

# Exp-Function Method for Finding Some Exact Solutions of Rosenau Kawahara and Rosenau Korteweg-de Vries Equations

Ehsan Mahdavi

**Abstract**—In this paper, we apply the Exp-function method to Rosenau-Kawahara and Rosenau-KdV equations. Rosenau-Kawahara equation is the combination of the Rosenau and standard Kawahara equations and Rosenau-KdV equation is the combination of the Rosenau and standard KdV equations. These equations are nonlinear partial differential equations (NPDE) which play an important role in mathematical physics. Exp-function method is easy, succinct and powerful to implement to nonlinear partial differential equations arising in mathematical physics. We mainly try to present an application of Exp-function method and offer solutions for common errors which occur during some of the recent works.

**Keywords**—Exp-function method, Rosenau Kawahara equation, Rosenau Korteweg-de Vries equation, nonlinear partial differential equation.

MSC Subject Classifications: 35D99, 65D25

## I. INTRODUCTION

THE study of nonlinear partial differential equations (NPDE) plays an important role in mathematical physics, engineering and the other sciences. In the past several decades, various methods for obtaining solutions of NPDEs and ODEs have been presented, such as, tanh-function method [1], [2], [3], Adomian decomposition method [4], [5], Homotopy perturbation method [6], [7], [8], variational iteration method [9], [10], [11], spectral method [12], [13], [14], sine-cosine method [15], [16] and so on. Recently, Ji-Huan He and Xu-Hong Wu [17], [18] have proposed a novel method called Exp-function method, which is easy, succinct and powerful to implement to nonlinear partial differential equations arising in mathematical physics. The Exp-function method has been successfully applied to many kinds of NPDEs, such as, KdV equation with variable coefficients [19], Maccari's system [20], Boussinesq equations [21], Burger's equations [22], [23], [24], Double Sine-Gordon equation [25], [26], Schrödinger equations [27], Jaulent-Miodek equations [28] and other important nonlinear differential equations [29], [30], [31], [32], [33]. However, common errors have occurred during the application of Exp-function method in several recent papers. Seven common errors are formulated and classified by [34]. In this paper, we attempt to rectify these common errors in order to obtain the exact solutions of two nonlinear partial differential equations, namely, Rosenau-Kawahara equation and Rosenau Korteweg-de Vries (Rosenau-KdV) equation

given by

$$u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0 ,$$

$$u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} = 0 ,$$

,respectively.

Evidently, the Rosenau-Kawahara equation can be considered as the combination of the following Rosenau equation and standard Kawahara equation

$$u_t + u_{xxxxx} + u_x + uu_x = 0 ,$$

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0 ,$$

,respectively and the Rosenau-KdV equation can be considered as the combination of the following Rosenau equation and standard KdV equation

$$u_t + u_{xxxxx} + u_x + uu_x = 0 ,$$

$$u_t + uu_x + u_{xxx} = 0 ,$$

respectively. Rosenau equation was proposed by Rosenau [35], [36] to describe the dynamics of dense discrete systems. Also, Kawahara equation is a model equation for plasma waves, capillary-gravity water waves [37].

The rest of the paper is organized as follows: Section II describes the Exp-function method for finding some exact solutions for the NPDEs. Afterwards, the applications of the proposed analytical scheme are presented in Section III. The conclusions and findings are discussed in the Section IV.

## II. BASIC IDEA OF EXP-FUNCTION METHOD

We consider a general nonlinear PDE in the following form

$$N(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0 , \quad (1)$$

where  $N$  is a polynomial function with respect to the indicated variables or some functions which can be reduced to a polynomial function by using some transformation. We introduce a complex variation as

$$u(x, t) = U(\eta) , \quad \eta = kx + \omega t . \quad (2)$$

where  $k$  and  $\omega$  are constants. We can rewrite (1) in the following nonlinear ordinary differential equations

$$N(U, kU', \omega U', k^2 U'', \dots) = 0 ,$$

where the prime denotes the derivation with respect to  $\eta$ . According to the Exp-function method [17], we assume that

E. Mahdavi is now with Department of Computer Engineering, Isfahan University of Technology, Isfahan, Iran (Phone:+98 - (0)31- 33913110; e-mail: (e.mahdavi@ec.iut.ac.ir)).

the solution can be expressed in the form

$$U(\eta) = \frac{\sum_{i=-d}^c a_i \exp(i\eta)}{\sum_{j=-q}^p b_j \exp(j\eta)}, \quad (3)$$

where  $c$ ,  $d$ ,  $p$  and  $q$  are positive integers which can be freely chosen and  $a_i$  and  $b_j$  are unknown constants to be determined. To determine the values of  $c$  and  $p$ , we balance the highest order linear term with the highest order nonlinear term in (3). Similarly to determine the values of  $d$  and  $q$ . So, by means of the Exp-function method, we obtain the generalized solitary solution and periodic solution for nonlinear evolution equations arising in mathematical physics.

