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An Optimal Bayesian Maintenance Policy for a Partially Observable System Subject to Two Failure Modes

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Abstract—In this paper, we present a new maintenance model for a partially observable system subject to two failure modes, namely a catastrophic failure and a failure due to the system degradation. The system is subject to condition monitoring and the degradation process is described by a hidden Markov model. A cost-optimal Bayesian control policy is developed for maintaining the system. The control problem is formulated in the semi-Markov decision process framework. An effective computational algorithm is developed, illustrated by a numerical example.

Keywords—Partially observable system, hidden Markov model, competing risks, multivariate Bayesian control.

I. Introduction

RECENTLY, due to the advances in sensor development and computer technology, it became possible to implement effective condition monitoring (CM) systems for critical equipment in many companies. This information can be utilized for the assessment of the actual condition of the operating equipment without any unwanted disruption or stopping of the operation, which result in a high cost. A maintenance strategy referred to as condition-based maintenance (CBM) can then be developed. Compared with the traditional maintenance techniques, CBM reduces the risk of catastrophic system failure as well as the maintenance cost. It is obvious that the collected data carries only partial information about the unknown, hidden state of the equipment and the dimensionality of such data is typically very large, with lots of redundancy, noise, and substantial cross and auto correlation present.

Recently, industrial practitioners and researchers have recognized the cost benefits obtained by applying the economically designed control charts to equipment with CM and maintenance decision making (see, e.g.[1], [2], [3]) and it has been shown that the multivariate Bayesian control charts (MVBCHs) are optimal tools to control the process compared with the non-Bayesian charts [4]. The main objective of this paper is to develop a Bayesian CBM policy for a system subject to two failure modes. Recently, several papers have been published which applied MVBCH tools for CBM decision making (see e.g.[4], [5]). One common assumption in these papers is that there are two system states, an in-control state and an out-of-control state. Although this is a reasonable assumption for quality control applications, such an assumptions is usually not appropriate for maintenance modeling. In their contribution to the statistical design of the MVBCH tools for CBM applications, [6] considered an observable failure state. Using the fixed sampling interval, they applied average run length (ARL) criterion to obtain the optimal control limit. Later, including an observable failure state, [7] in the same framework developed the optimization models for the economic and economic-statistical design of the MVBCH for a three-state CBM model. The authors showed that the MVBCH performs better than the CBM chi-square chart. Similar development of the design of a MVBCH for CBM model can be found in [8], [9], [10], [3]. In all existing CBM models, a failure can only occur when system state degradation exceed the failure threshold and maintenace is performed when the system state degradation exceed the failure threshold. However, in reality a system may fail suddenly during the operation of the system due to to equipment design deficiency, manufacturing defects, etc., even when its degradation has not yet reached the maintenance threshold. In fact, [11] is the only reference where CBM with multiple failure modes was developed for continuously monitored degrading systems. However, this assumption is no longer valid when the system state is monitored at discrete times, which is the usual practice. Such a drawback of existing models motivates us to consider a realistic scenario and develop a CBM model with two modes of failures (competing risks) i.e., a catastrophic and degradation failures which arise quite naturally and are of much interest in the reliability area.

Various approaches for processing and modeling of such information have been proposed in the literature which can be generally classified as nonparametric and parametric techniques (see e.g. [12], [13], [14], [9], [15]). In this paper, we focus on the application of a parametric technique which can be used to extract useful information for early fault detection of a technical system subject to both deterioration and sudden failures. The system is subject to CM and data collection at regular times. We assume that the degradation process evolves as a continuous-time homogeneous Markov chain $(X_t : t \in \mathbb{R}^+)$ with state space $\mathcal{X} = \{0, 1, 2\}$, where states 0 and 1 are unobservable, representing the healthy and unhealthy operational states respectively, and state 2 represents the observable failure state.

In this paper, we formulate a new model assumptions for a

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system with two failure modes in Section II. The maintenance control problem in the semi-Markov decision process (SMDP) framework is presented in Section III, where the observation process is defined as the residual process obtained after data pre-processing and fitting the reference model to the in-control portion of data histories. The whole procedure is illustrated by an example in Section IV, followed by conclusions in Section V.

