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On Algebraic Structure of Improved Gauss-Seidel Iteration

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Abstract—Analysis of real life problems often results in linear systems of equations for which solutions are sought. The method to employ depends, to some extent, on the properties of the coefficient matrix. It is not always feasible to solve linear systems of equations by direct methods, as such the need to use an iterative method becomes imperative. Before an iterative method can be employed to solve a linear system of equations there must be a guaranty that the process of solution will converge. This guaranty, which must be determined apriori, involve the use of some criterion expressible in terms of the entries of the coefficient matrix. It is, therefore, logical that the convergence criterion should depend implicitly on the algebraic structure of such a method. However, in deference to this view is the practice of conducting convergence analysis for Gauss-Seidel iteration on a criterion formulated based on the algebraic structure of Jacobi iteration. To remedy this anomaly, the Gauss-Seidel iteration was studied for its algebraic structure and contrary to the usual assumption, it was discovered that some property of the iteration matrix of Gauss-Seidel method is only diagonally dominant in its first row while the other rows do not satisfy diagonal dominance. With the aid of this structure we herein fashion out an improved version of Gauss-Seidel iteration with the prospect of enhancing convergence and robustness of the method. A numerical section is included to demonstrate the validity of the theoretical results obtained for the improved Gauss-Seidel method.

Keywords—Linear system of equations, Gauss-Seidel iteration, algebraic structure, convergence.

I. Introduction

LINEAR systems of equations can be expressed in matrix form as

$$Ax = b \tag{1}$$

where $A = (a_{ij}), i, j = 1, 2, \ldots, n$ is a square matrix while $b = (b_j), j = 1, 2, \ldots, n$, and $x = (x_j), j = 1, 2, \ldots, n$ are vectors. Such equations arise in electrical networks, economic modelling, optimization and approximation of ordinary differential equations, see [18]. Similarly, the solution of partial differential equations by finite difference discretisation often result in a set of linear equations [1]. Thus the solution of linear systems of equations is of great importance in mathematics.

The method employed in solving (1) depends, to some extent, on the properties of the coefficient matrix A. Linear systems of equations are generally solved by such direct methods as LU factorisation, Gaussian elimination, matrix inversion and Cramer's methods. For a number of reasons, it is not always feasible to solve (1) by any of the direct methods of solution. One major reason is the case where A is a singular matrix, [3]. Again, if the number of equations is large or the coefficient matrix A is a band matrix, the use of direct methods becomes inefficient, see Saad and van der Vorst [16]. Under these situations, recourse is made to iterative methods as solution techniques.

There are two classes of iterative methods: stationary and nonstationary iterative methods. Stationary iterative methods are older and simpler to implement in that the iteration matix remains unchanged in the course of implementation [13]. There are essentially two basic stationary iterative methods for the solution of linear systems of equations, namely, the Jacobi and the Gauss-Seidel. The other method, Successive Over Relaxation (SOR) method, is simply a modification of the latter [6].

Before an iterative method is used to solve a linear system of equations there must be a guaranty that the process of solution will converge. This guaranty, which must be determined apriori, involve the use of some criterion expressible in terms of the entries of the coefficient matrix A [13]. It is therefore logical that the convergence criterion should depend solely on the algebraic structure of such a method. However, in deference to this view is the practice of conducting convergence analysis for Gauss-Seidel iteration on a criterion formulated based on the algebraic structure of Jacobi iteration, see Demidovich [5]. To redress this misapplication, the Gauss-Seidel iteration has been studied for its algebraic structure [10] and contrary to the usual assumption, it was discovered that the algebraic structure of the iteration matrix of Gauss-Seidel method is only diagonally dominant in its first row while the other rows do not satisfy diagonal dominance. With the aid of this structure we fashion out an improved version of Gauss-Seidel iteration with the prospect of enhancing convergence and robustness of the method.

