International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:9, 2014

Fuzzy Multi-Component DEA with Shared and Undesirable Fuzzy Resources

Jolly Puri, Shiv Prasad Yadav

Abstract**—**Multi-component data envelopment analysis (MC-DEA) is a popular technique for measuring aggregate performance of the decision making units (DMUs) along with their components. However, the conventional MC-DEA is limited to crisp input and output data which may not always be available in exact form. In real life problems, data may be imprecise or fuzzy. Therefore, in this paper, we propose (i) a fuzzy MC-DEA (FMC-DEA) model in which shared and undesirable fuzzy resources are incorporated, (ii) the proposed FMC-DEA model is transformed into a pair of crisp models using $\alpha -$ cut approach, (iii) fuzzy aggregate performance of a DMU and fuzzy efficiencies of components are defined to be fuzzy numbers, and (iv) a numerical example is illustrated to validate the proposed approach.

Keywords**—**Multi-component DEA, fuzzy multi-component DEA, fuzzy resources.

I. Introduction

THE data envelopment analysis (DEA) is a non-parametric technique for evaluating the relative efficiencies of decision making units (DMUs) with multiple inputs and outputs [1]. It has been applied to wide range of organizations such as banks, hospitals, schools, etc. However, in many real life instances, DMUs can be separated into different components, also known as decision making sub-units (DMSUs). A DMU with such structure is known as multi-component DMU. The study of the aggregate performance of multi-component DMUs along with their components is known as multi-component DEA (MC-DEA) [2]-[4]. The standard DEA and MC-DEA models are typically based on the assumption that inputs have to be minimized and outputs have to be maximized. However, undesirable and shared resources can also be present in the production process which needs to be included while measuring aggregate and component-wise performances. Thus, in this study, both shared and undesirable resources are incorporated into the production process of MC-DEA.

The conventional MC-DEA is limited to crisp input and output data which may not always be available in exact form. In real life applications, data might be available in fuzzy or imprecise form. Therefore, in such situations, fuzzy MC-DEA (FMC-DEA) approach is more preferable as compared to traditional MC-DEA. In this paper, we extend traditional MC-DEA to FMC-DEA and propose FMC-DEA model. In order to evaluate fuzzy performance of DMUs along with their DMSUs in fuzzy environment, we use $\alpha -$ cut approach to solve FMC-DEA model. Further, proposed methodology is illustrated with a numerical example.

The paper is organized as follows: Section II presents an overview of DEA and MC-DEA with shared and undesirable resources. Section III presents the proposed FMC-DEA model with shared and undesirable fuzzy resources followed by the methodology to solve it. A numerical illustration is presented in Section IV. Section V concludes the findings of our study.

J. Puri is with the Indian Institute of Technology Roorkee, Roorkee-247667, India (corresponding author, Tel: +91-7500766529; e-mail: puri.jolly@ gmail.com).

S. P. Yadav is with the Indian Institute of Technology Roorkee, Roorkee-247667, India (e-mail: spyorfma@gmail.com).

II. DEA and Multi-Component DEA

A. DEA (Data Envelopment Analysis)

To describe DEA efficiency evaluation, assume that the performance of a set of n homogeneous DMUs be measured. The performance of DMU$_k$ is characterized by a production process of m inputs x_{ik} ; i = 1, …, m to yield s_1 desirable outputs $y^g_{rk};\ r=1,2,\dots,s_1$ and s_2 undesirable outputs $y^b_{pk};\ p=1,2,\dots,s_2$. Assume that input-output data are positive. In DEA, the efficiency E_k of DMU$_k$ in the presence of undesirable outputs is defined as

$$E_k = \left(\sum_{r=1}^{s_1} u^g_{rk} y^g_{rk} - \sum_{p=1}^{s_2} u^b_{pk} y^b_{pk} \right) \Bigg/ \sum_{i=1}^{m} v_{ik} x_{ik}.$$

Then, the relative efficiency of DMU$_k$ is evaluated from the following mathematical model presented by Puri and Yadav [5]:

$$\textbf{Model - 1} \qquad \text{Max } E_k$$
$$\text{subject to} \quad 0 \le E_j \le 1, \quad \forall\, j=1,\dots,n,$$
$$u^g_{rk} \ge \varepsilon \ \ \forall\, r,\ u^b_{pk} \ge \varepsilon \ \ \forall\, p,\ v_{ik} \ge \varepsilon \quad \forall\, i,$$

where v_{ik}, u^g_{rk} and u^b_{pk} are the weights corresponding to the i^{th} input, r^{th} desirable output and p^{th} undesirable output of DMU$_k$ respectively.

B. Multi-Component DEA with Shared and Undesirable Resources

Nomenclature: Let n: Number of DMUs, d: Number of DMSUs. For DMU$_k$ and $i=1,2,\dots,d$, let

- I^S : Number of shared inputs consumed by DMSU$_i$.
- I_i : Number of external inputs consumed by DMSU$_i$.
- K^g_i : Number of desirable outputs produced by DMSU$_i$.

- K_i^b : Number of undesirable outputs produced by DMSU$_i$.
- $X_k^S = (x_{1k}^S, x_{2k}^S, \dots, x_{I^S k}^S)^T$: Vector of shared inputs.
- $X_k^{(i)} = (x_{1k}^{(i)}, x_{2k}^{(i)}, \dots, x_{I_i k}^{(i)})^T$: Vector of external inputs consumed by DMSU$_i$.
- $Y_k^{g(i)} = (y_{1k}^{g(i)}, y_{2k}^{g(i)}, \dots, y_{K_i^g k}^{g(i)})^T$: Vector of desirable outputs produced by DMSU$_i$.
- $Y_k^{b(i)} = (y_{1k}^{b(i)}, y_{2k}^{b(i)}, \dots, y_{K_i^b k}^{b(i)})^T$: Vector of undesirable outputs produced by DMSU$_i$.

Let $\alpha_{ik} = (\alpha_{ik}^1, \alpha_{ik}^2, \dots, \alpha_{ik}^{I^S})^T$ be vector for DMSU$_i$ such that each α_{ik}^t corresponds to t^{th} shared input for $t = 1, 2, \dots, I^S$. Let $\alpha_{ik}^t x_{tk}^S$ be the portion of t^{th} shared input consumed by DMSU$_i$ such that $\sum_{i=1}^d \alpha_{ik}^t = 1, \forall t$. The production process of a multi-component DMU$_k$ with shared and undesirable resources is depicted in Fig. 1.

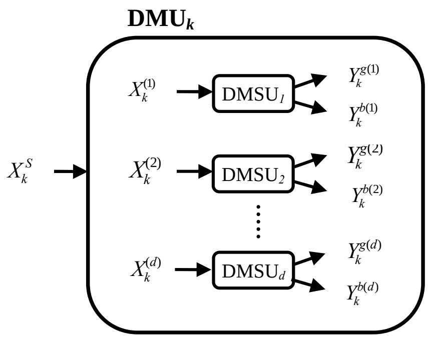

Fig. 1 Multi-component DMU in DEA

The aggregate performance $E_k^{(a)}$ of DMU$_k$ along with the component-wise efficiencies $E_k^{(i)}\,'s$ are given by

$$E_k^{(a)} = \frac{\sum_{i=1}^d U_k^{g(i)} Y_k^{g(i)} - \sum_{i=1}^d U_k^{b(i)} Y_k^{b(i)}}{\sum_{i=1}^d V_k^{(i)} X_k^{(i)} + \sum_{i=1}^d V_k^{S(i)} (\alpha_{ik} X_k^S)}$$

and

$$E_k^{(i)} = \frac{U_k^{g(i)} Y_k^{g(i)} - U_k^{b(i)} Y_k^{b(i)}}{V_k^{(i)} X_k^{(i)} + V_k^{S(i)} (\alpha_{ik} X_k^S)}, i = 1, 2, \dots, d.$$

where $U_k^{g(i)}, U_k^{b(i)}, V_k^{(i)}$ and $V_k^{S(i)}$ are the vectors and $\alpha_{ik} X_k^S = (\alpha_{ik}^1 x_{1k}^S, \alpha_{ik}^2 x_{2k}^S, \dots, \alpha_{ik}^{I^S} x_{I^S k}^S)^T$.