II. Model Formulation

Assume that the degradation process of the system evolves as a continuous-time homogeneous Markov chain $(X_t : t \in \mathbb{R}^+)$ with state space $\mathcal{X} = \{0, 1, 2\}$, where states 0 and 1 are unobservable, representing the healthy and unhealthy operational states respectively, and state 2 represents the observable failure state. The system is assumed to start in a healthy state and the sojourn times in states 0 and 1 have exponential distributions with parameters λ_0 and λ_1, respectively. The transitions can only take place from each state to the next higher state. We further assume that the sudden failures may also occur during the system's operational time even when the system is working in a good condition. Let the random variable $\xi_1 \geq 0$ denote failure time of a system if the sudden failure occurs, and $\xi_2 \geq 0$ represents the observable failure time if the system fails due to degradation. We note that only the smaller of the ξ_l, for $l = 1, 2$ is in fact observable, together with the actual mode of failure.

Suppose that at equidistance sampling times $\Delta, 2\Delta, \ldots$ for $\Delta \in (0, +\infty)$, vector data $Z_1, Z_2, \ldots \in \mathbb{R}^d$ is collected through condition monitoring, which gives partial information about the system state. We first identify the healthy portions of the data histories. There exists a variety of segmentation methods in the literature (see e.g.[16] and [17] for segmentation of short and long non-stationary time series). Next, a vector autoregressive (VAR) time series model is fitted using the healthy portion of the data to capture any dependencies among monitored variables. When vector AR model is identified, parameters are estimated, and the model adequacy is verified, the residuals of VAR model are calculated using all data histories and utilized to detect an early fault occurrence of an operating system. Residual monitoring has been proposed and studied by several authors (see e.g. [18], [19], [20]). The main advantage of the proposed approach is that the residuals of the fitted model are conditionally independent and normally distributed [18] which are essential properties for tractable maintenance modeling and fast parameter estimation. For successful application of the residual approach using real data see e.g. [14] and [9]. Therefore the observation process is represented by residuals $Y_1, Y_2, \ldots$, which are assumed to be conditionally independent given the state of the system, and for each $h \in \mathbb{N}$, we assume that Y_n given $X_{h\Delta} = i$ for $i = 0, 1$, has d-dimensional normal distribution $N_d(\mu_i, \Sigma_i)$.

III. Optimal Maintenance Control

In this section, we describe the Bayesian maintenance control in the SMDP framework. From the theory of partially observable Markov decision processes (see e.g. [21], [4], [5]) it is well known that the posterior probability statistics that the system is in a warning state is sufficient for decision making. Therefore at each decision epoch $h\Delta$ upon collecting a sample, the posterior probability that the system is in the unhealthy state Π_h is updated using Baye's rule. If Π_h exceeds a control limit $\bar{\Pi} \in [0, 1]$, a system inspection is performed to check whether the system is in the healthy state or in the unhealthy state. If the system is found to be in healthy state, it will be left operational without further repairs or replacement. Otherwise preventive maintenance is triggered. If a failure occurs, failure replacement is carried out immediately. In addition, we assume that the sudden failure may occur which is affected by the age of the system. Therefore, we also keep recording the age information of the system. If the age of the system exceeds a threshold $\bar{T}$ for $0 < \bar{T}\Delta \leq M\Delta$ for a fixed value of M, preventive maintenance is carried out. The objective is to find the optimal values of the sampling interval $\Delta^\star$ and control limits $(\bar{\Pi}^\star, \bar{T}^\star)$ that minimize the long-run expected average cost per unit time. The following cost components are considered:

- C_S: Sampling cost incurred every time we take a sample,
- C_I: Inspection cost incurred when full system inspection is initiated which takes T_I time units,
- $C_{PM}(h)$: Preventive maintenance cost when the age of the system is h. It takes T_P time units,
- C_F: Failure (replacement) cost incurred when corrective maintenance is performed, which takes T_F time units,
- C_{LP}: Loss of production cost rate incurred whenever the system is stopped for performing preventive maintenance, failure replacement, or system inspection,
- C_{OM}: Cost rate (operating maintenance cost rate) in the warning state.