Iterative methods for solving linear systems of the form (1) usually involve splitting the coefficient matrix A into matrices P and Q such that $A = P - Q$ where P is nonsingular [4]. Thus, the linear system is transformed into the form

$$x = P^{-1}Qx + b. \tag{2}$$

$P^{-1}Q$ is referred to as the iteration matrix, see [17] and $P^{-1}b$, we call the iteration vector.

The stationary iterative methods for solving (1) is obtained by the splitting method [20] to give the coefficient matrix A as $A = L + D + U$, where L, D and U are the strictly lower triangular, diagonal and strictly upper triangular matrices.

There are only three feasible splittings, each of which must

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include the diagonal matrix D. Arising from these splittings, we have the Jacobi method as

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b, \tag{3}$$

the forward Gauss-Seidel method as

$$x^{(k+1)} = -(L+D)^{-1}Ux^{(k)} + (L+D)^{-1}b \tag{4}$$

and the backward Gauss-Seidel method [6], is given by

$$x^{(k+1)} = -(D+U)Lx^{(k)} + (D+U)^{-1}b \tag{5}$$

Since D, $(L+D)$ and $(D+U)$ are triangular matrices, then their inverses exist, even for singular linear systems, provided $a_{ii} \neq 0$ for all i, see [21].

The structure and properties of the Jacobi method are well known and are even commonly utilized in the convergence analysis of the Gauss-Seidel method, see Bagnara [2], Householder [9] and Richard [14]. As a result, there is a need to study the structure and properties of the iteration matrices of both the forward and backward Gauss-Seidel methods.

In the next section we characterise the algebraic structures of Gauss-Seidel iterations followed by the analysis of the improved Gauss-Seidel method. The section on numerical consideration precedes the discussion on results and the conclusion.

II. Characterisation of Gauss-Seidel iterations

For the forward Gauss-Seidel method, let
$(L+D)^{-1} = (p_{ij}), i,j = 1,2,\ldots,n$
and
$(L+D)^{-1}U = (m_{ij}), i,j = 1,2,\ldots,n.$
Then, its iteration matrix, M_F, and iteration vector, c_F, can be written respectively as

$$M_F = -(L+D)^{-1}U = -\begin{pmatrix} 0 & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & m_{n2} & \cdots & m_{nn} \end{pmatrix} \tag{6}$$

and

$$c_F = (L+D)^{-1}b = \begin{pmatrix} \frac{b_1}{a_{11}} + \sum_{j=2}^{n} q_{1j}b_j \\ p_{21}b_1 + \frac{b_2}{a_{22}} \\ \vdots \\ \sum_{j=1}^{n-1} p_{nj}b_j + \frac{b_n}{a_{nn}} \end{pmatrix} \tag{7}$$

where

$$m_{ij} = \begin{cases} 0 & j = 1, \\ \sum_{k=1}^{j-1} p_{ik}a_{kj} & 1 < j \leq i, \\ \sum_{k=1}^{i} p_{ik}a_{kj} & j > i \geq 1. \end{cases} \tag{8}$$

and

$$p_{ij} = \begin{cases} 0 & j > i, \\ \frac{1}{a_{ii}} & j = i, \\ -\frac{1}{a_{ii}} \sum_{k=j}^{i-1} a_{ik}p_{kj} & j < i. \end{cases} \tag{9}$$

For the backward Gauss-Seidel method, let
$(D+U)^{-1} = (q_{ij}), i,j = n, n-1, \ldots, 1$
and
$(D+U)^{-1}L = (r_{ij}), i,j = n, n-1, \ldots, 1.$
Then its iteration matrix, M_B, can be put in the form

$$M_B = -(D+U)^{-1}L = -\begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1(n-1)} & 0 \\ r_{21} & r_{22} & \cdots & r_{2(n-1)} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{n(n-1)} & 0 \end{pmatrix} \tag{10}$$

