underline{u}(t, r) & k(x, t) \geq 0 \\ k(x, t) \bar{u}(t, r) & k(x, t) < 0 \end{cases} \quad (22)$$

and

$$V_2(k(x, t) \bar{u}(t, r)) = \begin{cases} k(x, t) \bar{u}(t, r) & k(x, t) \geq 0 \\ k(x, t) \underline{u}(t, r) & k(x, t) < 0 \end{cases} \quad (23)$$

Throughout this paper, we consider Fuzzy Fredholm integral equation (19), with  $a = 0$ ,  $b = 1$  and  $\lambda = 1$ , then we write (19), in the following form:

$$\underline{u}(x, r) = \underline{f}(x, r) + \lambda \int_0^1 k(x, t) \underline{u}(t, r) dt \quad (24)$$

$$\bar{u}(x, r) = \bar{f}(x, r) + \lambda \int_0^1 k(x, t) \bar{u}(t, r) dt \quad (25)$$

we know  $(\underline{u}(x, r), \bar{u}(x, r))$  is an unknown function with can be expanded into OBH function. Likewise,  $k(x, t)$ ,  $\underline{u}(x, r)$  and  $\bar{u}(x, r)$  are also expanded into the OBH functions, then we

must Let approximate  $\underline{u}(x, r)$ ,  $\bar{u}(x, r)$ ,  $\underline{f}(x, r)$ ,  $\bar{f}(x, r)$  and  $k(x, t)$  by (14) as following for:

$$k(x, t) = OBH^T(x)KOBH(t) \quad (26)$$

$$\underline{u}(x, r) = OBH^T(x)U1OBH(r) \quad (27)$$

$$\bar{u}(x, r) = OBH^T(x)U2OBH(r) \quad (28)$$

$$\underline{f}(x, r) = OBH^T(x)F1OBH(t) \quad (29)$$

$$\bar{f}(x, r) = OBH^T(x)F2OBH(t) \quad (30)$$

With substituting above equations into (24) and (25), we have:

$$\begin{aligned} OBH^T(x)U1OBH(r) &= OBH^T(x)F1OBH(t) + \\ &\int_0^1 OBH^T(x)KOBH(t)OBH^T(t)U1OBH(r)dt \\ OBH^T(x)U2OBH(r) &= OBH^T(x)F2OBH(t) + \\ &\int_0^1 OBH^T(x)KOBH(t)OBH^T(t)U2OBH(r)dt \end{aligned}$$

By substituting (12) into above equation we get

$$\begin{aligned} OBH^T(x)U1OBH(r) &= OBH^T(x)F1OBH(t) + \\ &OBH^T(x)KDU1OBH(r) \\ OBH^T(x)U2OBH(r) &= OBH^T(x)F2OBH(t) + \\ &OBH^T(x)KDU2OBH(r) \end{aligned}$$

Therefore,

$$U1 = F1 + KDU1 \quad (31)$$

$$U2 = F2 + KDU2 \quad (32)$$

where, the dimensional subscripts have been dropped to simplify the notation. Rewriting (31) and (32), we have

$$U1 = (I - KD)^{-1}F1 \quad (33)$$

$$U2 = (I - KD)^{-1}F2 \quad (34)$$

where,  $I$  is  $nM \times nM$ -identity matrix. The unknowns matrix  $U1$  and  $U2$  can be obtained by solving (33) and (34). Thus the solutions  $\underline{u}(x, r)$  and  $\bar{u}(x, r)$  can be calculated in the OBH function expansion by using  $U1$ ,  $U2$  and (27)-(28).

## V. THE CONVERGENCE OF THE METHOD

In this section, we obtain error estimate for the numerical method proposed in previous section.