### III. APPLICATIONS OF THE EXP-FUNCTION METHOD

#### A. Rosenau-Kawahara equation

In this section, we detailed the steps of the Exp-function method to construct the exact solutions of Rosenau-Kawahara (RK) equation

$$u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0. \quad (4)$$

Making the travelling wave transformation

$$u(x, t) = U(\eta), \quad \eta = kx + \omega t, \quad (5)$$

and integrating with respect to  $\eta$ , (4) becomes an ordinary differential equation in the form

$$(\omega + k)U + \frac{k}{2}U^2 + k^3U'' + k^4(\omega - k)U'''' = 0, \quad (6)$$

where the prime denotes the derivative with respect to  $\eta$  and also where the integration constant is chosen as zero. In other words, we solved this problem for the case when integration constant is zero.

According to the Exp-function method [26], [17], [38], we assume that the solution of (6) can be expressed in the form

$$U(\eta) = \frac{a_c \exp(c\eta) + \dots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \dots + b_{-q} \exp(-q\eta)}, \quad (7)$$

where  $c$ ,  $d$ ,  $p$  and  $q$  are positive integers which are unknown to be determined later. In order to determine the values of  $c$  and  $p$ , we balance the linear term of the highest order with the highest order nonlinear terms in (6), i.e.  $U''''$  and  $U^2$ . we have

$$U'''' = \frac{c_1 \exp[(c + 15p)\eta] + \dots}{c_2 \exp[16p\eta] + \dots}, \quad (8)$$

and

$$U^2 = \frac{c_3 \exp[2c\eta] + \dots}{c_4 \exp[2p\eta] + \dots} = \frac{c_3 \exp[(2c + 14p)\eta] + \dots}{c_4 \exp[16p\eta] + \dots}, \quad (9)$$

where  $c_i$  are coefficients. By balancing the highest order of Exp-function in (8) and (9), we derive

$$c + 15p = 2c + 14p,$$

which leads to the following result

$$p = c.$$

Similarly to determine the values of  $d$  and  $q$ , we balance the linear term of lowest order in (6)

$$U'''' = \frac{\dots + d_1 \exp[-(15q + d)\eta]}{\dots + d_2 \exp[-16q\eta]}, \quad (10)$$

and

$$U^2 = \frac{\dots + d_3 \exp[-2d\eta]}{\dots + d_4 \exp[-2q\eta]} = \frac{\dots + d_3 \exp[-(2d + 14q)\eta]}{\dots + d_4 \exp[-16q\eta]}, \quad (11)$$

where  $d_i$  are determined coefficients, we obtain

$$-(15q + d) = -(2d + 14q),$$

which leads to the result  $q = d$ .

1) Case I:  $p = c = 1$ ,  $q = d = 1$ : Based on this selection, (3) reduces to

$$U(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{\exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \quad (12)$$

Substituting (12) into (6) and equating to zero the coefficients of each  $\exp(n\eta)$  yield a set of algebraic equations for  $a_0$ ,  $b_0$ ,  $a_{-1}$ ,  $a_1$ ,  $b_{-1}$ ,  $k$  and  $\omega$ . By solving the system of algebraic equations with a professional mathematical software, we obtain

case 1.

$$\begin{cases} a_{-1} = -b_0^2, & b_{-1} = \frac{1}{4}b_0^2, \\ a_0 = 8b_0, & b_0 = b_0, \\ a_1 = -4, & k = \omega = \sqrt{2}. \end{cases} \quad (13)$$

inserting these results into (12), we obtain

$$U(\eta) = \frac{-4 \exp(\eta) + 8b_0 - b_0^2 \exp(-\eta)}{\exp(\eta) + b_0 + \frac{1}{4}b_0^2 \exp(-\eta)}. \quad (14)$$

where  $\eta = \sqrt{2}(x + t)$  and  $b_0$  is a free parameter which can be determined by initial or boundary conditions. We rewrite (14) by using (5)

$$u(x, t) = \frac{-4 \exp(\sqrt{2}(x + t)) + 8b_0 - b_0^2 \exp(-\sqrt{2}(x + t))}{\exp(\sqrt{2}(x + t)) + b_0 + \frac{1}{4}b_0^2 \exp(-\sqrt{2}(x + t))}. \quad (15)$$

These results cover some of the special solutions of (4) regarding to the initial value conditions. By considering  $u(x, 0) = -4 + 6 \operatorname{csch}^2(\frac{\sqrt{2}}{2}x)$  as a initial value condition, we have

$$\begin{cases} u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = -4 + 6 \operatorname{csch}^2(\frac{\sqrt{2}}{2}x). \end{cases} \quad (16)$$

From (16) and (15), we obtain

$$b_0 = 2, \quad (17)$$

Thus, from substituting (17) into (15), we obtain

$$\begin{aligned} u(x, t) &= \frac{-4 \exp(\sqrt{2}(x + t)) + 16 - 4 \exp(-\sqrt{2}(x + t))}{\exp(\sqrt{2}(x + t)) + 2 + \exp(-\sqrt{2}(x + t))} \\ &= -4 + 6 \times \frac{4}{\exp(\sqrt{2}(x + t)) + 2 + \exp(-\sqrt{2}(x + t))} \\ &= -4 + 6 \operatorname{csch}^2 \left[ \frac{\sqrt{2}}{2}(x + t) \right]. \end{aligned}$$

which is the solution obtained by tanh method in [39].