Theorem 1. $E_k^{(a)}$ is a convex combination of $E_k^{(i)} s$.

Proof $E_k^{(a)} = \dfrac{\sum_{i=1}^d U_k^{g(i)} Y_k^{g(i)} - \sum_{i=1}^d U_k^{b(i)} Y_k^{b(i)}}{\sum_{i=1}^d V_k^{(i)} X_k^{(i)} + \sum_{i=1}^d V_k^{S(i)} (\alpha_{ik} X_k^S)}$

Let $H = \sum_{i=1}^d V_k^{(i)} X_k^{(i)} + \sum_{i=1}^d V_k^{S(i)} (\alpha_{ik} X_k^S)$. Then

$$E_k^{(a)} = \frac{U_k^{g(1)} Y_k^{g(1)} - U_k^{b(1)} Y_k^{b(1)}}{H} + \frac{U_k^{g(2)} Y_k^{g(2)} - U_k^{b(2)} Y_k^{b(2)}}{H} + \dots + \frac{U_k^{g(d)} Y_k^{g(d)} - U_k^{b(d)} Y_k^{b(d)}}{H}$$

$$= \frac{U_k^{g(1)} Y_k^{g(1)} - U_k^{b(1)} Y_k^{b(1)}}{V_k^{(1)} X_k^{(1)} + V_k^{S(1)} \left(\alpha_{1k} X_k^S\right)} \times \frac{V_k^{(1)} X_k^{(1)} + V_k^{S(1)} \left(\alpha_{1k} X_k^S\right)}{H} + \frac{U_k^{g(2)} Y_k^{g(2)} - U_k^{b(2)} Y_k^{b(2)}}{V_k^{(2)} X_k^{(2)} + V_k^{S(2)} \left(\alpha_{2k} X_k^S\right)} \times \frac{V_k^{(2)} X_k^{(2)} + V_k^{S(2)} \left(\alpha_{2k} X_k^S\right)}{H} + \dots + \frac{U_k^{g(d)} Y_k^{g(d)} - U_k^{b(d)} Y_k^{b(d)}}{V_k^{(d)} X_k^{(d)} + V_k^{S(d)} \left(\alpha_{dk} X_k^S\right)} \times \frac{V_k^{(d)} X_k^{(d)} + V_k^{S(d)} \left(\alpha_{dk} X_k^S\right)}{H}$$

$$= E_k^{(1)} \times \frac{V_k^{(1)} X_k^{(1)} + V_k^{S(1)} \left(\alpha_{1k} X_k^S\right)}{H} + \dots + E_k^{(d)} \times \frac{V_k^{(d)} X_k^{(d)} + V_k^{S(d)} \left(\alpha_{dk} X_k^S\right)}{H}.$$

Let $\lambda_i = \dfrac{V_k^{(i)} X_k^{(i)} + V_k^{S(i)} \left(\alpha_{ik} X_k^S\right)}{H}, \ i = 1, 2, 3, \dots, d.$

Then $E_k^{(a)} = \lambda_1 \times E_k^{(1)} + \lambda_2 \times E_k^{(2)} + \dots + \lambda_d \times E_k^{(d)}$.

Hence, $E_k^{(a)} = \sum_{i=1}^d \lambda_i E_k^{(i)}$ and $\sum_{i=1}^d \lambda_i = 1$. (1)

completes the proof.

To derive $E_k^{(a)}, E_k^{(1)}, E_k^{(2)}, \dots, E_k^{(d)}$, we solve the following mathematical model:

Model - 2 Max $E_k^{(a)}$

subject to $0 \le E_j^{(a)} \le 1, \ \forall j = 1, 2, \dots, n,$

$0 \le E_j^{(i)} \le 1, \ \forall i = 1, 2, \dots, d; \ \forall j = 1, 2, \dots, n,$

$\sum_{i=1}^d \alpha_{ik}^t = 1, \ \forall t = 1, 2, \dots, I^S,$

$\left(U_k^{g(i)}, \ U_k^{b(i)}\right) \in \Omega_1, V_k^{(i)} \in \Omega_2, V_k^{S(i)} \in \Omega_3, \ \forall i = 1, 2, \dots, d;$

$\alpha_{ik} \in \Omega_4, \ \forall i = 1, 2, \dots, d.$

The sets Ω_1, Ω_2, Ω_3 and Ω_4 are assurance regions defined by any restrictions imposed on multipliers [6]. In Model-2, the constraints $E_j^{(a)} \ge 0 \ (j = 1, 2, \dots, n)$ are redundant and this can be shown by using (1) and the constraints $E_j^{(i)} \ge 0 \ (\forall i = 1, 2, \dots, d; \forall j = 1, 2, \dots, n)$. From (1), we have $E_k^{(a)} = \sum_{i=1}^d \lambda_i E_k^{(i)} \ge 0$ as $E_k^{(i)} \ge 0$ and $\lambda_i \ge 0, \ \forall i$. Therefore, Model-2 reduces to following model after removing the redundant constraints $E_j^{(a)} \ge 0 \ \forall j$:

Model - 3 Max $E_k^{(a)}$

subject to $E_j^{(a)} \le 1, \ \forall j = 1,2,\dots,n,$

$$0 \le E_j^{(i)} \le 1, \ \forall i = 1,2,\dots,d; \ \forall j = 1,2,\dots,n,$$

$$\sum_{i=1}^{d} \alpha_{ik}^{t} = 1, \ \forall t = 1,2,\dots,I^S,$$

$$\left(U_k^{g(i)},\ U_k^{b(i)}\right) \in \Omega_1,\ V_k^{(i)} \in \Omega_2,\ V_k^{S(i)} \in \Omega_3,\ \forall i = 1,2,\dots,d;$$

$$\alpha_{ik} \in \Omega_4,\ \forall i = 1,2,\dots,d.$$

Model–3 is a fractional model and can be reduced to the linear form using Charnes-Cooper transformation [1] and the variable substitution: $\alpha_{ik}^{t} v_{tk}^{S(i)} = \overline{v}_{tk}^{S(i)}, \ \forall t = 1,2,\dots,I^S; \forall i:$

