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Exponentiated Transmuted Weibull Distribution A Generalization of the Weibull Distribution

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Abstract**—**This paper introduces a new generalization of the two parameter Weibull distribution. To this end, the quadratic rank transmutation map has been used. This new distribution is named exponentiated transmuted Weibull (ETW) distribution. The ETW distribution has the advantage of being capable of modeling various shapes of aging and failure criteria. Furthermore, eleven lifetime distributions such as the Weibull, exponentiated Weibull, Rayleigh and exponential distributions, among others follow as special cases. The properties of the new model are discussed and the maximum likelihood estimation is used to estimate the parameters. Explicit expressions are derived for the quantiles. The moments of the distribution are derived, and the order statistics are examined.

Keywords**—**Exponentiated, Inversion Method, Maximum Likelihood Estimation, Transmutation Map.

I. Introduction

FOR more than half a century the Weibull distribution has attracted the attention of statisticians working on theory and methods as well as in various fields of applied statistics. Thousands of papers have been written on this distribution.

It is of utmost interest to theory orientated statisticians because of its great number of special features and to practitioners because of its ability to fit to data from various fields, ranging from life data to weather data or observations made in economics and business administration, in hydrology, in biology or in the engineering sciences.

When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes.

However, the Weibull distribution does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates may be bathtub or unimodal shapes.

An interesting idea of generalizing a distribution, known in the literature by transmution, is derived by using the quadratic rank transmutation map [1].

Recently, various generalizations have been introduced based on the transmuted generalization included the transmuted extreme value distribution [2], transmuted Weibull distribution [3], transmuted modified Weibull distribution [4] and transmuted log-logistic distribution [5].

A random variable T is said to have a two parameter Weibull [6], [7] with parameters $\alpha > 0$ and $\beta > 0$ if its

$$g(t) = \frac{\beta\, t^{\beta-1}}{\alpha^{\beta}} e^{-\left(\frac{t}{\alpha}\right)^{\beta}} \ , t \geq 0 \ , \tag{1}$$

$$G(t) = 1 - e^{-\left(\frac{t}{\alpha}\right)^{\beta}}. \tag{2}$$

The rest of the article is organized as follows. In Section II, the pdf and cdf of the subject distribution and some special sub-models are derived. In Section III, the statistical properties including quantiles, moments and moment generating function etc. are studied. The reliability analyses of the subject model are given in Section IV. Section V presents the maximum likelihood estimates and the asymptotic confidence intervals of the unknown parameters. The order statistics from the distribution are discussed in Section VI.

II. Exponentiated Transmuted Weibull Distribution

A random variable T is said to have transmuted distribution if its cdf , $Z(t)$, is given by:

$$Z(t) = (1+\lambda)G(t) - \lambda\,[G(t)]^2, \tag{3}$$

where $G(t)$ is the cdf of the base distribution.

The cdf, $F(t)$, of the exponentiated transmuted distribution is given by:

$$F(t) = Z^{v}(t) = \{(1+\lambda)G(t) - \lambda\,[G(t)]^2\}^{v} \ , |\lambda| \leq 1. \tag{4}$$

Combining (2) and (4), gives the cdf of the ETW distribution as:

$$F(t) = \left\{1 + (\lambda - 1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v} , t \geq 0, \tag{5}$$

where $\alpha > 0$, $\beta > 0$ and $|\lambda| \leq 1$ are the scale, shape and transmuted parameters, respectively.

Differentiating (5) with respect to t, and doing the necessary simplifications, gives the pdf as:

$$f(t) = \frac{v\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}} \left[1 - \lambda + 2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \left\{1 + (\lambda - 1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v-1} \tag{6}$$

It is clear that the exponentiated transmuted Weibull (ETW) distribution is very flexible (as seen from Table I). This is so since several other distributions follow as special cases from the ETW, by selecting the appropriate values of the parameters. These special cases include eleven distributions (as shown in Fig. 1). Namely; the exponentiated transmuted Rayleigh (ETR), exponentiated transmuted exponential (ETE), transmuted Weibull (TW) [3], transmuted Rayleigh

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(TR) [5], transmuted exponential (TE) [1], exponentiated Weibull (EW) [8], exponentiated Rayleigh (ER) [9], exponentiated exponential (EW) [10], Weibull (W), Rayleigh (R), and the exponential (E).

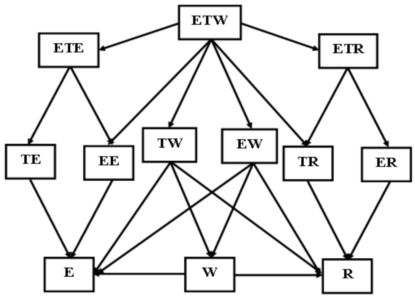

Fig. 1 Submodels of the ETW distribution

TABLE I
THE ETW DISTRIBUTION SUBMODELS

Submodels	Parameters of ETW				Cumulative distribution function
	α	β	λ	v	
ETR	-	2	-	-	$\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right\}^v$
ETE	-	1	-	-	$\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)}-\lambda e^{-2\left(\frac{t}{\alpha}\right)}\right\}^v$
TW	-	-	-	1	$1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^\beta}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^\beta}$
TR	-	2	-	1	$1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^2}$
TE	-	1	-	1	$1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)}-\lambda e^{-2\left(\frac{t}{\alpha}\right)}$
EW	-	-	0	-	$\left\{1-e^{-\left(\frac{t}{\alpha}\right)^\beta}\right\}^v$
ER	-	2	0	-	$\left\{1-e^{-\left(\frac{t}{\alpha}\right)^2}\right\}^v$
EE	-	1	0	-	$\left\{1-e^{-\left(\frac{t}{\alpha}\right)}\right\}^v$
W	-	-	0	1	$1-e^{-\left(\frac{t}{\alpha}\right)^\beta}$
R	-	2	0	1	$1-e^{-\left(\frac{t}{\alpha}\right)^2}$
E	-	1	0	1	$1-e^{-\left(\frac{t}{\alpha}\right)}$