If (4) be in the following form

$$\begin{cases} u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = -4 - 6 \operatorname{sech}^2(\frac{\sqrt{2}}{2}x) \end{cases} \quad (18)$$

then, by considering, (15), we can obtain

$$b_0 = -2, \quad (19)$$

Thus, substituting  $b_0 = -2$  into (15), we have

$$\begin{aligned} u(x, t) &= \frac{-4 \exp(\sqrt{2}(x+t)) - 16 - 4 \exp(-\sqrt{2}(x+t))}{\exp(\sqrt{2}(x+t)) - 2 + \exp(-\sqrt{2}(x+t))} \\ &= -4 - 6 \times \frac{4}{\exp(\sqrt{2}(x+t)) - 2 + \exp(-\sqrt{2}(x+t))} \\ &= -4 - 6 \sec h^2 \left[ \frac{\sqrt{2}}{2}(x+t) \right]. \end{aligned}$$

which is the solution obtained by tanh method in [39].

**case 2.**

$$\begin{cases} a_{-1} = 0, & b_{-1} = \frac{1}{4}b_0^2, \\ a_0 = -12b_0, & b_0 = b_0, \\ a_1 = 0, & k = \omega = i\sqrt{2}. \end{cases} \quad (20)$$

In this case,  $k$  and  $\omega$  are imaginary numbers. Substituting these results into (12), we obtain the following solution

$$U(\eta) = \frac{-12b_0}{\exp(\eta) + b_0 + \frac{1}{4}b_0^2 \exp(-\eta)}. \quad (21)$$

where  $\eta = i\sqrt{2}(x+t)$  and  $b_0$  is a free parameter which can be determined by initial or boundary conditions. We rewrite (21) by using (5)

$$u(x, t) = \frac{-12b_0}{\exp(i\sqrt{2}(x+t)) + b_0 + \frac{1}{4}b_0^2 \exp(-i\sqrt{2}(x+t))}. \quad (22)$$

If (4) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = -6 \csc^2\left(\frac{\sqrt{2}}{2}x\right). \end{cases} \quad (23)$$

then, by considering (22), we can obtain

$$b_0 = 2, \quad (24)$$

Thus, substituting  $b_0 = 2$  into (22), we have:

$$\begin{aligned} u(x, t) &= \frac{-24}{\exp(i\sqrt{2}(x+t)) + 2 + \exp(-i\sqrt{2}(x+t))} \\ &= -6 \csc^2 \left[ \frac{\sqrt{2}}{2}(x+t) \right]. \end{aligned}$$

which is the same as Zuo's solutions [39].

Also if (4) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = -6 \sec^2\left(\frac{\sqrt{2}}{2}x\right). \end{cases} \quad (25)$$

then, by considering (22), we can obtain

$$b_0 = -2, \quad (26)$$

Thus, substituting  $b_0 = -2$  into (22), we have:

$$\begin{aligned} u(x, t) &= \frac{24}{\exp(i\sqrt{2}(x+t)) - 2 + \exp(-i\sqrt{2}(x+t))} \\ &= -6 \sec^2 \left[ \frac{\sqrt{2}}{2}(x+t) \right]. \end{aligned}$$

which is the exact solution given by Zuo in[39].

2) *Case II:*  $p = c = 2, q = d = 2$ : In this case (7) becomes

$$U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + \dots}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + \dots} \quad (27)$$

$$\frac{\dots + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\dots + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)},$$

There are some free parameters in this equation. For simplicity, we study the two different cases:

**A.**  $a_{-1} = a_1 = 0$

In this case (27) can be expressed as

$$U(\eta) = \frac{a_2 \exp(2\eta) + a_0 + \dots}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + \dots} \quad (28)$$

$$\frac{\dots + a_{-2} \exp(-2\eta)}{\dots + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}$$

Substituting (28) into (6) and equating to zero, the coefficients of each  $\exp(n\eta)$  yield a set of algebraic equations for  $a_0, b_0, a_{-1}, a_1, b_1, k$  and  $\omega$ . By solving the system of algebraic equations with a professional mathematical software, we obtain the coefficients