Model - 4 Max $E_k^{(a)} = \sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rk}^{g(i)} - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pk}^{b(i)}$

subject to $\sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lk}^{(i)} + \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tk}^{S} = 1,$

$$\sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} - \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lj}^{(i)} -$$

$$\sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tj}^{S} \le 0, \ \forall j = 1,2,\dots,n,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} - \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lj}^{(i)} - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tj}^{S} \le 0,$$

$$\forall i = 1,2,\dots,d; \ \forall j = 1,2,\dots,n,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} \ge 0, \forall i = 1,2,\dots,d; \forall j = 1,2,\dots,n,$$

$$\sum_{i=1}^{d} \alpha_{ik}^{t} = 1, \ \forall t = 1,2,\dots,I^S,$$

$$\left(\left(u_{1k}^{g(i)}, u_{2k}^{g(i)},\dots,u_{K_i^g k}^{g(i)}\right),\left(u_{1k}^{b(i)}, u_{2k}^{b(i)},\dots,u_{K_i^b k}^{b(i)}\right)\right) \in \overline{\Omega}_1, \forall i = 1,2,\dots,d,$$

$$\left(v_{1k}^{(i)}, v_{2k}^{(i)},\dots,v_{I_i k}^{(i)}\right) \in \overline{\Omega}_2, \left(\overline{v}_{1k}^{S(i)}, \overline{v}_{2k}^{S(i)},\dots,\overline{v}_{I^S k}^{S(i)}\right) \in \overline{\Omega}_3, \forall i = 1,2,\dots,d,$$

$$\left(\alpha_{ik}^{1}, \alpha_{ik}^{2},\dots,\alpha_{ik}^{I^S}\right) \in \overline{\Omega}_4, \ \forall i = 1,2,\dots,d.$$

The form of $\overline{\Omega}_1$, $\overline{\Omega}_2$ and $\overline{\Omega}_4$ will depend upon the structure of Ω_1, Ω_2 and Ω_4 respectively. The form of $\overline{\Omega}_3$ depends upon how Ω_3 and Ω_4 are structured. The optimal objective function value of Model – 4 will give $E_k^{(a)}$ for DMU$_k$ and the optimal solution (weights) obtained from Model – 4 are used to evaluate $E_k^{(1)}, E_k^{(2)}, \dots, E_k^{(d)}$.

Theorem 2. A DMU$_k$ is said to be overall efficient if and only if each DMSU of DMU$_k$ is efficient. Equivalently $E_k^{(a)} = 1$ if and only if each $E_k^{(i)} = 1 \ \forall i = 1,2,\dots,d.$

III. Fuzzy Multi-Component DEA

A. Fuzzy MC-DEA with Shared and Undesirable Fuzzy Resources

The production process of a fuzzy multi-component DMU$_k$ is depicted in Fig. 2.

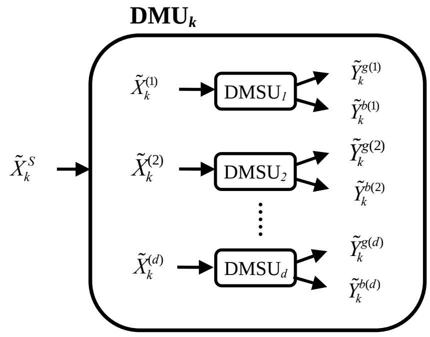

Fig. 2 Multi-component DMU in fuzzy DEA

Nomenclature: For DMU$_k$, let n: Number of DMUs, d: Number of DMSUs. For $i = 1,2,\dots,d$, let

- I^S : Number of shared fuzzy inputs used by DMSU$_i$.
- I_i : Number of external fuzzy inputs used by DMSU$_i$.
- K_i^g : Number of desirable fuzzy outputs produced by DMSU$_i$.
- K_i^b :Number of undesirable fuzzy outputs produced by DMSU$_i$.
- $\tilde{X}_k^S = (\tilde{x}_{1k}^S, \tilde{x}_{2k}^S, \dots, \tilde{x}_{I^S k}^S)^T$: Vector of shared fuzzy inputs.
- $\tilde{X}_k^{(i)} = (\tilde{x}_{1k}^{(i)}, \tilde{x}_{2k}^{(i)}, \dots, \tilde{x}_{I_i k}^{(i)})^T$: Vector of external fuzzy inputs consumed by DMSU$_i$.
- $\tilde{Y}_k^{g(i)} = (\tilde{y}_{1k}^{g(i)}, \tilde{y}_{2k}^{g(i)}, \dots, \tilde{y}_{K_i^g k}^{g(i)})^T$: Vector of desirable fuzzy outputs produced by DMSU$_i$.
- $\tilde{Y}_k^{b(i)} = (\tilde{y}_{1k}^{b(i)}, \tilde{y}_{2k}^{b(i)}, \dots, \tilde{y}_{K_i^b k}^{b(i)})^T$: Vector of undesirable fuzzy outputs produced by DMSU$_i$.

Let $\alpha_{ik} = (\alpha_{ik}^1, \alpha_{ik}^2, \dots, \alpha_{ik}^{I^S})^T$ be vector for DMSU$_i$. Let $\alpha_{ik}^t \tilde{x}_{tk}^S$ be the portion of t^{th} shared fuzzy input consumed by DMSU$_i$ such that $\sum_{i=1}^{d} \alpha_{ik}^t = 1, \forall t.$ Then, the fuzzy aggregate performance $\tilde{E}_k^{(a)}$ and component-wise fuzzy efficiencies $\tilde{E}_k^{(i)}\,'s$ are given by

$$\tilde{E}_k^{(a)} = \frac{\sum_{i=1}^{d} U_k^{g(i)}\, \tilde{Y}_k^{g(i)} - \sum_{i=1}^{d} U_k^{b(i)}\, \tilde{Y}_k^{b(i)}}{\sum_{i=1}^{d} V_k^{(i)}\, \tilde{X}_k^{(i)} + \sum_{i=1}^{d} V_k^{S(i)}\,(\alpha_{ik} \tilde{X}_k^S)} \text{ and}$$

$$\tilde{E}_k^{(i)} = \frac{U_k^{g(i)} \tilde{Y}_k^{g(i)} - U_k^{b(i)} \tilde{Y}_k^{b(i)}}{V_k^{(i)} \tilde{X}_k^{(i)} + V_k^{S(i)} (\alpha_{ik} \tilde{X}_k^S)}, i = 1,2,\dots,d.$$

where $U_k^{g(i)}, U_k^{b(i)}, V_k^{(i)}$ and $V_k^{S(i)}$ are the vectors and $\alpha_{ik}\tilde{X}_k^S = (\alpha_{ik}^1 \tilde{x}_{1k}^S, \alpha_{ik}^2 \tilde{x}_{2k}^S, \ldots, \alpha_{ik}^{I^S} \tilde{x}_{I^S k}^S)^T$. To derive $\tilde{E}_k^{(a)}, \tilde{E}_k^{(1)}, \tilde{E}_k^{(2)}, \ldots, \tilde{E}_k^{(d)}$, we solve the following mathematical model:

Model - 5 Max $\tilde{E}_k^{(a)} = \sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} \tilde{y}_{rk}^{g(i)} - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)} \tilde{y}_{pk}^{b(i)}$

subject to $\sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)} \tilde{x}_{lk}^{(i)} + \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} \tilde{x}_{tk}^S = 1,$

$$\sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} \tilde{y}_{rj}^{g(i)} - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)} \tilde{y}_{pj}^{b(i)} - \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)} \tilde{x}_{lj}^{(i)} - \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} \tilde{x}_{tj}^S \le 0, \ \forall j = 1,2,\ldots,n,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} \tilde{y}_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} \tilde{y}_{pj}^{b(i)} - \sum_{l=1}^{I_i} v_{lk}^{(i)} \tilde{x}_{lj}^{(i)} - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} \tilde{x}_{tj}^S \le 0, \quad \forall i = 1,2,\ldots,d;\ \forall j = 1,2,\ldots,n,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)} \tilde{y}_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} \tilde{y}_{pj}^{b(i)} \ge 0, \forall i = 1,2,\ldots,d; \forall j = 1,2,\ldots,n,$$

$$\sum\nolimits_{i=1}^{d} \alpha_{ik}^t = 1, \ \forall t = 1,2,\ldots,I^S,$$

$$\left(\left(u_{1k}^{g(i)}, u_{2k}^{g(i)}, \ldots, u_{K_i^g k}^{g(i)}\right), \left(u_{1k}^{b(i)}, u_{2k}^{b(i)}, \ldots, u_{K_i^b k}^{b(i)}\right)\right) \in \overline{\Omega}_1, \forall i = 1,2,\ldots,d,$$

$$\left(v_{1k}^{(i)}, v_{2k}^{(i)}, \ldots, v_{I_i k}^{(i)}\right) \in \overline{\Omega}_2, \left(\overline{v}_{1k}^{S(i)}, \overline{v}_{2k}^{S(i)}, \ldots, \overline{v}_{I^S k}^{S(i)}\right) \in \overline{\Omega}_3, \forall i = 1,2,\ldots,d,$$

$$\left(\alpha_{ik}^1, \alpha_{ik}^2, \ldots, \alpha_{ik}^{I^S}\right) \in \overline{\Omega}_4, \ \forall i = 1,2,\ldots,d.$$

Model – 5 is known as FMC-DEA model.

B. Methodology to Solve FMC-DEA Model

Model-5 is a fuzzy linear programming problem (FLPP) which can be solved using $\alpha -$cut approach given in [5], [7]-[9]. In this approach, a FLPP model is transformed to a pair of crisp LPP models by applying $\alpha -$cuts and extension principle. The procedure to convert Model-5 into crisp models is discussed below.

Let $S(\tilde{x}_{lk}^{(i)})\ \forall l = 1,2,\ldots,I_i; S(\tilde{x}_{tk}^S)\ \forall t = 1,2,\ldots,I^S;\ S(\tilde{y}_{rk}^{g(i)})\ \forall r = 1,2,\ldots,K_i^g$ and $S(\tilde{y}_{pk}^{b(i)})\ \forall p = 1,2,\ldots,K_i^b$ be the supports of I_i external fuzzy inputs, I^S shared fuzzy inputs, K_i^g desirable fuzzy outputs and K_i^b undesirable fuzzy outputs of DMSU$_i$ of DMU$_k$ respectively, given by

$$S(\tilde{x}_{lk}^{(i)}) = \{x_{lk}^{(i)} \mid \mu_{\tilde{x}_{lk}^{(i)}}(x_{lk}^{(i)}) > 0\}\ \forall l = 1,2,\ldots,I_i;\ S(\tilde{x}_{tk}^S) = \{x_{tk}^S \mid \mu_{\tilde{x}_{tk}^S}(x_{tk}^S) > 0\}$$

$$\forall t = 1,2,\ldots,I^S;\ S(\tilde{y}_{rk}^{g(i)}) = \{y_{rk}^{g(i)} \mid \mu_{\tilde{y}_{rk}^{g(i)}}(y_{rk}^{g(i)}) > 0\}\ \forall r = 1,2,\ldots,K_i^g$$

and

$$S(\tilde{y}_{pk}^{b(i)}) = \{y_{pk}^{b(i)} \mid \mu_{\tilde{y}_{pk}^{b(i)}}(y_{pk}^{b(i)}) > 0\}\ \forall p = 1,2,\ldots,K_i^b$$

The $\alpha -$cuts of $\tilde{x}_{lk}^{(i)}$, $\tilde{x}_{tk}^S$, $\tilde{y}_{rk}^{g(i)}$ and $\tilde{y}_{pk}^{b(i)}$ are respectively defined as

$$(\tilde{x}_{lk}^{(i)})_\alpha = \{x_{lk}^{(i)} \in S(\tilde{x}_{lk}^{(i)}) \mid \mu_{\tilde{x}_{lk}^{(i)}}(x_{lk}^{(i)}) \ge \alpha\} = [(x_{lk}^{(i)})_\alpha^L, (x_{lk}^{(i)})_\alpha^U] \quad \forall l = 1,2,\ldots,I_i, \forall i = 1,2,\ldots,d,\ \forall k = 1,2,\ldots,n \tag{2}$$

$$= [\min_{x_{lk}^{(i)}}\{x_{lk}^{(i)} \in S(\tilde{x}_{lk}^{(i)}) \mid \mu_{\tilde{x}_{lk}^{(i)}}(x_{lk}^{(i)}) \ge \alpha\}, \max_{x_{lk}^{(i)}}\{x_{lk}^{(i)} \in S(\tilde{x}_{lk}^{(i)}) \mid \mu_{\tilde{x}_{lk}^{(i)}}(x_{lk}^{(i)}) \ge \alpha\}] \quad \forall l,i,k$$

$$(\tilde{x}_{tk}^S)_\alpha = \{x_{tk}^S \in S(\tilde{x}_{tk}^S) \mid \mu_{\tilde{x}_{tk}^S}(x_{tk}^S) \ge \alpha\} = [(x_{tk}^S)_\alpha^L, (x_{tk}^S)_\alpha^U] \quad \forall t = 1,2,\ldots,I^S,\ \forall k = 1,2,\ldots,n \tag{3}$$

$$= [\min_{x_{tk}^S}\{x_{tk}^S \in S(\tilde{x}_{tk}^S) \mid \mu_{\tilde{x}_{tk}^S}(x_{tk}^S) \ge \alpha\}, \max_{x_{tk}^S}\{x_{tk}^S \in S(\tilde{x}_{tk}^S) \mid \mu_{\tilde{x}_{tk}^S}(x_{tk}^S) \ge \alpha\}] \quad \forall t,k$$

$$(\tilde{y}_{rk}^{g(i)})_\alpha = \{y_{rk}^{g(i)} \in S(\tilde{y}_{rk}^{g(i)}) \mid \mu_{\tilde{y}_{rk}^{g(i)}}(y_{rk}^{g(i)}) \ge \alpha\} = [(y_{rk}^{g(i)})_\alpha^L, (y_{rk}^{g(i)})_\alpha^U] \quad \forall r = 1,2,\ldots,K_i^g, \forall i = 1,2,\ldots,d,\ \forall k = 1,2,\ldots,n \tag{4}$$

$$= [\min_{y_{rk}^{g(i)}}\{y_{rk}^{g(i)} \in S(\tilde{y}_{rk}^{g(i)}) \mid \mu_{\tilde{y}_{rk}^{g(i)}}(y_{rk}^{g(i)}) \ge \alpha\}, \max_{y_{rk}^{g(i)}}\{y_{rk}^{g(i)} \in S(\tilde{y}_{rk}^{g(i)}) \mid \mu_{\tilde{y}_{rk}^{g(i)}}(y_{rk}^{g(i)}) \ge \alpha\}] \quad \forall r,i,k$$

$$(\tilde{y}_{pk}^{b(i)})_\alpha = \{y_{pk}^{b(i)} \in S(\tilde{y}_{pk}^{b(i)}) \mid \mu_{\tilde{y}_{pk}^{b(i)}}(y_{pk}^{b(i)}) \ge \alpha\} = [(y_{pk}^{b(i)})_\alpha^L, (y_{pk}^{b(i)})_\alpha^U] \quad \forall p = 1,2,\ldots,K_i^b, \forall i = 1,2,\ldots,d,\ \forall k = 1,2,\ldots,n \tag{5}$$