By renewal theory, the cost minimization problem is equivalent to finding the optimal control limits such that:

$$g(\bar{\Pi}^\star, \bar{T}^\star, \Delta^\star) = \inf_{\substack{(\bar{\Pi}, \bar{T}) \\ \Delta > 0}} \left(\frac{E_{(\bar{\Pi}, \bar{T})}(CC)}{E_{(\bar{\Pi}, \bar{T})}(CL)} \right)$$

where CL and CC denote the cycle length and cycle cost, respectively and $(\bar{\Pi}, \bar{T})$ is a fixed control limit pair. We assume that a cycle is completed when the system is brought back to the healthy state and machine condition is as good as new.

Next, we develop an efficient computational algorithm in the semi-Markov decision process (SMDP) framework to determine the optimal control limits. Typically, computing the long-run average cost in the SMDP framework requires discretization of the state space of the posterior probability process. For fixed sampling interval Δ and control limit pair $(\bar{\Pi}, \bar{T})$, we first define the state space of the SMDP. For a fixed large L, the SMDP is defined to be in state (l, h) if the $\Pi_h \in [\frac{l-1}{L}, \frac{l}{L})$ and $h\Delta$ represents the age of the system. We denote set $K_1 = \{(l, h) | 1 \leq l \leq L, h \in \mathbb{N}^+\}$. If the posterior probability of being in the warning state is above the control limit, and upon full system inspection the system is found to be in healthy state, the SMDP is defined to be in state $(0, h)$ for $h \in \mathbb{N}^+$ otherwise the SMDP is defined to be in state (PM, h). We denote set $K_2 = \{(0, h), (PM, h)\}$.

Finally, when the machine has just started working in a good condition the SMDP is defined to be in state $(0,0)$ and we denote $K_3 = \{(0,0)\}$. Thus, the state space for the SMDP is given by $\mathbf{K} = \{K_1 \cup K_2 \cup K_3\}$. With this definition of the state space, for the long run average cost criterion, the SMDP is determined by the following quantities:
$p_{(i,h)(i',h+1)}$ = the probability that at the next decision epoch the system will be in state $(i', h+1) \in \mathbf{K}$ given the current state is $(i,h) \in \mathbf{K}$.
$\tau_{(i,h)}$ = the expected sojourn time until the next decision epoch given the current state is $(i,h) \in \mathbf{K}$.
$C_{(i,h)}$ = the expected cost incurred until the next decision epoch given the current state is $(i,h) \in \mathbf{K}$.
With the quantities defined above, the long-run expected average cost $g(\bar{\Pi}, \bar{T}, \Delta)$ can be obtained by solving the following system of linear equations,

$$\begin{aligned} v_{(i,h)} &= C_{(i,h)} - g(\bar{\Pi}, \bar{T}, \Delta)\tau_{(i,h)} \\ &\quad + \sum_{(i',h+1)\in \mathbf{K}} P_{(i,h)(i',h+1)} v_{(i',h+1)} \quad \forall (i,h) \in \mathbf{K} \\ v_{(s_1,s_2)} &= 0 \qquad \text{for any single } (s_1, s_2) \in \mathbf{K} \end{aligned}$$

so that the optimal control limits $(\bar{\Pi}^\star, \bar{T}^\star)$, sampling epoch $\Delta^\star$, and the corresponding optimal average cost $g(\bar{\Pi}^\star, \bar{T}^\star, \Delta^\star)$ can be computed using Eq. (1). The remainder of the mathematical analysis in this section is devoted to deriving closed form expressions for the SMDP quantities i.e. $P_{(i,h)(i',h+1)}, \tau_{(i,h)}, C_{(i,h)}$ for $(i,h), (i', h+1) \in \mathbf{K}$.