**case A.1.**

$$a_2 = a_{-2} = 0, \quad a_0 = \frac{105}{13}b_1^2 \frac{\sqrt{205}-13}{36}, \quad b_0 = \frac{3}{8}b_1^2, \quad (29)$$

$$b_{-1} = \frac{1}{16}b_1^3, \quad b_{-2} = \frac{1}{256}b_1^4$$

$$k = \frac{1}{6}\sqrt{\sqrt{205}-13}, \quad \omega = -\frac{1}{6}\sqrt{\sqrt{205}-13} \times \frac{\sqrt{205}}{13}$$

where  $b_1$  is a free parameter. Inserting (29) into (28), we have

$$u(x, t) = \frac{\frac{105}{13}b_1^2 \frac{\sqrt{205}-13}{36}}{\exp(2\eta) + b_1 \exp(\eta) + \frac{3}{8}b_1^2 + \dots} \quad (30)$$

$$\frac{\dots}{\dots + \frac{1}{16}b_1^3 \exp(-\eta) + \frac{1}{256}b_1^4 \exp(-2\eta)}$$

where  $\eta = \frac{\sqrt{\sqrt{205}-13}}{6} \left[ x - \frac{\sqrt{205}}{13}t \right]$ . If (4) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = \left( -\frac{35}{12} + \frac{35}{156}\sqrt{205} \right) \csc h^4 \left[ \frac{1}{12}\sqrt{\sqrt{205}-13}x \right]. \end{cases} \quad (31)$$

then, by considering, (30), we can obtain

$$b_1 = 4. \quad (32)$$

Thus, substituting  $b_1 = 4$  into (36), we have:

$$\begin{aligned} u(x, t) &= \frac{\frac{140}{39}(\sqrt{205}-13)}{\exp(2\eta) + 4 \exp(\eta) + 6 + 4 \exp(-\eta) + \exp(-2\eta)} \\ &= \left( -\frac{35}{12} + \frac{35}{156}\sqrt{205} \right) * \dots \\ &\dots * \csc h^4 \left[ \frac{1}{12}\sqrt{\sqrt{205}-13} \left( x - \frac{\sqrt{205}}{13}t \right) \right]. \end{aligned}$$

which is the exact solution given by Zuo in [39] obtained via sine-cosine method.

Also, if (4) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = \left( -\frac{35}{12} + \frac{35}{156}\sqrt{205} \right) * \dots \\ \dots * \sec h^4 \left[ \frac{1}{12}\sqrt{\sqrt{205}-13}x \right]. \end{cases} \quad (33)$$

then, by considering (36), we can obtain

$$b_1 = -4. \quad (34)$$

Thus, substituting  $b_1 = -4$  into (36), we have:

$$\begin{aligned} u(x, t) &= \frac{\frac{140}{39}(\sqrt{205}-13)}{\exp(2\eta) - 4\exp(\eta) + 6 - 4\exp(-\eta) + \exp(-2\eta)} \\ &= \left( -\frac{35}{12} + \frac{35}{156}\sqrt{205} \right) * \dots \\ &\dots * \sec h^4 \left[ \frac{1}{12}\sqrt{\sqrt{205}-13} \left( x - \frac{\sqrt{205}}{13}t \right) \right]. \end{aligned}$$

Comparing our result and Zuo's result [39] show that the results are the same.

**case A.2.**

$$\begin{aligned} a_2 = a_{-2} = 0, \quad a_0 &= -\frac{105}{13}b_1^2 \frac{\sqrt{205}+13}{36}, \quad b_0 = \frac{3}{8}b_1^2, \quad (35) \\ b_{-1} &= \frac{1}{16}b_1^3, \quad b_{-2} = \frac{1}{256}b_1^4 \\ k &= \frac{1}{6}i\sqrt{\sqrt{205}+13}, \quad \omega = \frac{1}{6}i\sqrt{\sqrt{205}+13} \times \frac{\sqrt{205}}{13} \end{aligned}$$

where  $b_1$  is a free parameter. Substituting these results into (28), we obtain the following exact solution

$$u(x, t) = -\frac{\frac{105}{13}b_1^2 \frac{\sqrt{205}+13}{36}}{\exp(2\eta) + b_1 \exp(\eta) + \frac{3}{8}b_1^2 + \dots} \quad (36)$$

$$\frac{\dots + \frac{1}{16}b_1^3 \exp(-\eta) + \frac{1}{256}b_1^4 \exp(-2\eta)}{\dots}$$

where  $\eta = i\frac{\sqrt{\sqrt{205}+13}}{6} \left[ x + \frac{\sqrt{205}}{13}t \right]$ . If (4) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = \\ -\left( \frac{35}{12} + \frac{35}{156}\sqrt{205} \right) \csc^4 \left[ \frac{1}{12}\sqrt{\sqrt{205}+13}x \right] \end{cases} \quad (37)$$

Then, by considering (36), we can obtain

$$b_1 = 4. \quad (38)$$

Thus, substituting  $b_1 = 4$  into (36), we would have the same solution obtained by sine-cosine method [39]. Also, if (4) be

in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} - u_{xxxxx} = 0, \\ u(x, 0) = \\ -\left( \frac{35}{12} + \frac{35}{156}\sqrt{205} \right) \sec^4 \left[ \frac{1}{12}\sqrt{\sqrt{205}+13}x \right] \end{cases} \quad (39)$$

Then, by considering (36), we can obtain

$$b_1 = -4. \quad (40)$$

Thus, substituting  $b_1 = -4$  into (36), we would have the same solution obtained by sine-cosine method [39].