$$= [\min_{y_{pk}^{b(i)}}\{y_{pk}^{b(i)} \in S(\tilde{y}_{pk}^{b(i)}) \mid \mu_{\tilde{y}_{pk}^{b(i)}}(y_{pk}^{b(i)}) \ge \alpha\}, \max_{y_{pk}^{b(i)}}\{y_{pk}^{b(i)} \in S(\tilde{y}_{pk}^{b(i)}) \mid \mu_{\tilde{y}_{pk}^{b(i)}}(y_{pk}^{b(i)}) \ge \alpha\}] \quad \forall p,i,k$$

where $\alpha \in (0,1]$. Further, FMC-DEA model can easily be transformed into a pair of crisp models by using $\alpha -$cuts given in (2), (3), (4) and (5). Since the input-output data are in terms of fuzzy numbers, $\tilde{E}_k^{(a)}$ should also be a fuzzy number. Let $\tilde{E}_k^{(a)}$ be a fuzzy number with membership function $\mu_{\tilde{E}_k^{(a)}}$. Let $S(\tilde{E}_k^{(a)})$ be the support of $\tilde{E}_k^{(a)}$ given by $S(\tilde{E}_k^{(a)}) = \{E_k^{(a)} \mid \mu_{\tilde{E}_k^{(a)}}(E_k^{(a)}) > 0\}$. The $\alpha -$cut of $\tilde{E}_k^{(a)}$ is denoted by $(\tilde{E}_k^{(a)})_\alpha$, $\forall \alpha \in (0,1]$ and is defined as

$$(\tilde{E}_k^{(a)})_\alpha = \{E_k^{(a)} \in S(\tilde{E}_k^{(a)}) \mid \mu_{\tilde{E}_k^{(a)}}(E_k^{(a)}) \ge \alpha\} = [(E_k^{(a)})_\alpha^L, (E_k^{(a)})_\alpha^U]\ \forall k$$
$$= [\min_{E_k^{(a)}}\{E_k^{(a)} \in S(\tilde{E}_k^{(a)}) \mid \mu_{\tilde{E}_k^{(a)}}(E_k^{(a)}) \ge \alpha\}, \max_{E_k^{(a)}}\{E_k^{(a)} \in S(\tilde{E}_k^{(a)}) \mid \mu_{\tilde{E}_k^{(a)}}(E_k^{(a)}) \ge \alpha\}]\ \forall k$$

where $(E_k^{(a)})_\alpha^L$ and $(E_k^{(a)})_\alpha^U$ are given by the Models-6a and 6b respectively.

Further, for finding '***minimum aggregate efficiency***' of a targeted DMU, we use (i) lower bound desirable outputs for each DMSU$_i$ of the targeted DMU and upper bound desirable outputs for each DMSU$_i$ of the other DMUs, (ii) upper bound

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:9, 2014

undesirable outputs for each DMSU$_i$ of the targeted DMU and lower bound undesirable outputs for each DMSU$_i$ of the other DMUs, (iii) upper bound inputs (external and shared) for each DMSU$_i$ of the targeted DMU and lower bound inputs (external and shared) for each DMSU$_i$ of the other DMUs. For finding '***maximum aggregate efficiency***' of a targeted DMU, we use (i) upper bound desirable outputs for each DMSU$_i$ of the targeted DMU and lower bound desirable outputs for each DMSU$_i$ of the other DMUs, (ii) lower bound undesirable outputs for each DMSU$_i$ of the targeted DMU and upper bound undesirable outputs for each DMSU$_i$ of the other DMUs, (iii) lower bound inputs (external and shared) for each DMSU$_i$ of the targeted DMU and upper bound inputs (external and shared) for each DMSU$_i$ of the other DMUs.

Model - 6a

$$\left(E_k^{(a)}\right)_\alpha^L = \min_{\substack{\left(x_{lj}^{(i)}\right)_\alpha^L \le x_{lj}^{(i)} \le \left(x_{lj}^{(i)}\right)_\alpha^U \\ \left(x_{tj}^{S}\right)_\alpha^L \le x_{tj}^{S} \le \left(x_{tj}^{S}\right)_\alpha^U \\ \left(y_{rj}^{g(i)}\right)_\alpha^L \le y_{rj}^{g(i)} \le \left(y_{rj}^{g(i)}\right)_\alpha^U \\ \left(y_{pj}^{b(i)}\right)_\alpha^L \le y_{pj}^{b(i)} \le \left(y_{pj}^{b(i)}\right)_\alpha^U \\ \forall l,t,r,p,i,j}} \left\{ \begin{array}{l} \text{Max } E_k^{(a)} = \sum_{i=1}^{d} \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rk}^{g(i)} - \sum_{i=1}^{d} \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pk}^{b(i)} \\ \text{subject to } \sum_{i=1}^{d} \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lk}^{(i)} + \sum_{i=1}^{d} \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tk}^{S} = 1, \\ \sum_{i=1}^{d} \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{i=1}^{d} \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} - \sum_{i=1}^{d} \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lj}^{(i)} - \sum_{i=1}^{d} \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tj}^{S} \le 0, \forall j \\ \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} - \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lj}^{(i)} - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tj}^{S} \le 0, \forall i, j, \\ \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} \ge 0, \forall i, j, \\ \sum_{i=1}^{d} \alpha_{ik}^{t} = 1, \ \forall t = 1, \dots, I^S, \\ \left(\left(u_{1k}^{g(i)}, \dots, u_{K_i^g k}^{g(i)}\right), \left(u_{1k}^{b(i)}, \dots, u_{K_i^b k}^{b(i)}\right)\right) \in \overline{\Omega}_1, \forall i, \left(v_{1k}^{(i)}, \dots, v_{I_i k}^{(i)}\right) \in \overline{\Omega}_2, \forall i, \\ \left(\overline{v}_{1k}^{S(i)}, \dots, \overline{v}_{I^S k}^{S(i)}\right) \in \hat{\Omega}_3, \left(\alpha_{ik}^{1}, \alpha_{ik}^{2}, \dots, \alpha_{ik}^{I^S}\right) \in \overline{\Omega}_4, \ \forall i. \end{array} \right\}$$