Transition probabilities: The SMDP transition probabilities $p_{(i,h)(i',h+1)}$ for $i < \bar{\Pi}, h < \bar{T}$, $\theta_1 = \frac{i'-1}{L}$, and $\theta_2 = \frac{i'}{L}$ when the system is $h\Delta$ units old can be computed as follows:

$$\begin{aligned} P_{(i,h)(i',h+1)} &= P\Big(\theta_1 \le \Pi_{h+1} < \theta_2, \xi_1 > (h+1)\Delta, \xi_2 > \\ &(h+1)\Delta \Big| \Pi_h, \xi_1 > h\Delta, \xi_2 > h\Delta\Big) = P\Big(\theta_1 \le \Pi_{h+1} \\ &< \theta_2 \Big| \xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, \Pi_h, \xi_1 > h\Delta, \\ &\xi_2 > h\Delta\Big) \times P\Big(\xi_1 > (h+1)\Delta \Big| \xi_2 > (h+1)\Delta, \Pi_h, \\ &\xi_1 > h\Delta, \xi_2 > h\Delta\Big) \times P\Big(\xi_2 > (h+1)\Delta \Big| \Pi_h, \xi_1 > \\ &h\Delta, \xi_2 > h\Delta\Big) = P\Big(\theta_1 \le \Pi_{h+1} < \theta_2 \Big| \xi_1 > (h+1) \\ &\Delta, \xi_2 > (h+1)\Delta, \Pi_h\Big) \frac{R_1((h+1)\Delta)}{R_1(h\Delta)} R_2(\Delta|\Pi_h) \end{aligned}$$

where,

$$\begin{aligned} &P(\theta_1 \le \Pi_{h+1} < \theta_2 \big| \xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, \Pi_h) \\ &\quad = P\Big(\theta_1 \le \Pi_{h+1} < \theta_2 \big| \xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, \\ &\quad \Pi_h, X_{(h+1)\Delta} = 0\Big) \times P\big(X_{(h+1)\Delta} = 0 \big| \xi_1 > (h+1)\Delta, \\ &\quad \xi_2 > (h+1)\Delta, \Pi_h\big) + P\Big(\theta_1 \le \Pi_{h+1} < \theta_2 \big| \xi_1 > (h+1) \\ &\quad \Delta, \xi_2 > (h+1)\Delta, \Pi_h, X_{(h+1)\Delta} = 1\Big) \times P\big(X_{(h+1)\Delta} = 1 \\ &\quad \big| \xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, \Pi_h\big) \end{aligned}$$

Where Π_{h+1} represents the posterior probability that the system with age $(h+1)\Delta$ is in warning state. It can be expressed recursively as:

$$\begin{aligned} \Pi_{h+1} &= P(X_{(h+1)\Delta} = 1 | Y_{h+1}, \xi_2 > (h+1)\Delta, \Pi_h) \\ &= \frac{f(Y_{h+1}|1)(P_{01}(\Delta)(1-\Pi_h) + P_{11}(\Delta)\Pi_h)}{f(Y_{h+1}|1)(P_{01}(\Delta)(1-\Pi_h)+P_{11}(\Delta)\Pi_h) + f(Y_{h+1}|0)P_{00}(\Delta)(1-\Pi_{n-1})} \end{aligned}$$

where transition probability matrix for the state process is given by:

$$P(t) = (p_{ij}(t)) = \begin{pmatrix} e^{-\lambda_0 t} & \frac{\lambda_0(e^{-\lambda_1 t} - e^{-\lambda_0 t})}{\lambda_0 - \lambda_1} & 1 - e^{-\lambda_0 t} - \frac{\lambda_0(e^{-\lambda_1 t} - e^{-\lambda_0 t})}{\lambda_0 - \lambda_1} \\ 0 & e^{-\lambda_1 t} & 1 - e^{-\lambda_1 t} \\ 0 & 0 & 1 \end{pmatrix}$$