Table I shows the specific values of the parameters used to generate the abovementioned eleven special cases.

Figs. 2 (a)-(d) illustrate some of the possible shapes of the pdf of the ETW distribution for different values of the parameters $\alpha > 0$, $\beta > 0$, $v > 0$ and $|\lambda| \leq 1$.

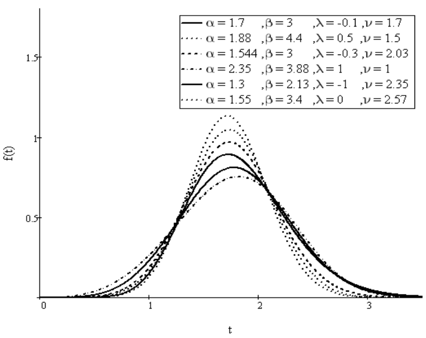

Fig. 2 (a) The behavior of the pdf of ETW

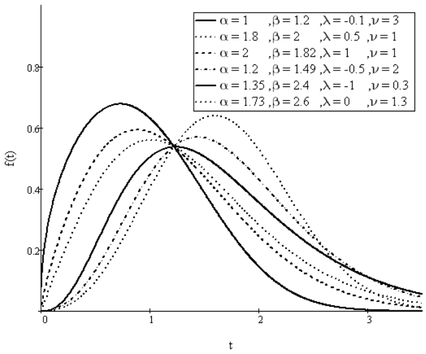

Fig. 2 (b) The behavior of the pdf of ETW

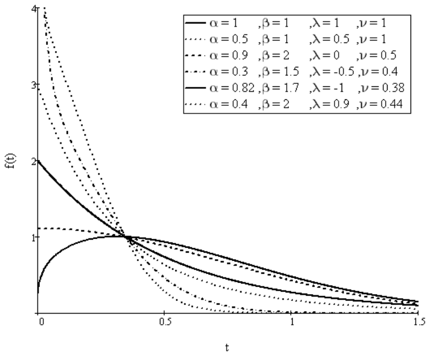

Fig. 2 (c) The behavior of the pdf of ETW

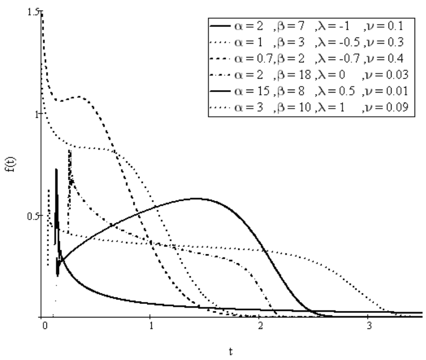

Fig. 2 (d) The behavior of the pdf of ETW

Figs. 3 (a)-(d) illustrate some of the possible shapes of the cdf of the ETW distribution for different values of the parameters $\alpha > 0$, $\beta > 0$, $v > 0$ and $|\lambda| \leq 1$.