and

Model - 6b

$$\left(E_k^{(a)}\right)_\alpha^U = \max_{\substack{\left(x_{lj}^{(i)}\right)_\alpha^L \le x_{lj}^{(i)} \le \left(x_{lj}^{(i)}\right)_\alpha^U \\ \left(x_{tj}^{S}\right)_\alpha^L \le x_{tj}^{S} \le \left(x_{tj}^{S}\right)_\alpha^U \\ \left(y_{rj}^{g(i)}\right)_\alpha^L \le y_{rj}^{g(i)} \le \left(y_{rj}^{g(i)}\right)_\alpha^U \\ \left(y_{pj}^{b(i)}\right)_\alpha^L \le y_{pj}^{b(i)} \le \left(y_{pj}^{b(i)}\right)_\alpha^U \\ \forall l,t,r,p,i,j}} \left\{ \begin{array}{l} \text{Max } E_k^{(a)} = \sum_{i=1}^{d} \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rk}^{g(i)} - \sum_{i=1}^{d} \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pk}^{b(i)} \\ \text{subject to } \sum_{i=1}^{d} \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lk}^{(i)} + \sum_{i=1}^{d} \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tk}^{S} = 1, \\ \sum_{i=1}^{d} \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{i=1}^{d} \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} - \sum_{i=1}^{d} \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lj}^{(i)} - \sum_{i=1}^{d} \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tj}^{S} \le 0, \forall j \\ \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} - \sum_{l=1}^{I_i} v_{lk}^{(i)} x_{lj}^{(i)} - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)} x_{tj}^{S} \le 0, \forall i, j, \\ \sum_{r=1}^{K_i^g} u_{rk}^{g(i)} y_{rj}^{g(i)} - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)} y_{pj}^{b(i)} \ge 0, \forall i, j, \\ \sum_{i=1}^{d} \alpha_{ik}^{t} = 1, \ \forall t = 1, \dots, I^S, \\ \left(\left(u_{1k}^{g(i)}, \dots, u_{K_i^g k}^{g(i)}\right), \left(u_{1k}^{b(i)}, \dots, u_{K_i^b k}^{b(i)}\right)\right) \in \overline{\Omega}_1, \forall i, \left(v_{1k}^{(i)}, \dots, v_{I_i k}^{(i)}\right) \in \overline{\Omega}_2, \forall i, \\ \left(\overline{v}_{1k}^{S(i)}, \dots, \overline{v}_{I^S k}^{S(i)}\right) \in \hat{\Omega}_3, \left(\alpha_{ik}^{1}, \alpha_{ik}^{2}, \dots, \alpha_{ik}^{I^S}\right) \in \overline{\Omega}_4, \ \forall i. \end{array} \right\}$$

Thus, Models-6a and 6b reduce to the following models:

Model - 7a

$$\left(E_k^{(a)}\right)_\alpha^L = \text{ Max } \sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^U$$

$$\text{subject to } \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lk}^{(i)})_\alpha^U + \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tk}^{S})_\alpha^U = 1,$$

$$\sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lk}^{(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tk}^{S})_\alpha^U \le 0,$$

$$\sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rj}^{g(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pj}^{b(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lj}^{(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tj}^{S})_\alpha^L \le 0, \ \forall j, \ j \ne k,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^L - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^U - \sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lk}^{(i)})_\alpha^U - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tk}^{S})_\alpha^U \le 0, \ \forall i,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rj}^{g(i)})_\alpha^U - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pj}^{b(i)})_\alpha^L - \sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lj}^{(i)})_\alpha^L - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tj}^{S})_\alpha^L \le 0, \ \forall i, j, \ j \ne k,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^L - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^U \ge 0, \ \forall i,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rj}^{g(i)})_\alpha^U - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pj}^{b(i)})_\alpha^L \ge 0, \ \forall i, j, j \ne k,$$

$$\sum\nolimits_{i=1}^{d} \alpha_{ik}^{t} = 1, \ \forall t,$$

$$\left(\left(u_{1k}^{g(i)}, u_{2k}^{g(i)}, \ldots, u_{K_i^g k}^{g(i)}\right), \left(u_{1k}^{b(i)}, u_{2k}^{b(i)}, \ldots, u_{K_i^b k}^{b(i)}\right)\right) \in \overline{\Omega}_1, \forall i,$$

$$\left(v_{1k}^{(i)}, v_{2k}^{(i)}, \ldots, v_{I_i k}^{(i)}\right) \in \overline{\Omega}_2, \left(\overline{v}_{1k}^{S(i)}, \overline{v}_{2k}^{S(i)}, \ldots, \overline{v}_{I^S k}^{S(i)}\right) \in \hat{\Omega}_3, \forall i,$$

$$\left(\alpha_{ik}^{1}, \alpha_{ik}^{2}, \ldots, \alpha_{ik}^{I^S}\right) \in \overline{\Omega}_4, \ \forall i,$$

and

Model - 7b

$$\left(E_k^{(a)}\right)_\alpha^U = \text{ Max } \sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^L$$

$$\text{subject to } \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lk}^{(i)})_\alpha^L + \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tk}^{S})_\alpha^L = 1,$$

$$\sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lk}^{(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tk}^{S})_\alpha^L \le 0,$$

$$\sum_{i=1}^{d}\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rj}^{g(i)})_\alpha^L - \sum_{i=1}^{d}\sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pj}^{b(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lj}^{(i)})_\alpha^U - \sum_{i=1}^{d}\sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tj}^{S})_\alpha^U \le 0, \ \forall j, \ j \ne k,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^U - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^L - \sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lk}^{(i)})_\alpha^L - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tk}^{S})_\alpha^L \le 0, \ \forall i,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rj}^{g(i)})_\alpha^L - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pj}^{b(i)})_\alpha^U - \sum_{l=1}^{I_i} v_{lk}^{(i)}(x_{lj}^{(i)})_\alpha^U - \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)}(x_{tj}^{S})_\alpha^U \le 0, \ \forall i, j, \ j \ne k,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rk}^{g(i)})_\alpha^U - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pk}^{b(i)})_\alpha^L \ge 0, \ \forall i,$$

$$\sum_{r=1}^{K_i^g} u_{rk}^{g(i)}(y_{rj}^{g(i)})_\alpha^L - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)}(y_{pj}^{b(i)})_\alpha^U \ge 0, \ \forall i, j, j \ne k,$$

$$\sum\nolimits_{i=1}^{d} \alpha_{ik}^{t} = 1, \ \forall t,$$

$$\left(\left(u_{1k}^{g(i)}, u_{2k}^{g(i)}, \ldots, u_{K_i^g k}^{g(i)}\right), \left(u_{1k}^{b(i)}, u_{2k}^{b(i)}, \ldots, u_{K_i^b k}^{b(i)}\right)\right) \in \overline{\Omega}_1, \forall i,$$

$$\left(v_{1k}^{(i)}, v_{2k}^{(i)}, \ldots, v_{I_i k}^{(i)}\right) \in \overline{\Omega}_2, \left(\overline{v}_{1k}^{S(i)}, \overline{v}_{2k}^{S(i)}, \ldots, \overline{v}_{I^S k}^{S(i)}\right) \in \hat{\Omega}_3, \forall i,$$

$$\left(\alpha_{ik}^{1}, \alpha_{ik}^{2}, \ldots, \alpha_{ik}^{I^S}\right) \in \overline{\Omega}_4, \ \forall i,$$

Theorem 3. If $(E_k^{(a)})_\alpha^{L*}$ and $(E_k^{(a)})_\alpha^{U*}$ are the optimum objective function values of Model-7a and Model-7b respectively at any $\alpha \in (0,1]$, then $(E_k^{(a)})_\alpha^{L*} \le (E_k^{(a)})_\alpha^{U*}$.

Definition 1. TFN [8] $\tilde{A} = (a_1, a_2, a_3)$ is defined by membership function $\mu_{\tilde{A}}$ given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 < x \le a_2, \\ \dfrac{x - a_3}{a_2 - a_3}, & a_2 \le x < a_3, \\ 0, & \textit{otherwise.} \end{cases}$$

Theorem 4. $(\tilde{E}_k^{(a)})_{\alpha_2} \subseteq (\tilde{E}_k^{(a)})_{\alpha_1}$ for any $\alpha_1, \alpha_2 \in (0,1]$ and $\alpha_1 \le \alpha_2$, i.e., $(E_k^{(a)})_{\alpha_2}^{U*} \le (E_k^{(a)})_{\alpha_1}^{U*}$ and $(E_k^{(a)})_{\alpha_1}^{L*} \le (E_k^{(a)})_{\alpha_2}^{L*}$ where $(E_k^{(a)})_{\alpha_1}^{U*}$ and $(E_k^{(a)})_{\alpha_2}^{U*}$ are the optimum objective function values of Model-7b at α_1 and α_2 respectively, and $(E_k^{(a)})_{\alpha_1}^{L*}$ and $(E_k^{(a)})_{\alpha_2}^{L*}$ are the optimum objective function values of Model-7a at α_1 and α_2 respectively.