**case A.3.**

$$\begin{aligned} a_2 = -4, \quad a_{-2} &= -\frac{1}{324}b_1^4, \quad a_0 = \frac{14}{9}b_1^2, \quad b_0 = \frac{5}{18}b_1^2, \quad (41) \\ b_{-1} &= \frac{1}{36}b_1^3, \quad b_{-2} = \frac{1}{1296}b_1^4, \quad k = \omega = \pm\sqrt{2} \end{aligned}$$

where  $b_1$  is a free parameter. Substituting these results into (28), we obtain the following exact solution

$$u(x, t) = \frac{-4\exp(2\eta) + \frac{14}{9}b_1^2 - \dots}{\exp(2\eta) + b_1 \exp(\eta) + \frac{5}{18}b_1^2 + \dots}$$

$$\frac{\dots - \frac{1}{324}b_1^4 \exp(-2\eta)}{\dots + \frac{1}{36}b_1^3 \exp(-\eta) + \frac{1}{1296}b_1^4 \exp(-2\eta)},$$

where  $\eta = \pm\sqrt{2}(x+t)$ .

**B.  $b_{-1} = b_1 = 0$**

In this case (27) can be expressed as

$$U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + \dots}{\exp(2\eta) + b_0 + \dots} \quad (42)$$

$$\frac{\dots + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\dots + b_{-2} \exp(-2\eta)},$$

Substituting (42) into (6) and equating to zero the coefficients of each  $\exp(n\eta)$  yield a set of algebraic equations for  $a_0, b_0, a_{-1}, a_1, b_1, k$  and  $\omega$ . By solving the system of algebraic equations with a professional mathematical software, we obtain the coefficients

**case B.1.**

$$\begin{aligned} a_2 &= -4, \quad a_0 = -\frac{5}{72}a_1^2, \quad b_0 = -\frac{1}{288}a_1^2, \quad (43) \\ b_{-2} &= \frac{1}{331776}a_1^4, \quad a_{-1} = \frac{1}{576}a_1^3 \\ a_{-2} &= -\frac{1}{82944}a_1^4, \quad k = \omega = \pm\sqrt{2} \end{aligned}$$

where  $a_1$  is a free parameter. Inserting (43) into (42), we have

$$u(x, t) = \frac{-4\exp(2\eta) + a_1 \exp(\eta) - \frac{5}{72}a_1^2 + \dots}{\exp(2\eta) - \frac{1}{288}a_1^2 + \dots}$$

$$\frac{\dots + \frac{1}{576}a_1^3 \exp(-\eta) - \frac{1}{82944}a_1^4 \exp(-2\eta)}{\dots + \frac{1}{331776}a_1^4 \exp(-2\eta)},$$

where  $\eta = \pm\sqrt{2}(x+t)$ .

### case B.2.

$$\begin{aligned} a_2 = a_{-2} = 0, \quad a_0 = \pm 48b_{-1}^{1/2}, \quad b_0 = \mp 2b_{-1}^{1/2}, \quad (44) \\ a_1 = \pm 24b_{-1}^{1/4} \text{ or } \pm 24ib_{-1}^{1/4}, \\ a_{-1} = \pm 24b_{-1}^{3/4} \text{ or } \pm 24ib_{-1}^{3/4}, \\ k = \omega = \pm i\sqrt{2} \end{aligned}$$

where  $b_{-1}$  is a free parameter. Substituting (44) into (42), we get the exact solitary wave solutions of (4).

### B. Rosenau-KdV equation

In this section, we consider a generalized form of the Rosenau-KdV equation, which reads

$$u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} = 0, \quad (45)$$

Using the transformation (5) and integrating with respect to  $\eta$ , then (45) becomes

$$(\omega + k)U + \frac{k}{2}U^2 + k^3U'' + k^4\omega U'''' = 0, \quad (46)$$

where the prime denotes the derivative with respect to  $\eta$  and also where the integration constant is chosen as zero. In other words, we are solved this problem for the case when the integration constant is zero. Substituting (3) into (46) and then balancing the linear term of the highest order  $U''''$  with the highest order nonlinear term  $U^2$  in (46), we have

$$c + 15p = 2c + 14p, \quad -(15q + d) = -(2d + 14q),$$

This leads to the result

$$c = p, \quad q = d.$$

We can choose the values of  $c$  and  $d$ , we set  $p = c = 2$  and  $q = d = 2$ , the trial function, (3) becomes

$$U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + \dots}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + \dots} \quad (47)$$

$$\frac{\dots + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\dots + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)},$$

There are some free parameters in (47), we set  $a_1 = a_{-1} = 0$  for simplicity, the trial function is simplified as follows

$$U(\eta) = \frac{a_2 \exp(2\eta) + a_0 + \dots}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + \dots} \quad (48)$$

$$\frac{\dots + a_{-2} \exp(-2\eta)}{\dots + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)},$$