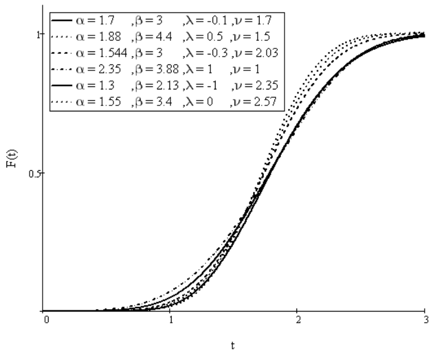

Fig. 3 (a) The behavior of the cdf of ETW

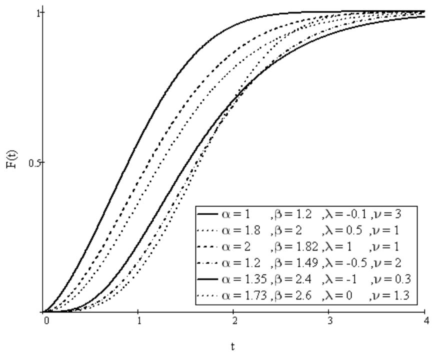

Fig. 3 (b) The behavior of the cdf of ETW

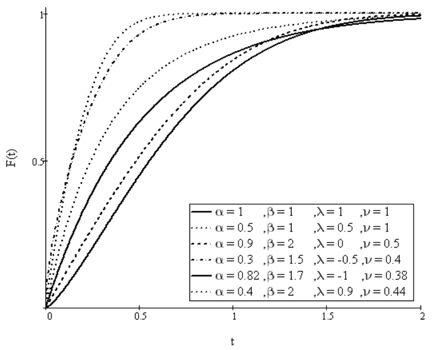

Fig. 3 (c) The behavior of the cdf of ETW

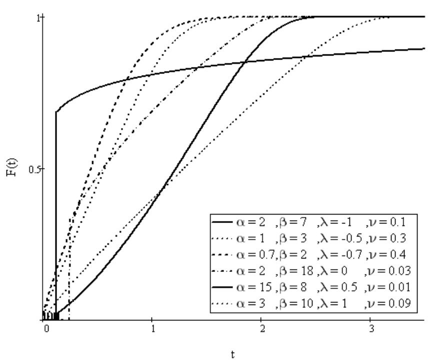

Fig. 3 (d) The behavior of the cdf of ETW

III. Statistical Properties

This section explains statistical properties of the ETW distribution including the quantiles, the median, random number generation, the central and non-central moments and the moment generating function.

A. Quantiles of the Distribution

The p^{th} quantile, t_p, of the ETW distribution is the real solution of the equation

$$F(t_p) = p,$$

and is given by:

$$t_p = \alpha\left\{-\ln\left[\left[\frac{1}{\lambda}\left(1-q^{\frac{1}{v}}\right)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right]\right\}^{\frac{1}{\beta}}. \quad (7)$$

In particular the ETW median is:

$$t_{0.5} = \alpha\left\{-\ln\left[\left[\frac{1}{\lambda}\left(1-\left(\frac{1}{2}\right)^{\frac{1}{v}}\right)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right]\right\}^{\frac{1}{\beta}}. \quad (8)$$

It follows, from (7) that, the p^{th} quantiles of the following special cases of the ETW distribution are:

1) The p^{th} quantile of the ETR , substituting $\beta = 2$, is

$$t_p = \alpha\left\{-\ln\left[\left[\frac{1}{\lambda}\left(1-p^{\frac{1}{v}}\right)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right]\right\}^{\frac{1}{2}}.$$

2) The p^{th} quantile of the ETE, substituting $\beta = 1$, is

$$t_p = -\alpha\ \ln\left[\left[\frac{1}{\lambda}\left(1-p^{\frac{1}{v}}\right)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right].$$

3) The p^{th} quantile of the TW, substituting $v = 1$, is

$$t_p = \alpha\left\{-\ln\left[\left[\frac{1}{\lambda}(1-p)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right]\right\}^{\frac{1}{\beta}}.$$

4) The p^{th} quantile of the TR, substituting $v = 1$ and $\beta = 2$, is

$$t_p = \alpha\left\{-\ln\left[\left[\frac{1}{\lambda}(1-p)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right]\right\}^{\frac{1}{2}}.$$

5) The p^{th} quantile of the TE by substituting $v = 1$ and $\beta = 1$

$$t_p = -\alpha\ \ln\left[\left[\frac{1}{\lambda}(1-p)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^2\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right].$$

B. Random Numbers Generation

Using the method of inversion in [11], random numbers from the ETW distribution can be generated with $u \sim U(0,1)$ as the solution of following equation

$$\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v}=u\,.$$

This yield

$$t=\alpha\left\{-\ln\left[\left[\frac{1}{\lambda}\left(1-(u)^{\frac{1}{v}}\right)+\frac{1}{4}\left(\frac{1-\lambda}{\lambda}\right)^{2}\right]^{\frac{1}{2}}-\frac{1}{2}\left(\frac{1-\lambda}{\lambda}\right)\right]\right\}^{\frac{1}{\beta}}. \qquad (9)$$

Moreover, (9) may be used to generate random numbers from the ETW distribution when the parameters α, β, v and λ are known.

C. Central and Non-Central Moments

The r^{th} noncentral moment, $\mu_r{}' = E(T^r)$ of the ETW distribution is given by:

$$\mu_r{}' = v\,\alpha^r\,\Gamma\left(\frac{r}{\beta}+1\right)\sum_{i=0}^{v-1}\sum_{j=0}^{i}\binom{v-1}{i}\binom{i}{j}(-\lambda)^j \times (\lambda-1)^{i-j}\left[\frac{(1-\lambda)}{(i+j+1)^{\frac{r}{\beta}+1}}+\frac{2\lambda}{(i+j+2)^{\frac{r}{\beta}+1}}\right]. \qquad (10)$$

For a positive integer v, the summation $\sum_{i=0}^{\infty}$ stops at $(v-1)$. Then the expected value, $E(T)$, and the variance, $Var(T)$, of the exponentiated transmuted Weibull random variable T are, respectively, given by:

$$E(T) = v\,\alpha\,\Gamma\left(\frac{1}{\beta}+1\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{v-1}{i}\binom{i}{j}(-\lambda)^j \times (\lambda-1)^{i-j}\left[\frac{(1-\lambda)}{(i+j+1)^{\frac{1}{\beta}+1}}+\frac{2\lambda}{(i+j+2)^{\frac{1}{\beta}+1}}\right], \qquad (11)$$

$$Var(T) = E(T^2) - E^2(T)\,, \qquad (12)$$

where $E(T^2)$ is given by:

$$E(T^2) = v\,\alpha^2\,\Gamma\left(\frac{2}{\beta}+1\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{v-1}{i}\binom{i}{j}(-\lambda)^j \times (\lambda-1)^{i-j}\left[\frac{(1-\lambda)}{(i+j+1)^{\frac{2}{\beta}+1}}+\frac{2\lambda}{(i+j+2)^{\frac{2}{\beta}+1}}\right].$$