and the iteration vector, c_B, as

$$c_B = (D+U)^{-1}b = \begin{pmatrix} \frac{b_1}{a_{11}} + \sum_{j=2}^{n} q_{1j}b_j \\ \frac{b_2}{a_{22}} + \sum_{j=3}^{n} q_{2j}b_j \\ \vdots \\ \frac{b_n}{a_{nn}} \end{pmatrix} \tag{11}$$

where the relationship between r_{ij} and m_{ij} as well as between q_{ij} and p_{ij}, based on the structural patterns of $(L+D)^{-1}$, $(L+D)^{-1}U$, $(D+U)^{-1}$ and $(D+U)^{-1}L$ are given by

$$r_{ij} = m_{(n-i+1)(n-j+1)}, i,j = n, n-1, \ldots, 1 \tag{12}$$

and

$$q_{ij} = p_{(n-i+1)(n-j+1)}, i,j = n, n-1, \ldots, 1. \tag{13}$$

A. Analysis of Algebraic Structures of Gauss-Seidel Iterations

We begin the analysis of Gauss-Seidel iterations by stating a result that specify some relationship between forward Gauss-Seidel method and its backward variant.

Theorem 1
For the linear system (2), both the backward Gauss-Seidel and forward Gauss-Seidel generate the same iterates, starting at the same initial vector.
Proof: The backward Gauss-Seidel iteration for the linear system (2) is

$$x^{(k+1)} = M_B x^{(k)} + C_B.$$

In line with Saad ([15], pg. 97), the above can be written in expanded form as

$$\begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{pmatrix}^{(k+1)} = -\begin{pmatrix} r_{11} & \cdots & 0 \\ r_{21} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ r_{(n-1)1} \cdots & 0 & \\ r_{n1} & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{pmatrix}^{(k)}$$

$$+ \begin{pmatrix} \frac{b_1}{a_{11}} + \sum_{j=2}^{n} q_{1j}b_j \\ \frac{b_2}{a_{22}} + \sum_{j=3}^{n} q_{2j}b_j \\ \vdots \\ \frac{b_{n-1}}{a_{(n-1)(n-1)}} + \sum_{j=n}^{n} q_{(n-1)j}b_j \\ \frac{b_n}{a_{nn}} \end{pmatrix} \tag{14}$$

Using (12) and (13) in (14) results in

$$\begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{pmatrix}^{(k+1)} = -\begin{pmatrix} m_{nn} & \cdots & 0 \\ m_{(n-1)n} & \cdots & 0 \\ \vdots & \vdots & \vdots \\ m_{2n} & \cdots & 0 \\ m_{1n} & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_2 \\ x_1 \end{pmatrix}^{(k)}$$

$$+ \begin{pmatrix} \frac{b_n}{a_{nn}} + \sum_{n-j+1=n-1}^{1} p_{n(n-j+1)} b_{(n-j+1)} \\ \frac{b_{(n-1)}}{a_{(n-1)(n-1)}} + \sum_{n-j+1=n-2}^{1} p_{(n-1)(n-j+1)} b_{(n-j+1)} \\ \vdots \\ \frac{b_2}{a_{22}} + \sum_{n-j+1=1}^{1} p_{2(n-j+1)} b_{n-j+1} \\ \frac{b_1}{a_{11}} \end{pmatrix}$$

which on rearranging yields

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}^{(k+1)} = -\begin{pmatrix} 0 & m_{12} & \cdots & m_{1n} \\ 0 & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & m_{(n-1)2} & \cdots & m_{(n-1)n} \\ 0 & m_{n2} & \cdots & m_{nn} \end{pmatrix} .$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}^{(k)} + \begin{pmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} + p_{(n-1)n} b_n \\ \vdots \\ \frac{b_{(n-1)}}{a_{(n-1)(n-1)}} + \sum_{k=1}^{n-2} p_{(n-1)k} b_k \\ \frac{b_n}{a_{nn}} + \sum_{k=1}^{n-1} p_{nk} b_k \end{pmatrix}$$

i.e.,

$$x^{(k+1)} = M_F x^{(k)} + c_F.$$

This is the forward Gauss-Seidel iteration for the same linear system (2). Hence, the result is established.