Substituting (48) into (46) and equating to zero the coefficients of each  $\exp(n\eta)$  yield a set of algebraic equations for  $a_0$ ,  $b_0$ ,  $a_{-1}$ ,  $a_1$ ,  $b_1$ ,  $k$  and  $\omega$ . By solving the system of algebraic equations with a professional mathematical software, we obtain the coefficients

### case 1.

$$\begin{aligned} a_2 = a_{-2} = 0, \quad a_0 = \frac{105}{13}b_1^2 \frac{\sqrt{313}-13}{72}, \quad (49) \\ b_0 = \frac{3}{8}b_1^2, b_{-1} = \frac{1}{16}b_1^3, \quad b_{-2} = \frac{1}{256}b_1^4 \\ k = \frac{1}{12}\sqrt{2\sqrt{313}-26}, \\ \omega = -\frac{1}{12}\sqrt{2\sqrt{313}-26} \times \left( \frac{1}{2} + \frac{\sqrt{313}}{26} \right) \end{aligned}$$

where  $b_1$  is free parameter which can be determined by the initial or boundary conditions. So, substituting (49) into (48) we have:

$$u(x, t) = \frac{\frac{105}{13}b_1^2 \frac{\sqrt{313}-13}{72}}{\exp(2\eta) + b_1 \exp(\eta) + \frac{3}{8}b_1^2 + \dots} \quad (50)$$

$$\frac{\dots + \frac{1}{16}b_1^3 \exp(-\eta) + \frac{1}{256}b_1^4 \exp(-2\eta)}{\dots},$$

where  $\eta = \frac{1}{12}\sqrt{2\sqrt{313}-26} \left[ x - \left( \frac{1}{2} + \frac{\sqrt{313}}{26} \right) t \right]$ . If (45) be in the following form

$$\begin{cases} u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} = 0, \\ u(x, 0) = \left( -\frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \csc h^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313}-26} x \right] \end{cases} \quad (51)$$

then, by considering (50), we have

$$b_1 = 4. \quad (52)$$

Thus, substituting  $b_1 = 4$  into (50), we have:

$$u(x, t) = \frac{\frac{70}{39}(\sqrt{313}-13)}{\exp(2\eta) + 4 \exp(\eta) + 6 + 4 \exp(-\eta) + \exp(-2\eta)}$$

$$= \left( -\frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots$$

$$\dots * \csc h^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313}-26} \left( x - \left( \frac{1}{2} + \frac{\sqrt{313}}{26} \right) t \right) \right].$$

which is the exact solution given by Zuo in [39] obtained via sine-cosine method.

Also, if (45) be in the following form

$$\begin{cases} u_t + u_{xxxxx} + u_x + uu_x + u_{xxx} = 0, \\ u(x, 0) = \left( -\frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \sec h^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313}-26} x \right]. \end{cases} \quad (53)$$

then taking into account (50), we obtain

$$b_1 = -4. \quad (54)$$

Thus, substituting  $b_1 = -4$  into (50), we have:

$$u(x, t) = \frac{\frac{70}{39}(\sqrt{313} - 13)}{\exp(2\eta) - 4\exp(\eta) + 6 - 4\exp(-\eta) + \exp(-2\eta)} \\ = \left( -\frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \sec^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313} - 26} \left( x - \left( \frac{1}{2} + \frac{\sqrt{313}}{26} \right) t \right) \right] .$$

which is the same as Zuo's solution[39].

#### case 2.

$$a_2 = a_{-2} = 0, \quad a_0 = -\frac{105}{13}b_1^2 \frac{\sqrt{313} + 13}{72}, \quad (55) \\ b_0 = \frac{3}{8}b_1^2, \quad b_{-1} = \frac{1}{16}b_1^3, \quad b_{-2} = \frac{1}{256}b_1^4 \\ k = i \frac{1}{12}\sqrt{2\sqrt{313} + 26}, \\ \omega = -i \frac{1}{12}\sqrt{2\sqrt{313} + 26} \times \left( \frac{1}{2} - \frac{\sqrt{313}}{26} \right)$$

where  $b_1$  is a free parameter. Substituting (55) into (48) we obtain the following solution:

$$u(x, t) = \frac{-\frac{105}{13}b_1^2 \frac{\sqrt{313} + 13}{72}}{\exp(2\eta) + b_1 \exp(\eta) + \frac{3}{8}b_1^2 + \dots} \quad (56) \\ \dots + \frac{1}{16}b_1^3 \exp(-\eta) + \frac{1}{256}b_1^4 \exp(-2\eta) ,$$

where  $\eta = \frac{i}{12}\sqrt{2\sqrt{313} + 26} \left[ x - \left( \frac{1}{2} - \frac{\sqrt{313}}{26} \right) t \right]$ . If (45) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0 , \\ u(x, 0) = -\left( \frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \csc^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313} + 26} x \right] . \end{cases}$$

then, taking into account (50), we obtain

$$b_1 = 4 . \quad (57)$$

Thus, substituting  $b_1 = 4$  into (50), we have:

$$u(x, t) = \frac{-\frac{70}{39}(\sqrt{313} + 13)}{\exp(2\eta) + 4\exp(\eta) + 6 + 4\exp(-\eta) + \exp(-2\eta)} \\ = -\left( \frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \csc^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313} + 26} \left( x - \left( \frac{1}{2} - \frac{\sqrt{313}}{26} \right) t \right) \right] .$$

which is the exact solution given by Zuo in [39] obtained via sine-cosine method.