The n^{th} central moment, m_n can be obtained easily from the r^{th} noncentral moments through the relation:

$$m_n = E(T-\mu)^n = \sum_{r=0}^{n}\binom{n}{r}(-\mu)^{n-r}E(T^r).$$

Thus, the n^{th} central moment of the ETW distribution is given by:

$$E(T-\mu)^n = v\sum_{r=0}^{n}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{v-1}{i}\binom{i}{j}\binom{n}{r}(-\lambda)^j \times (\lambda-1)^{i-j}(-\mu)^{n-r}\alpha^r\,\Gamma\left(\frac{r}{\beta}+1\right)\times\left[\frac{(1-\lambda)}{(i+j+1)^{\frac{r}{\beta}+1}}+\frac{2\lambda}{(i+j+2)^{\frac{r}{\beta}+1}}\right]. \qquad (13)$$

It follows that the coefficient of variation (ρ), the coefficient of skewness(γ_1), and the coefficient of kurtosis (γ_2) of ETW distribution are, respectively, obtained by

$$\rho = \frac{\sqrt{m_2}}{m_1} = \frac{\sqrt{\mu_2{}'-\mu_1{}'^2}}{\mu_1{}'}\,, \qquad (14)$$

$$\gamma_1 = \frac{m_3}{[m_2]^{3/2}} = \frac{\mu_3{}'-3\mu_2{}'\mu_1{}'+2\mu_1{}'^3}{\left[\mu_2{}'-\mu_1{}'^2\right]^{3/2}}\,, \qquad (15)$$

$$\gamma_2 = \frac{m_4}{[m_2]^2} = \frac{\mu_4{}'-4\mu_3{}'\mu_1{}'+6\mu_2{}'\mu_1{}'^2-3\mu_1{}'^2}{\left[\mu_2{}'-\mu_1{}'^2\right]^2}. \qquad (16)$$

The moment generating function of the ETW distribution is given by

$$M(t) = E(e^{tT}) = \sum_{r=0}^{\infty}\frac{t^r}{r!}E(T^r) = v\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{v-1}{i}\binom{i}{j}(-\lambda)^j(\lambda-1)^{i-j}\times\frac{(\alpha\,t)^r}{r!}\Gamma\left(\frac{r}{\beta}+1\right)\left[\frac{(1-\lambda)}{(i+j+1)^{\frac{r}{\beta}+1}}+\frac{2\lambda}{(i+j+2)^{\frac{r}{\beta}+1}}\right]. \qquad (17)$$

IV. Reliability Analysis

In this section the survival, the hazard rate, the cumulative hazard rate and the mean residual lifetime functions for the exponentiated transmuted Weibull distribution are presented.

A. The Survival Function

The exponentiated transmuted Weibull distribution provides a useful tool for modeling the lifetime data analysis of a given system. To this end, the survival function, $R(t)$, of the exponentiated transmuted Weibull distribution is given by:

$$R(t) = 1-\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v}. \qquad (18)$$

Figs. 4 (a)-(d) illustrate the behavior of the survival function of the ETW distribution for some selected values of the parameters.

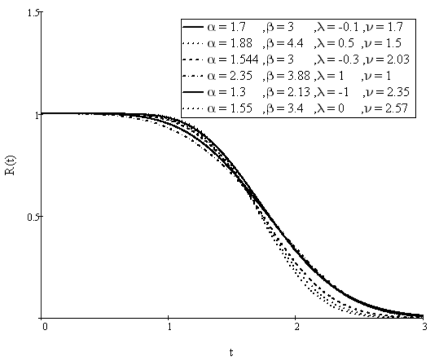

Fig. 4 (a) The behavior of the survival function of the ETW

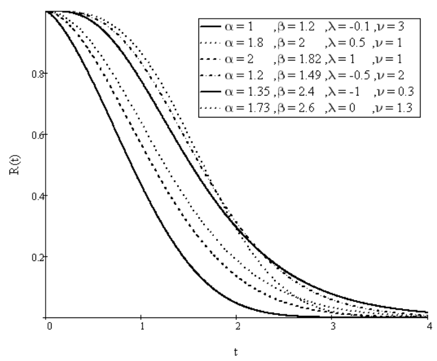

Fig. 4 (b) The behavior of the survival function of the ETW

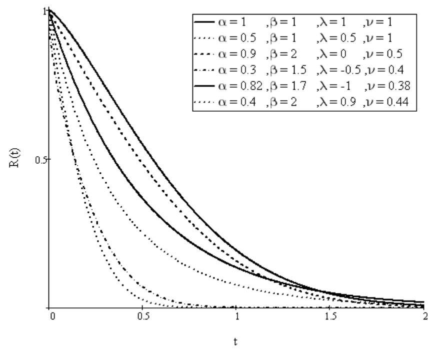

Fig. 4 (c) The behavior of the survival function of the ETW

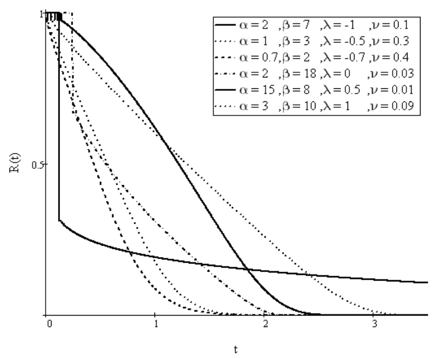

Fig. 4 (d) The behavior of the survival function of the ETW

B. The Hazard Rate Function

The other characteristic of interest of a random variable is the hazard rate function $h(t) = \frac{f(t)}{R(t)}$ and defined by:

$$h(t) = \frac{v\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \frac{\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v-1}}{1-\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v}} .(19)$$