Based on **Theorem 1**, we shall henceforth restrict our discussion to the forward Gauss-Seidel method.

Lemma 1

The Gauss-Seidel method converges if its iteration matrix satisfies

$$\|M_F\| < 1. \tag{15}$$

Proof: We shall prove this result for the general iterative method. Now, (2) can be re-written as

$$x = Tx + c \tag{16}$$

where $T = P^{-1}Q, \qquad c = P^{-1}b.$

Employing iteration in (16) yields

x$^{(k+1)} = Tx^{(k)} + c$. Let e$^{(k)} = x^{(k)} - x$ be the error of the k^{th} iteration. Then e$^{(k+1)} = Tx^{(k)} - Tx = T(x^{(k)} - x)$

i.e.,

e$^{(k+1)} = Te^{(k)} \qquad k = 0, 1 \cdots$

and thus,

$\|e^{(k+1)}\| = \|Te^{(k)}\| \leq \|T\|\|e^{(k)}\|$.

If the iterative method converges, then

$\|e^{(k+1)}\| < \|e^{(k)}\|$. Thus,

$\|e^{(k+1)}\| \leq \|T\|.\|e^{(k)}\| < \|e^{(k)}\|$,

which implies that

$$\|T\| < 1, \tag{17}$$

and this is the required convergence criterion for any stationary iterative method [11].

As a prelude to the next result, we give the following definition:

Definition 1: A square matrix $A = (a_{ij}), i, j = 1, 2 \cdots, n$ is said to be strictly (or strongly) diagonally dominant if the sum of the moduli of the off-diagonal elements is less than the modulus of the diagonal element for every row or column [7] i.e, $\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}| \ \forall i = 1, 2, \cdots, n$

We now seek to determine the convergence criterion specifically for Gauss-Seidel iteration.

Applying the row-sum criterion [12] to M_F, we have the following computations

Row 1:

$|\frac{a_{12}}{a_{11}}| + |\frac{a_{13}}{a_{11}}| + |\frac{a_{14}}{a_{14}}| + \cdots + |\frac{a_{1n}}{a_{11}}| < 1$

i.e.,

$|\frac{1}{a_{11}}|\{|a_{12}| + |a_{13}| + |a_{14}| + \cdots + |a_{1n}|\} < 1$

implying that

$|\frac{1}{a_{11}}| \sum_{k=2}^{n} |a_{ik}| < 1$

i.e.,

$$\sum_{2}^{n} |a_{ik}| < |a_{11}|. \tag{18}$$

Hence, the first row is diagonally dominant.

Row 2:

$$\sum_{k=1}^{1} \frac{|a_{2k}|}{|a_{kk}|}.|a_{k2}| + \sum_{k=3}^{n} |a_{2k}| < |a_{22}| \tag{19}$$

If Row 1 is substituted in Row 2, it indicates that Row 2 is also diagonally dominant.

Row 3:

$$\sum_{k=1}^{2} \frac{|a_{3k}|}{|a_{kk}|}.|a_{k3}| + \sum_{k=4}^{n} |a_{3k}| < |a_{33}| \tag{20}$$

If Rows 1 and 2 are put in Row 3, we have that Row 3 also satisfy diagonal dominance.

Row 4:

$$\sum_{k=1}^{3} \frac{|a_{4k}|}{|a_{kk}|}.|a_{k4}| + \sum_{k=5}^{n} |a_{4k}| < |a_{44}| \tag{21}$$

Substituting Rows 1, 2 and 3 in Row 4, diagonal dominance is attained.

Row 5:

$$\sum_{k=1}^{4} \frac{|a_{5k}|}{|a_{kk}|}.|a_{k5}| + \sum_{k=6}^{n} |a_{5k}| < |a_{55}| \tag{22}$$

Again, substituting Rows 1, 2, 3 and 4 into Row 5, diagonal dominance is attained.