The sets of intervals $\{[(E_k^{(a)})_\alpha^L, (E_k^{(a)})_\alpha^U] \mid \alpha \in (0,1]\}$ reveal the shape of $\mu_{\tilde{E}_k^{(a)}}$, although the exact form of membership function is not known explicitly [8]. If fuzzy inputs and outputs are taken as TFNs, $\mu_{\tilde{E}_k^{(a)}}$ can be approximated by the triangular membership function. Thus, the fuzzy efficiency $\tilde{E}_k^{(a)}$ can be defined as:

Definition 2. The fuzzy aggregate efficiency $\tilde{E}_k^{(a)}$ of DMU$_k$ is defined by its $\alpha-$cut $(\tilde{E}_k^{(a)})_\alpha = [(E_k^{(a)})_\alpha^L, (E_k^{(a)})_\alpha^U]$, $\alpha \in (0,1]$, where $(E_k^{(a)})_\alpha^L$ and $(E_k^{(a)})_\alpha^U$ are obtained from the optimal values of Model-7a and 7b respectively.

Definition 3. The fuzzy efficiency $\tilde{E}_k^{(i)}$ of DMSU$_i$ for DMU$_k$ is defined by its $\alpha-$cut $(\tilde{E}_k^{(i)})_\alpha = [(E_k^{(i)})_\alpha^L, (E_k^{(i)})_\alpha^U]$, $\alpha \in (0,1]$, where $(E_k^{(i)})_\alpha^L$ and $(E_k^{(i)})_\alpha^U$ are obtained from the optimal solutions of Model-7a and 7b respectively and are given by

$$(E_k^{(i)})_\alpha^L = \frac{\sum_{r=1}^{K_i^g} u_{rk}^{g(i)*} (y_{rk}^{g(i)})_\alpha^L - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)*} (y_{pk}^{b(i)})_\alpha^U}{\sum_{l=1}^{I_i} v_{lk}^{(i)*} (x_{lk}^{(i)})_\alpha^U + \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)*} (x_{tk}^S)_\alpha^U} \quad \text{and}$$

$$(E_k^{(i)})_\alpha^U = \frac{\sum_{r=1}^{K_i^g} u_{rk}^{g(i)\prime} (y_{rk}^{g(i)})_\alpha^U - \sum_{p=1}^{K_i^b} u_{pk}^{b(i)\prime} (y_{pk}^{b(i)})_\alpha^L}{\sum_{l=1}^{I_i} v_{lk}^{(i)\prime} (x_{lk}^{(i)})_\alpha^L + \sum_{t=1}^{I^S} \overline{v}_{tk}^{S(i)\prime} (x_{tk}^S)_\alpha^L} \forall i$$

where $(u_{1k}^{g(i)*}, u_{2k}^{g(i)*}, \ldots, u_{rk}^{g(i)*}; u_{1k}^{b(i)*}, u_{2k}^{b(i)*}, \ldots, u_{pk}^{b(i)*}; v_{1k}^{(i)*}, v_{2k}^{(i)*}, \ldots, v_{lk}^{(i)*}; \overline{v}_{1k}^{S(i)*}, \overline{v}_{2k}^{S(i)*}, \ldots, \overline{v}_{tk}^{S(i)*})$ and $(u_{1k}^{g(i)\prime}, u_{2k}^{g(i)\prime}, \ldots, u_{rk}^{g(i)\prime}; u_{1k}^{b(i)\prime}, u_{2k}^{b(i)\prime}, \ldots, u_{pk}^{b(i)\prime}; v_{1k}^{(i)\prime}, v_{2k}^{(i)\prime}, \ldots, v_{lk}^{(i)\prime}; \overline{v}_{1k}^{S(i)\prime}, \overline{v}_{2k}^{S(i)\prime}, \ldots, \overline{v}_{tk}^{S(i)\prime})$ are the optimal solutions of Model-7a and 7b respectively.

Definition 4. A DMU$_k$ is said to be overall efficient in FMC-DEA if each DMSU$_i$ of DMU$_k$ is efficient. Equivalently, $(E_k^{(a)})_\alpha^{L*} = (E_k^{(a)})_\alpha^{U*} = 1 \; \forall \alpha \in (0,1]$ if each $(E_k^{(i)})_\alpha^{L*} = (E_k^{(i)})_\alpha^{U*} = 1 \; \forall i = 1,2,\ldots,d,\; \forall \alpha \in (0,1]$.

IV. Numerical Illustration

To ensure the validity of the proposed methodology, we provide a numerical illustration. Consider a multi-component assessment problem of four DMUs in fuzzy environment. Each DMU consists of two DMSUs. The DMSU$_1$ consumes two external and one shared fuzzy inputs to produce one desirable and one undesirable fuzzy outputs. The DMSU$_2$ consumes one external and one shared fuzzy inputs to produce two desirable and one undesirable fuzzy outputs. The data of DMSUs for external input, desirable and undesirable outputs are shown in Table I. The data for shared input is shown in Table II. The $\alpha-$cuts $(\tilde{E}_k^{(a)})_\alpha$ and $(\tilde{E}_k^{(i)})_\alpha$, $\alpha \in (0,1]$, $\forall i = 1,2$ of aggregate fuzzy performance and component's fuzzy efficiencies of the four DMUs are evaluated by executing MATLAB programs of Models-7a and 7b at different values of α. The results are shown in Table III.

TABLE I
EXTERNAL INPUT AND OUTPUT DATA FOR DMSU$_1$ AND DMSU$_2$

DMSU$_1$				
DMU	$\tilde{x}_{1k}^{(1)}$	$\tilde{x}_{2k}^{(1)}$	$\tilde{y}_{1k}^{g(1)}$	$\tilde{y}_{1k}^{b(1)}$
1	(240,252.5,255)	(19,21,23)	(1247,1252,1255)	(206,210,213)
2	(145,149.77,152)	(10,13,16)	(1045, 1049,1052)	(135,139,142)
3	(87,90.19,92)	(12,14,16)	(900, 902,904)	(145,148,150)
4	(63,67.18,69.5)	(13,17,19)	(669, 672,676)	(175,176,178)
DMSU$_2$				
DMU	$\tilde{x}_{1k}^{(2)}$	$\tilde{y}_{1k}^{g(2)}$	$\tilde{y}_{2k}^{g(2)}$	$\tilde{y}_{1k}^{b(2)}$
1	(247,250,253)	(20,23,26)	(2545,2550,2556)	(233,235,237)
2	(140,144,146)	(19,21,24)	(1433,1440,1444)	(482,485,488)
3	(59,62,65)	(50,53,55)	(620,625,629)	(588,590,593)
4	(397,400,406)	(9,12,15)	(404,410,417)	(246,250,252)

TABLE II
SHARED INPUT DATA FOR EACH DMU

DMU	$\tilde{x}_{1k}^{S}$
1	(2240, 2250, 2256)
2	(2134, 2143, 2149)
3	(1376, 1382, 1390)
4	(2387, 2398, 2406)

The graphical representations of fuzzy aggregate efficiencies $\tilde{E}_k^{(a)}$ are shown in Fig. 3 along with the fuzzy efficiencies $\tilde{E}_k^{(i)}$ of each DMSU. It indicates that $\tilde{E}_k^{(a)}$ and $\tilde{E}_k^{(i)}$ are fuzzy numbers. Also shape of the membership functions of $\tilde{E}_k^{(a)}$ and $\tilde{E}_k^{(i)}$ can be approximated as triangular membership functions. Carefully observing Fig. 3, we can see that only DMU$_3$ is overall efficient using Definition 4. The DMUs 1st, 2nd and 4th are said to be FMC-DEA inefficient DMUs. The DMU$_1$ is efficient in case of DMSU$_1$ but inefficient in DMSU$_2$. Thus, it is inefficient in terms of aggregate performance. Similar is the case with DMU$_2$.