It can be seen that $h(t)$ might be constant, increasing, or decreasing depending on the values of the parameters involved. For example, if $\lambda = 0\,, v = 1$ and $\beta = 1$ then the $h(t) = \frac{1}{\alpha}$, a constant, and if $\lambda = 1\,, v = 1$ and $\beta = 1$ then $h(t) = \frac{2}{\alpha}$, which is also constant, while if $\lambda = 0$ and $v = 1$, $h(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}$ which is increasing for $\beta > 1$ and decreasing for $\beta < 1$, and if $\lambda = 1$ and $v = 1$, $h(t) = \frac{2\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}$ which is increasing for $\beta > 1$ and decreasing for $\beta < 1$.

Figs. 5 (a)-(d) illustrate the behavior of the hazard rate function of the ETW distribution for different choices of the parameters.

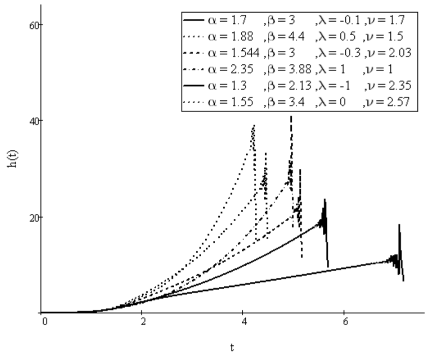

Fig. 5 (a) The hazard rate behavior

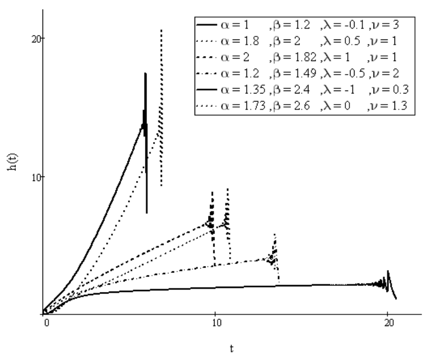

Fig. 5 (b) The hazard rate behavior

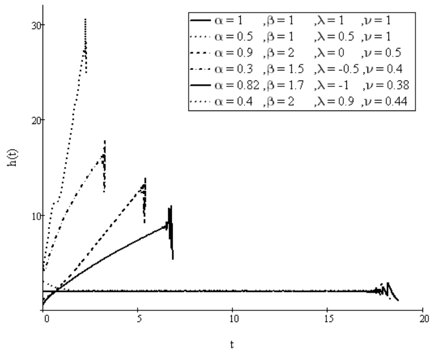

Fig. 5 (c) The hazard rate behavior

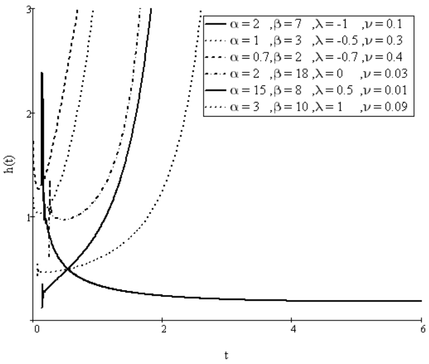

Fig. 5 (d) The hazard rate behavior of the ETW

As a consequence, the hazard rate functions of some special cases of the ETW distribution are:

1) The failure rate of the ETR distribution, substituting $\beta = 2$ is

$$h(t) = \frac{2vt}{\alpha^2} e^{-\left(\frac{t}{\alpha}\right)^2}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^2}\right] \times \frac{\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right\}^{v-1}}{1-\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right\}^{v}}.$$

2) The failure of the ETE distribution, substituting $\beta = 1$ is

$$h(t) = \frac{v}{\alpha} e^{-\left(\frac{t}{\alpha}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)}\right] \times \frac{\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)}-\lambda e^{-2\left(\frac{t}{\alpha}\right)}\right\}^{v-1}}{1-\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)}-\lambda e^{-2\left(\frac{t}{\alpha}\right)}\right\}^{v}}.$$

3) The failure rate of the TW distribution, substituting $v = 1$ is

$$h(t) = \frac{\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]}{(1-\lambda)+\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}}.$$

4) The failure rate of the TR distribution, substituting $v = 1$ and $\beta = 2$ is

$$h(t) = \frac{\frac{2t}{\alpha^2}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^2}\right]}{(1-\lambda)+\lambda e^{-\left(\frac{t}{\alpha}\right)^2}}.$$

5) The failure rate of the TE distribution, substituting $v = 1$ and $\beta = 1$ is

$$h(t) = \frac{\frac{1}{\alpha}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)}\right]}{(1-\lambda)+\lambda e^{-\left(\frac{t}{\alpha}\right)}}.$$

C. The Cumulative Hazard Rate Function

Many generalized Weibull models have been proposed in reliability literature through the relationship between the reliability function $R(t)$ and its cumulative hazard rate function $H(t)$ given by $H(t) = \int_0^t h(t)dt = -\ln R(t)$. Here, the cumulative hazard rate function of the ETW distribution is given by:

$$H(t) = -\ln\left[1-\left\{1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}}-\lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v}\right], \qquad (20)$$

where $H(t)$ is the total number of failures or deaths over an interval of time, which describes how the risk of a particular outcome changes with time for an ETW distribution.