The solution of Fredholm fuzzy integral equation (19), by using OBH converges if  $M < 1$ , where

$$M = \max_{0 \leq x, t \leq 1} |k(x, t)|$$

**Proof.** Assume  $\tilde{u}(x)$  and  $\tilde{u}_{NM}(x)$  show approximate and exact solution of (19) respectively. then

$$\begin{aligned} D(\tilde{u}(x), \tilde{u}_{NM}(x)) &= D\left(\int_0^1 k(x, t)\tilde{u}(t)dt, \right. \\ &\left. \int_0^1 k(x, t)\sum_{i=1}^N \sum_{j=0}^M {}^*c_{ij}OBH_{ij}(t)dt\right) \\ &\leq M \int_0^1 D\left(\tilde{u}(t), \sum_{i=1}^N \sum_{j=0}^M {}^*c_{ij}OBH_{ij}(t)\right) dt \end{aligned}$$

therefore, we have

$$\begin{aligned} D(\tilde{u}(x), \tilde{u}_{NM}(x)) &\leq M \int_0^1 D(\tilde{u}(t), \tilde{u}_{NM}(t))dt \\ \sup_{x \in [0,1]} D(\tilde{u}(x), \tilde{u}_{NM}(x)) &\leq M \sup_{x \in [0,1]} D(\tilde{u}(t), \tilde{u}_{NM}(t)) \end{aligned}$$

Therefore, if  $M < 1$ , we will have:

$$\lim_{NM \rightarrow \infty} \sup_{x \in [0,1]} D(\tilde{u}(x), \tilde{u}_{NM}(x)) = 0$$

## VI. NUMERICAL EXAMPLE

Consider the following linear fuzzy fredholm integral equation with

$$\underline{f}(x, r) = rx - x^2\left[\frac{2}{3}rx^3 - \frac{4}{3}x^3 - \frac{1}{2}rx^2 + x^2 + \frac{1}{12}r - \frac{1}{12}\right]$$

$$\bar{f}(x, r) = (2-r)x + x^2\left[\frac{2}{3}rx^3 - \frac{1}{2}rx^2 + \frac{1}{12}r - \frac{1}{12}\right]$$

and

$$k(x, t) = x^2(1 - 2t), \quad 0 \leq x, t \leq 1 \text{ and } \lambda = 1$$

The exact solution in this case is given by

$$\underline{u}(x, r) = rx$$

$$\bar{u}(x, r) = (2-r)x.$$

Results are shown in Table 1.

## VII. CONCLUSION

In this paper, we presented a numerical method for solving the fuzzy Fredholm integral equation of second kind. We solved Fredholm integral equations by combination of orthonormal Bernstein and Block-Pulse functions. We have shown advantages of this numerical method by an example and calculate the error analysis.

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TABLE I  
COMPARING OF EXACT AND OBH METHOD SOLUTIONS

$r$	Exact solution ( $\underline{u}(x, r)$ )	OBH method for $x = 0.1$ and $M = 2, N = 1$	Exact solution ( $\bar{u}(x, r)$ )	OBH method for $x = 0.1$ and $M = 2, N = 1$
0.0	0.00000000	0.00000000	0.20000000	0.20000000
0.1	0.01000000	0.01000000	0.19000000	0.19000000
0.2	0.02000000	0.02000000	0.18000000	0.18000000
0.3	0.03000000	0.03000000	0.17000000	0.17000000
0.4	0.04000000	0.04000000	0.16000000	0.16000000
0.5	0.05000000	0.05000000	0.15000000	0.15000000
0.6	0.06000000	0.06000000	0.14000000	0.14000000
0.7	0.07000000	0.07000000	0.13000000	0.13000000
0.8	0.08000000	0.08000000	0.12000000	0.12000000
0.9	0.09000000	0.09000000	0.11000000	0.11000000

