

# Robust Stabilization against Unknown Consensus Network

Myung-Gon Yoon, Jung-Ho Moon, Tae Kwon Ha

**Abstract**—This paper studies a robust stabilization problem of a single agent in a multi-agent consensus system composed of identical agents, when the network topology of the system is completely unknown. It is shown that the transfer function of an agent in a consensus system can be described as a multiplicative perturbation of the isolated agent transfer function in frequency domain. From an existing robust stabilization result, we present sufficient conditions for a robust stabilization of an agent against unknown network topology.

**Keywords**—Multi-agent System, Robust Stabilization, Transfer Function.

## I. INTRODUCTION

**M**ULTI-AGENT consensus systems have attracted much attention in control community, due to the fact that many important physical systems appearing in diverse fields of science and engineering can be described as multi-agent systems and as a result control problems of those systems have significant importance.

One of the most popular topics in various control problems of multi-agent system is how to economically and effectively modify overall or collective behaviors of multi-agent systems by controlling a small number of agents, taking advantages of inter-agent network and certain communication protocols. For this challenging problem, the leader-follower approach [1]–[4], pinning control [5]–[9] and single agent control [10], [11] have been proposed in literature. Basically those approaches are motivated by some ideas about how to take advantage of inter-agent communication of multi-agent systems. In this sense, existing communication between agents of a multi-agent system provides a favorable condition from the viewpoint of a controller designer. This is particularly the case when multi-agent systems employ a consensus protocol with which every agent tries to follow the behaviours of its neighboring agents.

However when a controller designer has either little or wrong information about the network topology or the configuration of inter-agent communication, then the very existence of communication among agents turns into a hard

obstacle, making the task of a control synthesis for a particular single agent become much more difficult.

More specifically, suppose we are given a system to be controlled but have no reliable information on whether or not this system is a part of a multi-agent consensus system. Or we know it is an agent of a certain multi-agent system but we have no information on the topology of the network. A natural idea in this situation is to robustly stabilize the agent by regarding the changed dynamics of the agent due to its roles as an agent in a (unknown) network system as a sort of dynamic perturbation. This approach motivates a quantitative analysis on the perturbation caused by inter-agent communication.

In this paper we assume that unknown multi-agent systems employ a linear consensus protocol and that the dynamics of every agent in a given multi-agent system is the same as the agent that is to be controlled by an external controller.

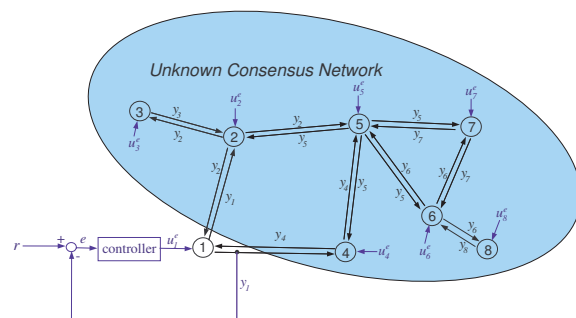


Fig. 1. Robust stabilization Against Unknown Consensus Network

For an illustration of our problem, let us consider a multi-agent system shown in Fig. 1. This multi-agent system consists of eight identical agents labelled  $1, \dots, 8$  and each agent has the same linear time-invariant dynamics characterized by a SISO (single-input single-output) transfer function  $g(s)$ .

Suppose that, without any knowledge on the network topology in Fig. 1, we choose a controller, call it  $c(s)$ , for agent 1 to achieve a stable closed loop system defined by  $\{g(s), c(s)\}$ . However existing consensus networking changes the dynamics of the agent 1 from the original dynamics  $g(s)$  to a perturbed one, call it  $\tilde{g}_1(s)$ . This paper will investigate the robust stability of the perturbed pair  $\{\tilde{g}_1(s), c(s)\}$  and provide sufficient conditions for the robust stability against unknown consensus networking.

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## II. MAIN RESULTS

### A. Multi-agent Consensus System

Consider a single-input single-output (SISO) time-invariant plant whose dynamics is given as a rational transfer function

$$g(s) = \frac{b(s)}{a(s)} \quad (1)$$

with two coprime polynomials  $a(s)$  and  $b(s)$ .