Under the assumption $\Sigma_0 \neq \Sigma_1$, we have,

$$\frac{f(Y_{h+1}|\mu_0, \Sigma_0)}{f(Y_{h+1}|\mu_1, \Sigma_1)} = \sqrt{\frac{|\Sigma_1|}{|\Sigma_0|}} e^{\frac{1}{2}(Y_{h+1}-B)'A(Y_{h+1}-B) + \frac{1}{2}C}$$

where constants $A = \Sigma_1^{-1} - \Sigma_0^{-1}, B = (\Sigma_1^{-1} - \Sigma_0^{-1})^{-1}(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0)$, and $C = (\mu_1'\Sigma_1^{-1}\mu_1 - \mu_0'\Sigma_0^{-1}\mu_0) - B'(\Sigma_1^{-1}\mu_1 - \Sigma_0^{-1}\mu_0)$. Thus for $V_{h+1} = (Y_{h+1} - B)'A(Y_{h+1} - B)$ we will have,

$$\Pi_{h+1} = \frac{c^1_{\Pi_h}}{c^1_{\Pi_h} + c^0_{\Pi_h}(|\Sigma_1|.|\Sigma_0|^{-1})^{1/2}\exp\left(\frac{1}{2}(V_{h+1} + C)\right)}$$

where $c^0_{\Pi_h} = P_{00}(\Delta)(1 - \Pi_h) + P_{10}(\Delta)\Pi_h$, and $c^1_{\Pi_h} = P_{01}(\Delta)(1 - \Pi_h) + P_{11}(\Delta)\Pi_h$. Therefore,

$$\begin{aligned} &P\big(\theta_1 \le \Pi_{h+1} < \theta_2 | \xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, \Pi_h\big) \\ &= P\left[2\ln\left(\frac{(1-\theta_2)c^1_{\Pi_h}}{\theta_2 c^0_{\Pi_h}}\sqrt{\frac{|\Sigma_0|}{|\Sigma_1|}}\right) - C < V_{h+1} \le \right. \\ &\left. 2\ln\left(\frac{(1-\theta_1)c^1_{\Pi_h}}{\theta_1 c^0_{\Pi_h}}\sqrt{\frac{|\Sigma_0|}{|\Sigma_1|}}\right) - C | X_{(h+1)\Delta} = 0\right] \times \\ &\left(\frac{c^0_{\Pi_h}}{c^0_{\Pi_h} + c^1_{\Pi_h}}\right) + P\left[2\ln\left(\frac{(1-\theta_2)c^1_{\Pi_h}}{\theta_2 c^0_{\Pi_h}}\sqrt{\frac{|\Sigma_0|}{|\Sigma_1|}}\right) - C\right. \\ &< V_{h+1} \le 2\ln\left(\frac{(1-\theta_1)c^1_{\Pi_h}}{\theta_1 c^0_{\Pi_h}}\sqrt{\frac{|\Sigma_0|}{|\Sigma_1|}}\right) - C | X_{(h+1)\Delta} \\ &\left. = 1\right] \times \left(\frac{c^1_{\Pi_h}}{c^0_{\Pi_h} + c^1_{\Pi_h}}\right) = T_0(\theta_1, \theta_2 | \xi_1 > (h+1)\Delta, \\ &\xi_2 > (h+1)\Delta, \Pi_h) \cdot \left(\frac{c^0_{\Pi_h}}{c^0_{\Pi_h} + c^1_{\Pi_h}}\right) + T_1(\theta_1, \theta_2 | \xi_1 > \\ &(h+1)\Delta, \xi_2 > (h+1)\Delta, \Pi_h) \cdot \left(\frac{c^1_{\Pi_h}}{c^0_{\Pi_h} + c^1_{\Pi_h}}\right) \end{aligned}$$