Thus from (18)-(22), we generalize for the ith row to obtain

$$\sum_{k=1}^{i-1} \frac{|a_{ik}|}{|a_{kk}|}.|a_{ki}| + \sum_{k=i+1}^{n} |a_{ik}| < |a_{ii}| \tag{23}$$

From the above we deduce as follows:

Theorem 2 [10]:
A necessary condition for the convergence of Gauss-Seidel iteration for (1) is strict diagonal dominance of the coefficient matrix A while a sufficient condition for its convergence is that
$\sum_{k=1}^{i-1} \frac{|a_{ik}|}{|a_{kk}|}.\,|a_{ki}| + \sum_{k=i+1}^{n} |a_{ik}| < |a_{ii}|\,\forall i = 1,\ 2, \ldots, n.$

III. Improved Gauss-Seidel iteration

We note that a square matrix A satisfying (23) has only its first-row in the form of (6) while the other rows are not. The new convergence criterion, i.e., (23) is stronger than the condition for weak diagonal dominance, see Varga [19]. Indeed, it is closer to non-diagonal dominance, and so we refer to it as the condition for weak non-diagonal dominance which is defined as follows:

Definition 2
The square matrix A is said to be weakly non-diagonally dominant if only one of its rows, the rth row in this case, satisfies

$$\sum_{\substack{j=1\\ j\neq r}}^{n} |a_{rj}| < |a_{rr}|, \quad 1 \leq r \leq n. \tag{24}$$

Based on the above observation, we can align every linear system of equations satisfying **Lemma 1** to a form that is weakly non-diagonally dominant with respect to its first row. The following result provides the theoretical justification for this assertion.

Theorem 3
The linear system (2) satisfying **Lemma 1** is transformable to a form

$$By = f, \tag{25}$$

where B is weakly non-diagonally dominant with respect to its first row.
Proof:
With $A = (a_{ij}); i, j = 1, 2, 3, ..., n$, we wish to express the first row in the form

$$\sum_{j=2}^{n} |b_{1j}| < |b_{11}|, \tag{26}$$

using the (row) transformation $\mathrm{B}_1 = A_1 + \sum_{j=2}^{n} \alpha_j A_j$, where A_j denotes the jth row of A. One way to achieve this is to set $\mathrm{b}_{1j} = 0$ for $j = 2, \ldots, n$ to have
$\mathrm{a}_{1k} + \sum_{j=2}^{n} \alpha_j a_{jk} = 0, \quad k = 2, \cdots, n$. This subsequently results in the nonhomogenous system

$$\begin{pmatrix} a_{22} & a_{32} & a_{42} & \cdots & a_{n2} \\ a_{23} & a_{33} & a_{43} & \cdots & a_{n3} \\ a_{24} & a_{34} & a_{44} & \cdots & a_{n4} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n} & a_{3n} & a_{4n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_n \end{pmatrix} = - \begin{pmatrix} 12 \\ a_{13} \\ a_{14} \\ \vdots \\ a_{1n} \end{pmatrix}. \tag{27}$$

The solution to (27) can be obtained analytically if A is positive definite; otherwise, it can be solved iteratively using (4). Hence, the system (2) is transformed into

$$\begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} \tag{28}$$

where

$b_{11} = a_{11} + \sum_{j=2}^{n} \alpha_j a_{j1}$,

$y_1 = \sum_{j=2}^{n} \alpha_j x_j$

and

$f_1 = \sum_{j=2}^{n} \alpha_j b_j$.

The conventional Gauss-Seidel iteration endowed with the above (row) transformation is herein called the improved Gauss-Seidel.

Theorem 4
The improved Gauss-Seidel iteration unconditionally converges for the class of linear algebraic systems satisfying the condition of **Lemma 1**.
Proof:
The proof follows from **Theorem** 2 and 3.