Also, if (45) be in the following form

$$\begin{cases} u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0 , \\ u(x, 0) = -\left( \frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \sec^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313} + 26} x \right] . \end{cases} \quad (58)$$

then taking into account (50), we obtain

$$b_1 = -4 . \quad (59)$$

Thus, substituting  $b_1 = -4$  into (50), we have:

$$u(x, t) = \frac{-\frac{70}{39}(\sqrt{313} + 13)}{\exp(2\eta) - 4\exp(\eta) + 6 - 4\exp(-\eta) + \exp(-2\eta)} \\ = -\left( \frac{35}{24} + \frac{35}{312}\sqrt{313} \right) * \dots \\ \dots * \sec^4 \left[ \frac{1}{24}\sqrt{2\sqrt{313} + 26} \left( x - \left( \frac{1}{2} - \frac{\sqrt{313}}{26} \right) t \right) \right] .$$

which is the same as Zuo's solution[39].

In the applications of Exp-function method in the pervious decade, common errors in finding the exact solutions of nonlinear problems have been omitted [34]. In this paper, we present an application of this method by tackling these common errors

- 1) In all the cases, we present the initial condition to obtain the special solution.
- 2) We substitute the obtained solutions into the equations to check if they satisfy the equations..
- 3) We simplify the solutions of differential equations.
- 4) We solve these problems for the case when the integration constant is zero.
- 5) We use the known general solutions of the ordinary differential equations.
- 6) We use the new method for solving Rosenau-Kawahara and Rosenau-KdV equations.

#### IV. CONCLUSIONS

In this paper, we considered two combined mathematical physics problems Rosenau-Kawahara and Rosenau-KdV for presenting an application of the Exp-function method. In the applications of Exp-function method in previous decade, common errors in finding the exact solutions of nonlinear problems have been omitted [34]. In this paper, we have presented an application of this method by tackling these common errors. Also, we showed that this method is easy, succinct and powerful to implement for nonlinear partial differential equations arising in mathematical physics.

#### REFERENCES

- [1] A.M. Wazwaz. The tanh method: exact solutions of the sine-Gordon and the sinh-Gordon equations. *Applied Mathematics and Computation*, 167:1196–1210, 2005.
- [2] A.M. Wazwaz. The variable separated ODE and the tanh methods for solving the combined and the double combined sinh-cosh-Gordon equations. *Applied Mathematics and Computation*, 177:745–754, 2006.