By regarding the plant  $g(s)$  to be controlled as a nominal plant and its correlation with other agents which are unknown to a controller designer, as a kind of dynamic perturbation, the stabilization problem of  $g(s)$  can be seen as a typical robust control problem. For an application of existing robust control technique, we firstly need an explicit description of the dynamics of agent when it is a member of a consensus system. Toward this, let us label agents in a consensus system with an index set  $\{1, 2, \dots, n\}$ , reserving the agent labelled 1 as the agent to which an external controller will be directly connected.

We suppose that the consensus system is equipped with a linear consensus protocol, that is, each agent  $i$  is subject to a consensus command  $\sum_{j \in N_i} (y_j - y_i)$  where  $y_i$  is the output of agent  $i$  and  $N_i \subset \{1, \dots, n\}$  denotes the set of agents which are directly connected to agent  $i$ . i.e., *neighbouring agents*. Then the overall dynamics of the multi-agent consensus system can be written as

$$\frac{a(s)}{b(s)} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = -L \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_1^e \quad (2)$$

where the *adjacency matrix*  $A$  and the *graph Laplacian*  $L$  are defined by

$$A := (a_{ij}) \in R^{n \times n}, \quad d_i := |N_i|, \\ D := \text{diag.}(d_i) \in R^{n \times n}, \quad L := D - A \quad (3)$$

and  $a_{ij}=1$  if agent  $i$  is directly connected to an agent  $j$  and  $a_{ij} = 0$  elsewhere. Note that the *degree*  $d_i$  denotes the number of connections that agent  $i$  has.

The multi-agent system (2) can be associated with a mathematical (undirected) graph where each agent represents a vertex and a bi-directional inter-agent connection is regarded as an edge.

A graph is called *connected* if every pair of vertices can be connected by a sequence of edges. In this paper we assume that graphs associated with multi-agent systems are connected. Moreover, for simplicity, we call a multi-agent system (2) connected if the corresponding graph is connected.

### B. Robust Stabilization

For an exogenous control input  $u_1^e = c(s)(r - y_1)$  with some controller  $c(s)$  and a measurement of agent output  $y_1$ ,

the transfer function  $\tilde{g}_1(s)$  from the input  $u_1^e$  to agent output  $y_1$  is given as

$$\tilde{g}_1(s) := e_1^T \left( \frac{1}{g(s)} I_{n \times n} + L \right)^{-1} e_1, \quad e_1 := \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (4)$$

Let  $\mu_1 > \mu_2 > \dots > \mu_m$  be the distinct eigenvalues of the Laplacian matrix  $L$ . Then the spectral factorization of  $L$  allows one to show that the transfer function  $\tilde{g}_1(s)$  has the following representation

$$\tilde{g}_1(s) = \frac{y_1(s)}{u_1^e(s)} = \sum_{k=1}^m \frac{b(s)}{a(s) + \mu_k b(s)} \alpha_k^2 \quad (5)$$

where

$$\alpha_k^2 := \langle e_1, P_k e_1 \rangle \quad (6)$$

and  $P_k$  denotes the orthogonal projection operator onto the  $\mu_k$ -eigenspace of the matrix  $L$ .

*Lemma 1:* For a connected graph, it holds that

$$\alpha_m^2 = \frac{1}{m} \quad \text{and} \quad \sum_{k=1}^m \alpha_k^2 = 1 \quad (7)$$

*Proof:* It is a standard fact in spectral graph theory that every connected graph has a smallest *simple* eigenvalue  $\mu_m = 0$  and a corresponding eigenvector  $v_m = [1 \ \dots \ 1] / \sqrt{m}$ . The first result comes from the fact  $P_m = v_m v_m^T$ . The second result is a consequence of the fact that  $P_k$  ( $k = 1, \dots, m$ ) is a projection operator. ■

Making use of the representation (5) with simple algebras, the agent transfer function  $\tilde{g}_1(s)$ , corresponding to an agent serving as a member of a certain consensus system, can be seen as a perturbation of the original (isolated) agent transfer function  $g(s)$  as follows;

$$\tilde{g}_1(s) = (1 + \Delta(s)) g(s) \quad (8)$$

where the multiplicative perturbation  $\Delta(s)$  is given by

$$\Delta(s) = \sum_{k=1}^m \frac{\mu_k g(s)}{1 + \mu_k g(s)} \alpha_k^2 \quad (9)$$

*Theorem 1:* Suppose that, for each  $k = 1, \dots, m-1$ , the transfer function  $\mu_k g / (1 + \mu_k g)$  is stable and the following condition holds

$$\left\| \frac{\mu_k g(s)}{1 + \mu_k g(s)} \right\|_{\infty} \leq 1. \quad (10)$$

If a controller  $c(s)$  stabilizing  $g(s)$ , satisfies the bound

$$\left\| \frac{c(s)g(s)}{1 + c(s)g(s)} \right\|_{\infty} \leq 1, \quad (11)$$

then the controller  $c(s)$  also stabilizes the agent transfer function  $\tilde{g}_1(s)$  of a consensus network.