for any $0 < \theta_0 < \theta_1 < 1$. Since $Y_h - B|X_{(h+1)\Delta} = 0 \sim N(\mu_0 - B, \Sigma_0)$ and $Y_h - B|X_{(h+1)\Delta} = 1 \sim N(\mu_1 - B, \Sigma_1)$, $T_0(\theta_1, \theta_2|\xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, Y_h, \Pi_h)$ and $T_1(\theta_1, \theta_2|\xi_1 > (h+1)\Delta, \xi_2 > (h+1)\Delta, Y_h, \Pi_h)$ can be computed using Theorem 3.1 in [22] which provided a closed-form expression for the cumulative distribution function of $V_{h+1}|X_{(h+1)\Delta}$. For a large value of L, Π_h can be computed using:

$$\Pi_h = \begin{cases} \frac{i-.5}{h} & \text{if } i > 0, \forall h, \\ 0 & \text{if } i = 0, \forall h. \end{cases}$$

And also,

$$P_{(i,h)(0,0)} = P\Big(\min(\xi_1,\xi_2) \leq (h+1)\Delta|\xi_1 > h\Delta, \xi_2 > h\Delta, \Pi_h\Big) = 1 - R_1((h+1)\Delta|h\Delta).R_2(\Delta|\Pi_h)$$

$$P_{(i,h)(PM,h)} = \frac{i-.5}{L} \qquad \text{for } i \geq \bar{\Pi} \text{ and } h < \bar{T}$$

$$P_{(i,h)(0,h)} = 1 - \frac{i-.5}{L} \qquad \text{for } i \geq \bar{\Pi} \text{ and } h < \bar{T}$$

$$P_{(PM,h)(0,0)} = 1$$

$$P_{(i,h)(0,0)} = 1 \qquad \text{for } h \geq \bar{T}$$

Expected sojourn times: The expected sojourn time for each state can be derived as follows:

$$\tau_{(i,h)} = E(T_{(i,h)}|\xi_1 > h\Delta, \xi_2 > h\Delta, \Pi_h)$$
$$= \Delta \sum_{i' \in \mathbf{K}_1} P_{(i,h)(i',h+1)} + \int_0^\Delta (T_F + t)\Big[-\frac{d}{dt}\Big(R_1(h\Delta + t|h\Delta)R_2(t|\Pi_h)\Big)\Big]dt = \Delta \sum_{i' \in \mathbf{K}_1} P_{(i,h)(i',h+1)} +$$
$$\Big[-(T_F+t)R_1(h\Delta+t|h\Delta)R_2(t|\Pi_h)\Big]_0^\Delta + \int_0^\Delta R_1(h\Delta + t|h\Delta)R_2(t|\Pi_h)dt \qquad \text{for } i < \bar{\Pi}, h < \bar{T}$$

$$\tau_{(PM,h)} = T_{PM}$$

$$\tau_{(i,h)} = T_I \qquad \text{for } i \geq \bar{\Pi},\ h < \bar{T}$$

$$\tau_{(i,h)} = T_{PM} \qquad \text{for } h \geq \bar{T}$$

Expected costs:: The average cost incurred until the next decision epoch for each state is given by:

$$C_{(i,h)} = E(Cost_{(i,h)}|\xi_1 > h\Delta, \xi_2 > h\Delta, \Pi_h)$$
$$= C_S \sum_{i' \in \mathbf{K}_1} P_{(i,h)(i',h+1)} + (C_{LP} \cdot T_F + C_F)P_{(i,h)(0,0)}$$
$$+ C_{OM} \int_0^\Delta P(X_t = 1|\xi_1 > h\Delta, \xi_2 > h\Delta, \Pi_h)dt$$
$$= C_S \sum_{i' \in \mathbf{K}_1} P_{(i,h)(i',h+1)} + (C_{LP} \cdot T_F + C_F)P_{(i,h)(0,0)}$$
$$+ C_{OM} \int_0^\Delta (P_{01}(t)(1-\Pi_h) + P_{11}(t)\Pi_h)\, dt \qquad \text{for } i < \bar{\Pi}, h < \bar{T}$$

$$C_{(PM,h)} = C_{PM}(h) + C_{LP} \cdot T_{PM}$$

$$C_{(i,h)} = C_I + C_{LP} \cdot T_I \qquad \text{for } i \geq \bar{\Pi},\ h < \bar{T}$$

$$C_{(i,h)} = C_{PM}(h) + C_{LP} \cdot T_{PM} \qquad \text{for } h \geq \bar{T}$$