Figs. 6 (a)-(d) illustrate the behavior of the cumulative hazard rate function for different values of the parameters.

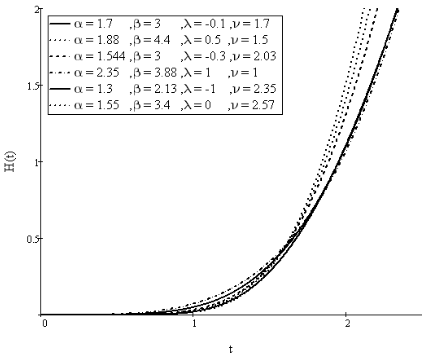

Fig. 6 (a) The behavior of the cumulative hazard rate function

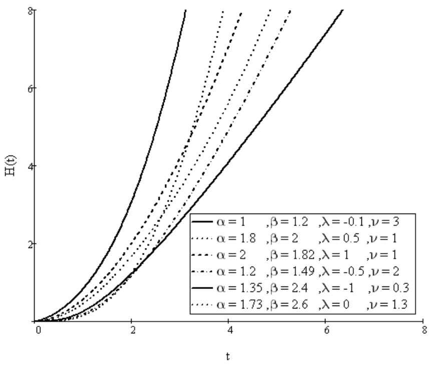

Fig. 6 (b) The behavior of the cumulative hazard rate function

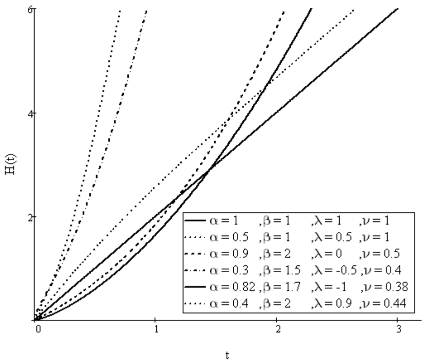

Fig. 6 (c) The behavior of the cumulative hazard rate function

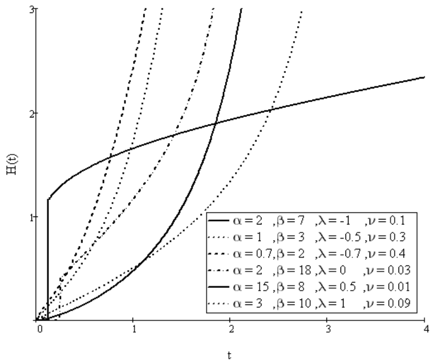

Fig. 6 (d) The behavior of the cumulative hazard rate function

D. The Mean Residual Lifetime Function

The additional lifetime given that the component has survived up to time t is called the residual life function of the component, then the expectation of the random variable T_t that represents the remaining lifetime is called the mean residual lifetime (MRL) and is given by:

$$m(t) = E(T - t | T > t) = \frac{1}{R(t)} \int_t^{\infty} u\, f(u) du - t.$$

The MRL represent the mean lifetime left for an item of age t. The failure rate function at t provides information on a random variable T about a small interval after t, whereas the MRL function at t considers information about the whole remaining interval (t, ∞).

The MRL function as well as the hazard rate function or the reliability function is very important as each of them can be used to characterize a unique corresponding lifetime distribution.

The MRL function $m(t)$ for ETW random variable is given by:

$$m(t) = -t + \frac{\alpha v}{1 - \left\{1 + (\lambda - 1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right\}^{v}} \times \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{v-1}{j} \binom{i}{j} (\lambda - 1)^{i+j} (-\lambda)^j \times \left\{ \frac{(1-\lambda).\Gamma\left(\frac{1}{\beta}+1, (i+j+1)\left(\frac{t}{\alpha}\right)^{\beta}\right)}{(i+j+1)^{\frac{1}{\beta}+1}} + \frac{2\lambda.\Gamma\left(\frac{1}{\beta}+1, (i+j+2)\left(\frac{t}{\alpha}\right)^{\beta}\right)}{(i+j+2)^{\frac{1}{\beta}+1}} \right\}. \quad (21)$$

V. Estimation of the Parameters

In this section, the method of maximum likelihood is used to estimate the parameters involved and hence build confidence intervals for the unknown parameters.