*Proof:* As  $\mu_k g / (1 + \mu_k g)$  is stable, so is  $\Delta(s)$  in (9). Moreover, from (7), it is clear that  $\Delta(s)$  is a convex

combination of a stable transfer functions  $\{\mu_k g/(1+\mu_k g) | k = 1, \dots, m\}$ , and thus we have

$$\begin{aligned} \|\Delta\|_\infty &= \sup_{\omega \geq 0} |\Delta(j\omega)| = \sup_{\omega \geq 0} \left| \sum_{k=1}^m \frac{g(j\omega)\mu_k}{1+g(j\omega)\mu_k} \alpha_k^2 \right| \\ &\leq \sum_{k=1}^m \left\| \frac{g(s)\mu_k}{1+g(s)\mu_k} \right\|_\infty \alpha_k^2 \leq \sum_{k=1}^m \alpha_k^2 \leq \frac{m-1}{m} < 1 \end{aligned} \quad (12)$$

From the strict inequality  $\|\Delta\|_\infty < 1$  and the hypothesis (11), it easily follows from the small-gain theorem, see e.g. Theorem 9.7 of [12], that the closed loop system defined by  $\{c(s), \tilde{g}_1(s)\}$  is stable. ■

### C. An Application of Spectral Graph Theory

In the field of spectral graph theory, there are several known bounds on the largest Laplacian eigenvalue, e.g. see [13]. For example, the simplest one might be that the largest eigenvalue is less than the number of agents in a consensus system, that is,

$$\mu_1 \leq n. \quad (13)$$

Another known bound of  $\mu_1$  is given in terms of the degrees (number of neighbours) of agents as follows;

$$\mu_1 \leq \max_{i,j} \{d_i + d_j; i \in N_j\} \quad (14)$$

which says that the largest eigenvalue is bounded by the maximal sum of two degrees of adjacent (directly connected) agent pair.

Now, for a given agent dynamics  $g(s)$ , define

$$\gamma_g^* := \sup \left\{ \gamma \geq 0 \mid \frac{g\gamma}{1+g\gamma} \text{ is stable, } \left\| \frac{g\gamma}{1+g\gamma} \right\|_\infty \leq 1 \right\}. \quad (15)$$

Note that if the quantity  $\gamma_g^*$  is greater than the right hand side of either (13) or (14), then the condition (10) in Theorem 1 holds. As a result, in this case, the sole condition (11) which is independent of the network topology of a consensus system, is sufficient for the robust closed loop stability against consensus network.

For example, let us consider an integrator agent  $g(s) = 1/s$  which has widely appeared as a model of agent dynamics in many important multi-agent systems. In this case, we have a stable transfer function  $g\gamma/(1+g\gamma) = \gamma/(s+\gamma)$  for all positive real number  $\gamma \geq 0$  and the bound (10) is satisfied as  $\|g\gamma/(1+g\gamma)\|_\infty = 1$  holds. If we choose a proportional controller  $c(s) = k > 0$ , for example, then the condition (11) holds as we have

$$\left\| \frac{g(s)c(s)}{1+g(s)c(s)} \right\|_\infty = \left\| \frac{k}{s+k} \right\|_\infty = 1 \quad (16)$$

Hence, from Theorem 1, we conclude that a proportional controller can stabilizes an integrator agent, irrespective of a consensus system in which the controlled agent is serving as a member, provided that the graph describing the network topology is connected.

### III. A NUMERICAL EXAMPLE

Let us consider a second order system

$$g(s) = \frac{2}{s^2 + 4s + 2}. \quad (17)$$

For this (isolated) system the following stabilizing controller has been designed;

$$c(s) = \frac{10}{s+10}. \quad (18)$$

Now suppose that the system  $g(s)$  is not isolated but it serves as an agent of the consensus system in Fig. 1. Our question then is whether or not the stability of the closed loop system defined by the pair  $\{g(s), c(s)\}$  is preserved against the un-modelled consensus behavior of the controlled agent.