IV. Numerical Consideration

The motivation for this section is to illustrate and validate the theoretical results reported in Sections II and III as well as to assess, comparatively, the performance of the improved Gauss-Seidel method. The numerical examples considered, as tabulated in Table I, are non-diagonally dominant linear systems whose iteration matrices satisfy the condition of **Lemma 1** and in particular of different dimensions including large-scale linear systems of equations.

A. Computational Details

The following prescibes the algorithm implemented for the improved Gauss-Seidel method:

Computational algorithm for improved Gauss-Seidel method
Step 1: Supply A, b, $x^{(0)}$, $\varepsilon > 0$ (a small number).
Step 2: Transform $Ax = b$ using (28) if coefficient matrix A satisfies **Lemma 1** , else stop.
Step 3: Compute the iterates $x^{(k)}, k = 1, 2, \cdots$.
Step 3: Terminate when convergence is achieved.

The performance of the improved Gauss-Seidel method was compared with six other iterative methods, namely, the forward Gauss-Seidel, its backward variant, Jacobi, SOR, conjugate gradient (CG) and generalised minimal residual (GMRE) methods.

The following program files: seidel.m, jacobi.m, sor.m and gmre.m of MATLAB 7.10 package were used for

TABLE I
LIST OF TEST PROBLEMS WITH FEATURES OF THE COEFFICIENT MATRIX A

b	A	Order
$\begin{pmatrix} 56 \\ 374 \\ 550 \\ 372 \\ 569 \\ 271 \\ 291 \\ 411 \\ 437 \\ 268 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & -1 & \cdots & 4 & -1 & 2 \\ 21 & 14 & 1 & \cdots & 1 & 2 & 1 \\ 7 & 30 & 22 & \cdots & 3 & 1 & 10 \\ 11 & 3 & 10 & \cdots & -1 & 1 & 1 \\ 8 & 30 & 1 & \cdots & -1 & 1 & 5 \\ 2 & 17 & 2 & \cdots & -1 & 2 & 2 \\ 14 & -1 & -1 & \cdots & 1 & 4 & 1 \\ 19 & 3 & 2 & \cdots & 18 & 7 & 1 \\ 6 & -2 & 35 & \cdots & 4 & 28 & 8 \\ 7 & 5 & 10 & \cdots & 11 & 2 & 9 \end{pmatrix}$	10x10
$\begin{pmatrix} -3 \\ -1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ -4 \end{pmatrix}$	$\begin{pmatrix} -1 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 4 & -8 & 3 & \cdots & 0 & 0 & 0 \\ 0 & 4 & -8 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 4 & -8 \end{pmatrix}$	500x500
$\begin{pmatrix} 263 \\ 486 \\ -97 \\ 303 \end{pmatrix}$	$\begin{pmatrix} 4 & -3 & 5 & 1 \\ 13 & 8 & -1 & 3 \\ 10 & 6 & 9 & 2 \\ 8 & 7 & 1 & 5 \end{pmatrix}$	4x4
$\begin{pmatrix} 14 \\ 7 \\ 7 \\ 7 \\ \vdots \\ 7 \\ 5 \end{pmatrix}$	$\begin{pmatrix} 2 & 8 & 4 & \cdots & 0 & 0 & 0 \\ 1 & 4 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 4 & & & & \end{pmatrix}$	600x600

implementing five of the methods to solve the numerical examples. A MATLAB version of CGM codes was used. It is to be noted that the optimum relaxation parameter [8], $\omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(M_J)}}$ was used for the SOR method.

The computer programs were run on a HP laptop with configurations 3 G RAM, 2.3 GHz processor and 300 GB hard disk setting the tolerance ε at 10^{-6}. The results obtained are tabulated in Tables II-V using the following notations: FGS - Forward Gauss-Seidel; BGS - Backward Gauss-Seidel; IGS - Improved Gauss-Seidel; SOR - Successive Over Relaxation; CG - Conjugate Gradient; GMRE - Generalised Minimal Residual; Iter - Iteration; CPUT - CPU Time (in seconds); NC - No Convergence; NA - Not Available; NS - Norm of Solution.