Let $T_1, T_2, \ldots, T_n$ be a sample of size n from an ETW distribution. Then the likelihood function (ℓ) is given by:

$$\ell = \prod_{i=1}^{n} f_i(t) = \left(\frac{v\beta}{\alpha^{\beta}}\right)^{n} \prod_{i=1}^{n} t_i^{\beta-1} e^{-\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^{\beta}} \times \prod_{i=1}^{n} \left\{1 + (\lambda - 1)e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}\right\}^{v-1} \times \prod_{i=1}^{n} \left[1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}\right] \quad (22)$$

Hence, the loglikelihood function $\mathcal{L} = \ln \ell$ becomes

$$\mathcal{L} = \ln \ell = n \ln(v) + n \ln(\beta) - n\beta \ln(\alpha) + (v - 1) \sum_{i=1}^{n} \ln\left\{1 + (\lambda - 1)e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}\right\} + (\beta - 1) \sum_{i=1}^{n} \ln(t_i) - \sum_{i=1}^{n} \left(\frac{t_i}{\alpha}\right)^{\beta} + \sum_{i=1}^{n} \ln\left[1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]. \quad (23)$$

Therefore, the MLEs of α, β, λ and v are derived from the derivatives of $\mathcal{L}$. They should satisfy the following equations:

$$\frac{\partial \mathcal{L}}{\partial \alpha} = -\frac{n\beta}{\alpha} + \frac{\beta}{\alpha} \sum_{i=1}^{n} \left(\frac{t_i}{\alpha}\right)^{\beta} + \frac{\beta}{\alpha} \sum_{i=1}^{n} \frac{2\lambda\left(\frac{t_i}{\alpha}\right)^{\beta} e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}}{\left[1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]} + (v - 1) \frac{\beta}{\alpha} \sum_{i=1}^{n} \frac{(\lambda-1)\left(\frac{t_i}{\alpha}\right)^{\beta} e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - 2\lambda\left(\frac{t_i}{\alpha}\right)^{\beta} e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}}{\left\{1 + (\lambda - 1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}\right\}} = 0, \quad (24)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \frac{n}{\beta} - n \ln(\alpha) + \ln(t_i) - \sum_{i=1}^{n} \left(\frac{t_i}{\alpha}\right)^{\beta} \ln\left(\frac{t_i}{\alpha}\right)^{\beta} - 2\lambda \sum_{i=1}^{n} \frac{e^{-\left(\frac{t}{\alpha}\right)^{\beta}} \left(\frac{t_i}{\alpha}\right)^{\beta} \ln\left(\frac{t_i}{\alpha}\right)^{\beta}}{\left[1 - \lambda + 2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right]} + (v - 1) \times \sum_{i=1}^{n} \frac{(1-\lambda)\left(\frac{t_i}{\alpha}\right)^{\beta} e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} \ln\left(\frac{t_i}{\alpha}\right)^{\beta} + 2\lambda\left(\frac{t_i}{\alpha}\right)^{\beta} e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}} \ln\left(\frac{t_i}{\alpha}\right)^{\beta}}{\left\{1 + (\lambda - 1)e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}\right\}} = 0, \quad (25)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{n} \frac{2e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - 1}{\left[1 - \lambda + 2\lambda e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]} + (v - 1) \times \sum_{i=1}^{n} \frac{e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}}{\left\{1 + (\lambda - 1)e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}\right\}} = 0. (27)$$

$$\frac{\partial \mathcal{L}}{\partial v} = \frac{n}{v} + \sum_{i=1}^{n} \ln\left\{1 + (\lambda - 1)e^{-\left(\frac{t_i}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t_i}{\alpha}\right)^{\beta}}\right\} = 0\,, \quad (26)$$

To solve (24) through (27), it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the loglikelihood function. In order to compute the standard errors and asymptotic confidence intervals the usual large sample approximation is used, in which the maximum likelihood estimators can be treated as being approximately trivariate normal [12]. Hence as $n \to \infty$, the asymptotic distribution of the MLE is given by,

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{v} \\ \hat{\lambda} \end{bmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \beta \\ v \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34} \\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & \hat{V}_{44} \end{pmatrix} \right],$$

where $\hat{V}_{ij} = V_{ij}\,|_{\theta=\hat{\theta}}$ and

$$\begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}^{-1},$$

is the approximate variance covariance matrix with its elements obtained from

$$A_{11} = -\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} \quad A_{12} = -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} \quad A_{13} = -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial v} \quad A_{14} = -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \lambda},$$

$$A_{22} = -\frac{\partial^2 \mathcal{L}}{\partial \beta^2} \quad A_{23} = -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial v} \quad A_{24} = -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial \lambda},$$

$$A_{33} = -\frac{\partial^2 \mathcal{L}}{\partial v^2} \quad A_{34} = -\frac{\partial^2 \mathcal{L}}{\partial v \partial \lambda},$$

$$A_{44} = -\frac{\partial^2 \mathcal{L}}{\partial \lambda^2}.$$

By solving this inverse of dispersion matrix, these solutions will yield the asymptotic variance and covariance of these MLEs for $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ and $\hat{v}$. Approximate $100(1-\alpha)\%$ confidence intervals for α, β, λ and v can be determined as

$$\hat{\alpha} \pm z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}\ ,\ \hat{\beta} \pm z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}},\ \hat{\lambda} \pm z_{\alpha/2}\sqrt{\hat{V}_{44}},\ \hat{v} \pm z_{\alpha/2}\sqrt{\hat{V}_{33}},$$

where $z_{\alpha/2}$ is the upper α^{th} percentile of the standard normal distribution.