Toward an application of Theorem 1, firstly, we note that the second order system  $g(s)$  has an infinity gain margin and hence the transfer function  $g\gamma/(1+g\gamma)$  is always stable for  $\gamma \geq 0$ . In addition, numerical computation revealed that

$$\left\| \frac{g(s)\gamma}{1+g(s)\gamma} \right\|_\infty \leq 1 \quad (19)$$

for all  $\gamma \leq 7$  as shown in Fig. 2, i.e., we have  $\gamma_g^* = 7$ .

In addition, the Laplacian matrix corresponding to consensus network of Fig. 1 has eight distinct eigenvalues and the coefficients  $\{\alpha_k^2\}$  of transfer functions are given in Table I.

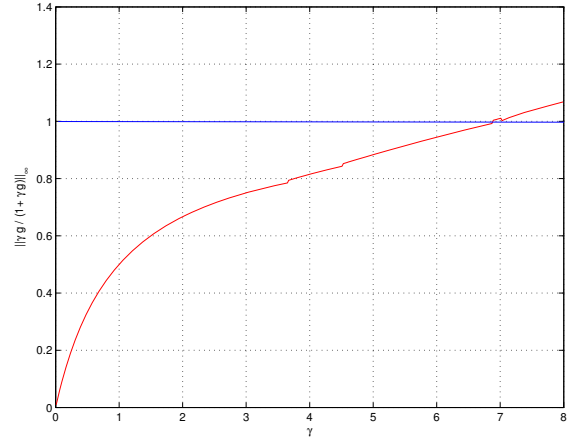


Fig. 2.  $H_\infty$  norm  $\left\| \frac{g(s)\gamma}{1+g(s)\gamma} \right\|_\infty$  versus  $\gamma \geq 0$

Now, since all conditions of Theorem 1 hold, we conclude that the closed loop stability is preserved even after the controlled agent becomes a member of the consensus network of Fig. 1.

Moreover, from the fact  $\gamma_g^* = 7$  and (13), it also follows that the closed loop stability is preserved not only for the particular network of Fig. 1, but also any arbitrary consensus system composed of less than six agents.

A numerical simulation of the step responses of the two closed loop systems: the nominal pair  $\{g(s), c(s)\}$  and the perturbed (because of a consensus network) pair  $\{\tilde{g}_1(s), c(s)\}$ , in Fig. 3 show the closed loop stability.

TABLE I  
TRANSFER FUNCTION PARAMETERS OF THE CONSENSUS SYSTEM OF FIG. 1

$k$	1	2	3	4	5	6	7	8
$\mu_k$	5.4740	3.8965	3.3682	2.5889	1.3485	0.8974	0.4265	0
$\alpha_k^2$	0.0417	0.0435	0.1715	0.2650	0.1315	0.1411	0.0806	0.1250

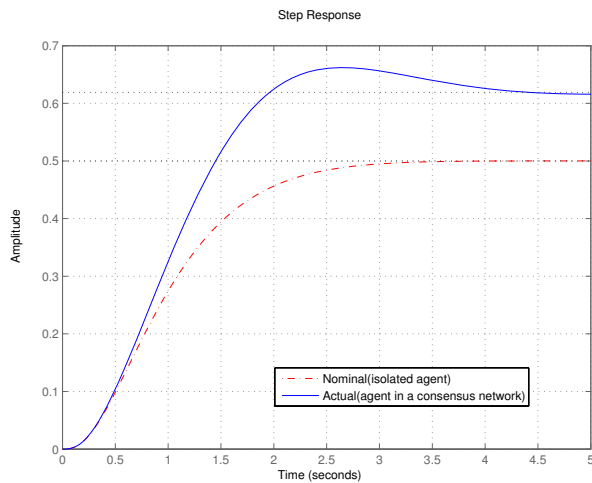


Fig. 3. Step responses of isolated and networked agent

#### IV. CONCLUSION

From an explicit representation of agent transfer function of a consensus multi-agent system, it was shown that the dynamic perturbation of an agent caused by consensus communication with other agents in a consensus system can be described as a multiplicative perturbation of an isolated original agent dynamics in frequency domain. This result, combined with existing theory on robust stability, allowed us to develop sufficient conditions for the robust stability of an agent serving as an agent in unknown consensus systems. From mathematical results in spectral graph theory, it was also found that the robust stability of an agent is closely related to the network topology of a consensus system such as the size and degree distribution of a graph corresponding to the consensus system.

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