TABLE II
NUMERICAL SOLUTIONS TO SOME LINEAR SYSTEMS USING FORWARD AND BACKWARD GAUSS-SEIDEL METHODS

FGS				BGS			
solution	Iter	CPUT	NS	solution	Iter	CPUT	NS
NC	NA	NA	NA	NC	NA	NA	NA
NC	NA	NA	NA	NC	NA	NA	NA
NC	NA	NA	NA	NC	NA	NA	NA
$\begin{pmatrix} 1.000 \\ 1.000 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	20	0.297	24.691	$\begin{pmatrix} 1.000 \\ 1.000 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	20	0.297	24.691

V. DISCUSSION ON RESULTS AND CONCLUSION

It can be observed from Tables II-V that only the improved Gauss-Seidel and GMRE methods solve all the numerical examples. Both the forward and backward Gauss-Seidel methods solve two of the numerical examples, the SOR method solves only one while the Jacobi and CG methods did not solve any. Comparing the average number of iterations and average time it took each of improved Gauss-Seidel (26.75 and 0.188 s) and GMRE (15.5 and 10.654 s) to solve each numerical example, the improved Gauss-Seidel iteration is adjudged to be more efficient than the GMRE since CPU time is a more valued computational measure. In guarantying the convergence of an iterative method for solving a linear system of equations, the convergence criterion must be an explicit expression of the algebraic structure of the method. With the use of its inherent structure, Gauss-Seidel is herein shown to be more robust than it was previously known to be. A new version of the method is obtained by exploiting the newly discovered algebraic structure.

TABLE III
NUMERICAL SOLUTIONS TO SOME LINEAR SYSTEMS USING IMPROVED GAUSS-SEIDEL METHOD

solution	Iter	CPUT	NS
$\begin{pmatrix} 10.000 \\ 9.000 \\ \vdots \\ 2.000 \\ 1.000 \end{pmatrix}$	36	0.009	19.621
$\begin{pmatrix} 1.000 \\ 1.000 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	45	0.444	22.384
$\begin{pmatrix} 65.000 \\ -79.918 \\ -47.005 \\ 77.886 \end{pmatrix}$	10	0.003	137.433
$\begin{pmatrix} 1.000 \\ 1.000 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	20	0.297	24.691

TABLE IV
NUMERICAL SOLUTIONS TO SOME LINEAR SYSTEMS USING JACOBI AND SOR METHODS

Jacobi				SOR			
solution	Iter	CPUT	NS	solution	Iter	CPUT	NS
NC	NA	NA	NA	NC	NA	NA	NA
NC	NA	NA	NA	NC	NA	NA	NA
NC	NA	NA	NA	NC	NA	NA	NA
NC	NA	NA	NA	$\begin{pmatrix} 1.000 \\ 1.000 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	18	0.364	24.690

TABLE V
NUMERICAL SOLUTIONS TO SOME LINEAR SYSTEMS USING CG AND GMRE METHODS

			CG				GMRE
solution	**Iter**	**CPUT**	**NS**	**solution**	**Iter**	**CPUT**	**NS**
NC	NA	NA	NA	$\begin{pmatrix} 10.000 \\ 9.000 \\ \vdots \\ 2.000 \\ 1.000 \end{pmatrix}$	10	0.223	19.621
NC	NA	NA	NA	$\begin{pmatrix} 1.000 \\ 1.000 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	31	2.001	22.361
NC	NA	NA	NA	$\begin{pmatrix} 65.000 \\ -80.000 \\ -47.000 \\ 78.000 \end{pmatrix}$	50	0.0003	137.543
NC	NA	NA	NA	$\begin{pmatrix} 1.631 \\ 0.815 \\ \vdots \\ 1.000 \\ 1.000 \end{pmatrix}$	18	1.907	24.691

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