VI. Order Statistics from the Distribution

Let $T_1, T_2, \ldots, T_n$ denotes n-independent random variables from a distribution function $F_T(t)$ with pdf $f_T(t)$, then $T_{(1)}, T_{(2)}, \ldots, T_{(n)}$ denote the order sample arrangement, and the pdf of $T_{(j)}$ is given by:

$$f_{T_{(j)}}(t) = \frac{n!}{(j-1)!\,(n-j)!} f_T(t)[F_T(t)]^{j-1}.[1-F_T(t)]^{n-j}$$

for $j = 1,2,\ldots,n$.

Then the pdf of the order statistic $T_{(j)}$ from the ETW distribution is given by:

$$f_{T_{(j)}}(t) = \frac{n!}{(j-1)!\,(n-j)!}\frac{v\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{vj-1} \times \left\{1-\left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{v}\right\}^{n-j} \quad (28)$$

Therefore, the pdf of the largest order statistic $T_{(n)}$ and the smallest order statistic $T_{(1)}$ are, respectively, given by:

$$f_{T_{(n)}}(t) = \frac{nv\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{nv-1}, \quad (29)$$

$$f_{T_{(1)}}(t) = \frac{nv\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{v-1} \times \left\{1-\left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{v}\right\}^{n-1}. \quad (30)$$

It is possible to express the pdf of the $(k+1)^{th}$ ordered statistic from the exponentiated transmuted Weibull in terms of the pdf of k^{th} ordered statistic from the exponentiated transmuted Weibull, using the following relationship:

$$f_{T_{(k+1)}}(t) = \frac{\left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{v}}{1-\left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{v}} \times \left(\frac{n-k}{k}\right) f_{T_{(k)}}(t).$$

Then, the special cases of the minimum and maximum order statistics of the derived submodels of the ETW distribution are mentioned in the following two cases as:

Case A: Maximum order statistics for submodels

1) The pdf of the maximum order statistic of the ETR, substituting $\beta = 2$, is

$$f_{T_{(n)}}(t) = \frac{2nvt}{\alpha^2} e^{-\left(\frac{t}{\alpha}\right)^2}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^2}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right]^{nv-1}.$$

2) The pdf of the maximum order statistic of the ETE, substituting $\beta = 1$, is

$$f_{T_{(n)}}(t) = \frac{nv}{\alpha} e^{-\left(\frac{t}{\alpha}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)} - \lambda e^{-2\left(\frac{t}{\alpha}\right)}\right]^{nv-1}.$$

3) The pdf of the maximum order statistic of the TW, substituting $v = 1$, is

$$f_{T_{(n)}}(t) = \frac{n\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{n-1}.$$

4) The pdf of the maximum order statistic of the TR, substituting $v = 1$ and $\beta = 2$, is

$$f_{T_{(n)}}(t) = \frac{2nt}{\alpha^2} e^{-\left(\frac{t}{\alpha}\right)^2}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^2}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right]^{n-1}.$$

5) The pdf of the maximum order statistic of the TE, substituting $v = 1$ and $\beta = 1$, is

$$f_{T_{(n)}}(t) = \frac{n}{\alpha} e^{-\left(\frac{t}{\alpha}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)} - \lambda e^{-2\left(\frac{t}{\alpha}\right)}\right]^{n-1}.$$

Case B: Minimum order statistic for submodels

1) The pdf of the minimum order statistic of the ETR, substituting $\beta = 2$, is

$$f_{T_{(1)}}(t) = \frac{2nvt}{\alpha^2} e^{-\left(\frac{t}{\alpha}\right)^2}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^2}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right]^{v-1}.\left\{1-\left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)^2} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^2}\right]^{v}\right\}^{n-1}.$$

2) The pdf of the minimum order statistic of the ETE, substituting $\beta = 1$, is

$$f_{T_{(1)}}(t) = \frac{nv}{\alpha} e^{-\left(\frac{t}{\alpha}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)}\right] \times \left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)} - \lambda e^{-2\left(\frac{t}{\alpha}\right)}\right]^{v-1}.\left\{1-\left[1+(\lambda-1)e^{-\left(\frac{t}{\alpha}\right)} - \lambda e^{-2\left(\frac{t}{\alpha}\right)}\right]^{v}\right\}^{n-1}.$$

3) The pdf of the minimum order statistic of the TW,

substituting $v = 1$, is

$$f_{T_{(1)}}(t) = \frac{n\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{\beta}}\right] \times \left[(1-\lambda)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{n-1}.$$

4) The pdf of the minimum order statistic of the TR, substituting $v = 1$ and $\beta = 2$, is

$$f_{T_{(1)}}(t) = \frac{2nt}{\alpha^2} e^{-\left(\frac{t}{\alpha}\right)^{2}}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)^{2}}\right] \times \left[(1-\lambda)e^{-\left(\frac{t}{\alpha}\right)^{\beta}} - \lambda e^{-2\left(\frac{t}{\alpha}\right)^{\beta}}\right]^{n-1}.$$

5) The pdf of the minimum order statistic of the TE by substituting $v = 1$ and $\beta = 1$

$$f_{T_{(1)}}(t) = \frac{n}{\alpha} e^{-\left(\frac{t}{\alpha}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{t}{\alpha}\right)}\right] \times \left[(1-\lambda)e^{-\left(\frac{t}{\alpha}\right)} - \lambda e^{-2\left(\frac{t}{\alpha}\right)}\right]^{n-1}.$$

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