IV. Numerical Example

We assume that the system deterioration follows a continuous-time homogenous Markov chain $(X_t, t \geq 0)$, with state space $\mathcal{X} = \{0, 1, 2\}$. Unobservable states 0 and 1 represent the healthy and unhealthy operational states respectively, and state 2 corresponds to the observable failure state. The transition rates of the state process are given by $\lambda_0 = .15$ and $\lambda_1 = .3$. We further assume that the observation process $(Y_h : h \in \mathbb{N})$ which represents the information collected through CM at equidistant sampling epochs Δ, follow 2-dimensional normal distributions $N_2(\mu_0, \Sigma_0)$ and $N_2(\mu_1, \Sigma_1)$ when the system is in the healthy and unhealthy state, respectively, where,

$$\mu_0 = \begin{pmatrix} .2 \\ -.1 \end{pmatrix} \quad \Sigma_0 = \begin{pmatrix} 1.5 & .5 \\ .5 & 1.5 \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} .8 \\ -.6 \end{pmatrix} \quad \Sigma_1 = \begin{pmatrix} 2.5 & 2.5 \\ 2.5 & 3 \end{pmatrix}$$

We also assume that the system is subject to sudden failure which is the second failure mode, and time to sudden failure follows Weibull distribution with scale parameter $\lambda = 10$ and shape parameter $k = 2$. It the age of a system exceeds a threshold $\bar{T}$, where $0 < \bar{T}\Delta \leq 10\Delta$, preventive maintenance activity is initiated.

Simultaneously, we apply the Bayesian control chart. The inspection, preventive maintenance, and replacement time parameters $T_I = 2, T_{PM} = 3, T_F = 10$, and the maintenance cost parameters $C_I = \$10, C_{PM}(h) = \$(500 + 10h), C_F = \$1500$ and $C_{LP} = 20, C_{OM} = 2$ dollars per unit time are considered. In order to define the state space of the SMDP, we have found that when $L \geq 25$, the partition leads to a high degree of precision, so that L does not need to be chosen very large. If the estimated posterior probability that the system is in the unhealthy state at a decision epoch is above the control limit $\bar{\Pi}$, the system full inspection is carried out, followed possibly by a preventive maintenance.

Recently, [9] showed that MVBCH with a single degradation failure mode achieved the highest number of predicted failures and lowest total maintenance cost when compared with other maintenance policies. We verify the superiority of the proposed MVBCH considering two failure modes and preventive maintenance age replacement by comparing its performance with previously developed MVBCH in [9] (see Table I).

TABLE I
Comparisons with other method.

	MVBCH with age replacement	MVBCH with no age replacement
$\bar{\Pi}^\star$	.25	.25
$\Delta^\star$	1	1
$g(\bar{\Pi}^\star, \bar{T}^\star, \Delta^\star)$	\$47.21	\$58.95

V. Conclusions

In this paper, we have proposed a maintenance policy based on a combination of a multivariate Bayesian control chart and a preventive age-based policy for a system subject to two failure modes, namely a sudden failure and a failure due to system degradation. It has been assumed that the failure time due to sudden failure follows a general type of distribution. The degradation state process has been modeled as a 3-state continuous time Markov chain, where only the failure state is observable. An efficient computational algorithm in the semi-Markov decision process (SMDP) framework has been developed to determine the optimal control limits. A comparison with a maintenance policy based on a multivariate Bayesian control chart with no age replacement has been given. It has been found that the new maintenance policy proposed in this paper considerably reduces the average maintenance cost.

By introducing the maintenance policy proposed in this paper which is easy to implement, the maintenance cost will be reduced substantially and also the machine safety and reliability will be improved. In our future work, the new maintenance policy will be applied to real data. We hope that the results obtained in this paper will motivate future research in this area by allowing the state sojourn times to have more general distributions.

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