- [3] A.M. Wazwaz. Exact solutions for the generalized sine-Gordon and the generalized sinh-Gordon equations. *Chaos, Solitons and Fractals*, 28:127–135, 2006.
- [4] I. Hashim, M. S. M. Noorani, and M. R. S. Hadidi. Solving the generalized Burgers-Huxley equation using the Adomian decomposition method. *Mathematical and Computer Modelling*, 43:1404–1411, 2006.
- [5] M. Tatari, M. Dehghan, and M. Razzaghi. Application of the Adomian decomposition method for the Fokker-Planck equation. *Mathematical and Computer Modelling*, 45:639–650, 2007.
- [6] M.M. Rashidi, D.D. Ganji, and S. Dinarvand. Explicit analytical solutions of the generalized burger and burger-fisher equations by homotopy perturbation method. *Numerical Methods for Partial Differential Equations*, 25:409–417, 2009.
- [7] J. Biazar, F. Badpeimaa, and F. Azimi. Application of the homotopy perturbation method to Zakharov-Kuznetsov equations. *Computers and Mathematics with Applications*, 58:2391–2394, 2009.
- [8] M.E. Berberler and A. Yildirim. He's homotopy perturbation method for solving the shock wave equation. *Applicable Analysis*, 88:997–1004, 2009.
- [9] F. Shakeri and M. Dehghan. Numerical solution of the Klein-Gordon equation via He's variational iteration method. *Nonlinear Dynamics*, 51:89–97, 2008.
- [10] A.A. Soliman and M.A. Abdou. Numerical solutions of nonlinear evolution equations using variational iteration method. *Journal of Computational and Applied Mathematics*, 207:111–120, 2007.
- [11] E.Yusufoglu and A. Bekir. The variational iteration method for solitary patterns solutions of gBBM equation. *Physics Letters A*, 367:461–464, 2007.
- [12] K. Parand and A. Taghavi. Rational scaled generalized Laguerre function collocation method for solving the Blasius equation. *Journal of Computational and Applied Mathematics*, 233:980–989, 2009.
- [13] K. Parand, M. Dehghan, and A. Pirkhedri. Sinc-collocation method for solving the Blasius equation. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 373:4060–4065, 2009.
- [14] K. Parand, M. Dehghan, A. R. Rezaei, and S. M. Ghaderi. An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method. *Comput Phys Commun*, 181:1096–1108, 2010.
- [15] F. Tascan and A. Bekir. Analytic solutions of the  $(2 + 1)$ -dimensional nonlinear evolution equations using the sine-cosine method. *Applied Mathematics and Computation*, 215:3134–3139, 2009.
- [16] A. M. Wazwaz. New solitary wave solutions to the modified Kawahara equation. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 360:588–592, 2007.
- [17] J. H. He and X. H. Wu. Exp-function method for nonlinear wave equations. *Chaos, Solitons and Fractals*, 30:700–708, 2006.
- [18] J.H. He. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering. *International journal of Modern Physics B*, 22:3487–3578, 2008.
- [19] S. Zhang. Application of Exp-function method to a KdV equation with variable coefficients. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 365:448–453, 2007.
- [20] S. Zhang. Exp-function method for solving Maccari's system. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 371:65–71, 2007.
- [21] M. A. Abdou, A. A. Soliman, and S. T. El-Basyony. New application of Exp-function method for improved Boussinesq equation. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 369:469–475, 2007.
- [22] A. Ebaid. Exact solitary wave solutions for some nonlinear evolution equations via Exp-function method. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 365:213–219, 2007.
- [23] J. Biazar and Z. Ayati. Extension of the Exp-function method for systems of two-dimensional Burger's equations. *Computers and Mathematics with Applications*, 58:2103–2106, 2009.
- [24] A.E.H. Ebaid. Generalization of He's Exp-Function method and new exact solutions for burgers equation. *Zeitschrift fur Naturforschung - Section A Journal of Physical Sciences*, 64:604–608, 2009.
- [25] G. Domairry, A. G. Davodi, and A. G. Davodi. Solutions for the Double Sine-Gordon equation by Exp-function, Tanh, and extended Tanh methods. *Numerical Methods for Partial Differential Equations*, 26:384–398, 2010.
- [26] J. H. He and M. A. Abdou. New periodic solutions for nonlinear evolution equations using Exp-function method. *Chaos, Solitons and Fractals*, 34:1421–1429, 2007.
- [27] F. Khani, S. Hamed-Nezhad, and A. Molabahrani. A reliable treatment for nonlinear schrödinger equations. *Physics Letters A*, 371:234–240, 2007.
- [28] J. H. He and L. Zhang. Generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations using the Exp-function method. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 372:1044–1047, 2008.
- [29] B.C. Shin, M.T. Darvishi, and A. Barati. Some exact and new solutions of the Nizhnik-Novikov-Vesselov equation using the Exp-function method. *Computers and Mathematics with Applications*, 58:2147–2151, 2009.
- [30] A. Bekir and A. Boz. Exact solutions for a class of nonlinear partial differential equations using exp-function method. *International Journal of Nonlinear Sciences and Numerical Simulation*, 8:505–512, 2007.
- [31] K. Parand and J. A. Rad. Exp-function method for some nonlinear PDE's and a nonlinear ODE's. *Journal of King Saud University-Science*, 24:1–10, 2012.
- [32] K. Parand and J. A. Rad. Some solitary wave solutions of generalized Pochhammer-Chree equation via Exp-function method. *International Journal of Computational and Mathematical Sciences*, 4:142–147, 2010.
- [33] J. A. Rad and K. Parand. Analytical solution of Gas Flow Through a Micro-Nano Porous Media by Homotopy Perturbation method. *International Journal of Mathematical and Computer Sciences*, 6:184–188, 2010.
- [34] N. A. Kudryashov. Seven common errors in finding exact solutions of nonlinear differential equations. *Commun Nonlinear Sci Numer Simulat*, 14:3507–3529, 2009.
- [35] P. Rosenau. A quasi-continuous description of a nonlinear transmission line. *Physica Scripta*, 34:827–829, 1986.
- [36] P. Rosenau. Dynamics of dense discrete systems. *Progress of Theoretical Physics*, 79:1028–1042, 1988.
- [37] T. Bridges and G. Derks. Linear instability of solitary wave solutions of the Kawahara equation and its generalizations. *SIAM J. Math. Anal.*, 33:1356–1378, 2002.
- [38] J. H. He and X. H. Wu. Exp-function method and its application to nonlinear equations. *Chaos, Solitons and Fractals*, 38:903–910, 2008.
- [39] J. M. Zuo. Solitons and periodic solutions for the Rosenau-KdV and Rosenau-Kawahara equations. *Applied Mathematics and Computation*, 215:835–840, 2009.