V. Conclusion

In this paper, we have extended MC-DEA with shared and undesirable resources to fuzzy environments as the precise input and output data are not always available in real life applications. We have proposed the FMC-DEA model and methodology to transform it into the crisp models by using $\alpha-$cut approach. The proposed methodology can easily be implemented to real life problems. The obtained fuzzy aggregate efficiencies and the fuzzy efficiencies of each component provide additional information to the planners and policy makers which help them to deal with uncertainties in real life problems.

TABLE III

THE $\alpha-$ CUTS OF $\tilde{E}_k^{(a)}, \tilde{E}_k^{(1)}$ AND $\tilde{E}_k^{(2)}$ AT DIFFERENT VALUES OF $\alpha \in (0,1]$

	DMU	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
$(E_k^{(a)})_\alpha^L$	1	0.9740	0.9745	0.9750	0.9754	0.9759	0.9764	0.9769	0.9774	0.9779	0.9784	0.9789
	2	0.9458	0.9468	0.9479	0.9490	0.9501	0.9512	0.9523	0.9534	0.9545	0.9556	0.9568
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.6636	0.6690	0.6745	0.6801	0.6857	0.6913	0.6970	0.7027	0.7085	0.7143	0.7202
	DMU	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
$(E_k^{(a)})_\alpha^U$	1	1.0000	1.0000	1.0000	0.9946	0.9876	0.9852	0.9829	0.9808	0.9799	0.9794	0.9789
	2	0.9926	0.9875	0.9825	0.9775	0.9727	0.9678	0.9631	0.9611	0.9597	0.9582	0.9568
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.7594	0.7589	0.7585	0.7580	0.7575	0.7557	0.7484	0.7412	0.7341	0.7271	0.7202
	DMU	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
$(E_k^{(1)})_\alpha^L$	1	0.6428	0.6494	0.6560	0.6625	0.6690	0.6756	0.6821	0.6885	0.6950	0.7015	0.7079
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.8826	0.8894	0.8963	0.9032	0.9102	0.9173	0.9243	0.9315	0.9387	0.9459	0.9535
	DMU	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
$(E_k^{(1)})_\alpha^U$	1	1.0000	1.0000	1.0000	0.9929	0.9186	0.9019	0.8858	0.8445	0.7215	0.7147	0.7079
	2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000	1.0000	1.0000	0.9982	0.9890	0.9799	0.9710	0.9622	0.9535
	DMU	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
$(E_k^{(2)})_\alpha^L$	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	2	0.6906	0.6958	0.7011	0.7065	0.7119	0.7173	0.7229	0.7285	0.7341	0.7398	07456
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.0404	0.0409	0.0414	0.0419	0.0425	0.0430	0.0435	0.0450	0.0465	0.0490	0.0521
	DMU	$\alpha=0$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$
$(E_k^{(2)})_\alpha^U$	1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	2	0.9909	0.9847	0.9785	0.9724	0.9664	0.9604	0.9546	0.7687	0.7609	0.7532	0.7456
	3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.0602	0.0594	0.0585	0.0577	0.0569	0.0561	0.0553	0.0545	0.0537	0.0529	0.0521

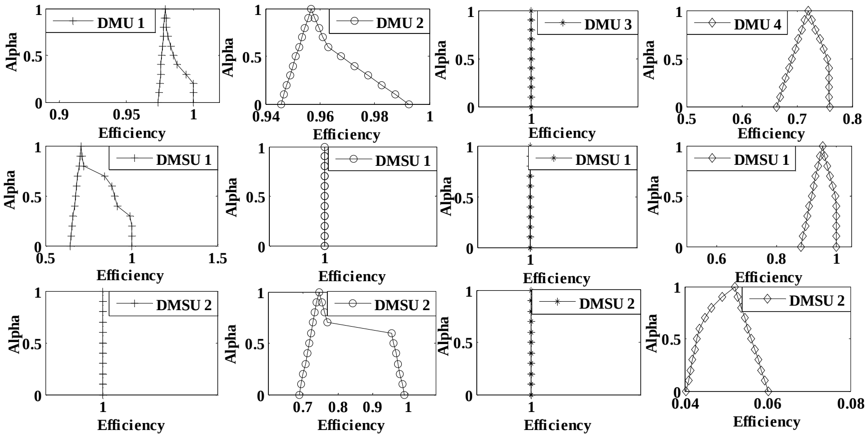

Fig. 3 Shape of membership functions of $\tilde{E}_k^{(a)}$ and $\tilde{E}_k^{(i)}$'s.

ACKNOWLEDGMENT

J. Puri is thankful to University Grants Commission (UGC), Government of India, New Delhi for financial support.

REFERENCES

[1] W.W. Cooper, L.M. Seiford, and K. Tone, *Data envelopment analysis: a comprehensive text with models, applications, references and DEA-solver software*. 2nd edn. New York: Springer, 2007.

[2] R. Fare, and S. Grosskopf, “Productivity and intermediate products: a frontier approach,” *Econ. Lett.*, vol. 50, pp. 65-70, 1996.

[3] W.W. Cook, M. Hababou, and H.J.H. Tuenter, “Multicomponent efficiency measurement and shared inputs in DEA: an application to sales and service performance in bank branches,” *J. Prod. Anal.*, vol. 14, pp. 209-224, 2000.

[4] G.R. Eslami, M. Mehralizadeh, and G.R. Jahanshahloo, “Efficiency measurement of multi-component decision making units using data envelopment analysis,” *Appl. Math. Sci.*, vol. 3, no. 52, pp. 2575-2594, 2009.

[5] J. Puri and S.P. Yadav, “A fuzzy DEA model with undesirable fuzzy outputs and its application to the banking sector in India ”, *Expert Syst. Appl.*, vol. 41, no. 14, pp. 6419-6432, 2014.

[6] R.G. Thompson, L.N. Langemeier, C.T. Lee, E. Lee, and R.M. Thrall, “The role of multiplier bounds in efficiency analysis with application to Kansas Framing,” *J. Econometrics*, vol. 46, pp. 91-108, 1990.

[7] C. Kao, and S.T. Liu, “Fuzzy efficiency measures in data envelopment analysis,” *Fuzzy Set. Syst.*, vol. 113, pp. 427–437, 2000.

[8] J. Puri and S.P. Yadav, “A concept of fuzzy input mix-efficiency in fuzzy DEA and its application in banking sector”, *Expert Syst. Appl.*, vol. 40, no. 5, pp. 1437-1450, 2013.

[9] A. hatami-Marbini, A. Emrouznejad, and M. Tavana, “A taxonomy and review of the fuzzy data envelopment analysis literature: Two decades in the making,” *Euro. J. Oper. Res.*, vol. 214, pp. 457-472, 2011.