## REFERENCES

- [1] L. A. Zadeh, "Linguistic variable approximate and disposition," in *Med. Inform.*, pp. 173-186, 1983.
- [2] S. S. L. Chang and L. Zadeh, "On fuzzy mapping and control," in *IEEE trans system Cybernet.*, vol. 2, pp. 30-34, 1972.
- [3] K. Maleknejad, M. Shahrezaee and H. Khatami, "Numerical solution of integral equations system of the second kind by Block-Pulse functions," in *Applied Mathematics and Computation*, vol. 166, pp. 15-24, 2005.
- [4] E. Babolian and Z. Masouri, "Direct method to solve Volterra integral equation of the first kind using operational matrix with block-pulse functions," in *Applied Mathematics and Computation*, vol. 220, pp. 51-57, 2008.
- [5] E. Babolian, K. Maleknejad, M. Roodaki and H. Almasieh, "Twodimensional triangular functions and their applications to nonlinear 2D Volterra-Fredholm integral equations," in *Computer and Mathematics with Applications*, vol. 60, pp. 1711-1722, 2010.
- [6] F. Mirzaee and S. Piroozfar, "Numerical solution of the linear two dimensional Fredholm integral equations of the second kind via two dimensional triangular orthogonal functions," in *Journal of King Saud University*, vol. 22, pp. 185-193, 2010.
- [7] Y. Ordokhani, "Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions," in *Applied Mathematics and Computation*, vol. 180, pp. 436-443, 2006.
- [8] K. Maleknejad and M. T. Kajani, "Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions," in *Applied Mathematics and Computation*, vol. 145, pp. 623-629, 2003.
- [9] E. Hashemzadeh, K. Maleknejad and B. Basirat, "Hybrid functions approach for the nonlinear Volterra-Fredholm integral equations," in *Procedia Computer Science*, vol. 3, pp. 1189-1194, 2011.
- [10] M. T. Kajani and A. H. Vencheh, "Solving second kind integral equations with Hybrid Chebyshev and Block-Pulse functions," in *Applied Mathematics and Computation*, vol. 163, pp. 71-77, 2005.
- [11] X. T. Wang and Y. M. Li, "Numerical solutions of integrodifferential systems by hybrid of general block-pulse functions and the second Chebyshev polynomials," in *Applied Mathematics and Computation*, vol. 209, pp. 266- 272, 2009.
- [12] K. Maleknejad and Y. Mahmoudi, "Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions," in *Applied Mathematics and Computation*, vol. 149, pp. 799-806, 2004.
- [13] B. Asady, M. T. Kajani, A. H. Vencheh and A. Heydari, "Solving second kind integral equations with hybrid Fourier and block-pulse functions," in *Applied Mathematics and Computation*, vol. 160, pp. 517-522, 2005.
- [14] G. P. Rao, "Piecewise Constant Orthogonal Functions and their Application to System and Control," in *Springer-Verlag*, 1983.
- [15] B. M. Mohan and K.B. Datta, "Orthogonal Function in Systems and Control," 1995.
- [16] K. Maleknejad, B. Basirat and E. Hashemizadeh, "A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm integro-differential equations," in *Math. Comput. Model.*, vol. 55, pp. 1363-1372, 2012.
- [17] T. J. Rivlin, "An introduction to the approximate of functions," in *New York, Dove Publications*, 1969.
- [18] S. A. yousefi, M. Behroozifar, "Operational matrices of Bernstein polynomials and their applications," in *International Journal of System Science*, vol. 41, pp. 709-716, 2010.
- [19] O. Kaleva, "Fuzzy differential equations," in *Fuzzy Sets Syst.*, vol. 24, pp. 301-317, 1987.
- [20] M. Ma, M. Friedman and A. Kandel, "Duality in fuzzy linear systems," *Fuzzy Sets and Systems*, vol. 109, pp. 55-58, 2000.
- [21] S. Gal, "Approximation theory in fuzzy setting," *Chapter 13 in Handbook of analytic-computational methods in applied mathematics*, pp. 617-666, 2000.
- [22] G. A. Anastassiou, "Fuzzy mathematics: Approximation theory," in *Springer, Heidelberg*, 2010.
- [23] R. Goetschel, W. Voxman, "Elementary fuzzy calculus," in *Fuzzy Sets and Systems*, vol. 18, pp. 31-43, 1986.
- [24] H. Wu, "The fuzzy Riemann integral and its numerical integration," in *Fuzzy Sets and Systems*, vol. 110, pp. 1-25, 2000.
- [25] M. Friedman, M. Ming and A. Kandel, "Fuzzy linear systems," in *FSS*, vol. 96, pp. 201-20, 1998.



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