

On the System of Nonlinear Rational Difference Equations

Qianhong Zhang, Wenzhuan Zhang

Abstract—This paper is concerned with the global asymptotic behavior of positive solution for a system of two nonlinear rational difference equations. Moreover, some numerical examples are given to illustrate results obtained.

Keywords—Difference equations, stability, unstable, global asymptotic behavior.

I. INTRODUCTION

DIFFERENCE equation or discrete dynamical system is a diverse field which impact almost every branch of pure and applied mathematics. Every dynamical system $x_{n+1} = f(x_n, x_{n-2}, \dots, x_{n-k})$ determines a difference equation and vice versa. Recently, there has been great interest in studying difference equations systems. One of the reasons for this is a necessity for some techniques whose can be used in investigating equations arising in mathematical models describing real life situations in population biology [1],[2], economic, probability theory, genetics, psychology, etc.

Marwan Aloqeili [3] investigated the stability character, semi-cycle behavior of the solution of the difference equation $x_{n+1} = x_{n-1}/(a - x_{n-1}x_n)$, where $n = 0, 1, \dots, x_{-1}, x_0 \in \mathbb{R}$ and $a > 0$. Cinar [4] gave the positive solution of the difference equation $x_{n+1} = x_{n-1}/(1 + x_{n-1}x_n)$. Other related difference equations where also considered by Cinar [5],[6], [7].

The study of properties of rational difference equations and systems of rational difference equations [8] has been an area of interest in recent years. There are many papers in which systems of difference equations have studied.

The theory of difference equations occupies a central position in applicable analysis. There is no doubt that the theory of difference equations will continue to play an important role in mathematics as a whole. Nonlinear difference equations of order greater than one are of paramount importance in applications. Such equations also appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations which model various diverse phenomena in biology, ecology, physiology, physics, engineering and economics. It is very interesting to investigate the behavior of solutions of a system of nonlinear

difference equations and to discuss the local asymptotic stability of their equilibrium points.

Papaschinopoulos and Schinas [9] studied the system of two nonlinear difference equation

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \quad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \dots,$$

where p, q are positive integers.

Clark and Kulenovic [10],[11] investigated the system of rational difference equations

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}, \quad n = 0, 1, \dots,$$

where $a, b, c, d \in (0, \infty)$ and the initial conditions x_0 and y_0 are arbitrary nonnegative numbers.

In 2012, Zhang, Yang and Liu [12] investigated the global behavior for a system of the following third order nonlinear difference equations.

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n},$$

where $A, B \in (0, \infty)$, and the initial values $x_{-i}, y_{-i} \in (0, \infty), i = 0, 1, 2$.

Although difference equations are sometimes very simple in their forms, they are extremely difficult to understand thoroughly the behavior of their solutions. In book [16], Kocic and Ladas have studied global behavior of nonlinear difference equations of higher order. Similar nonlinear systems of rational difference equations were investigated (see [13],[14],[15], [17],[18],[19], [20],[21],[22],[23]).

Motivated by above discussion, our goal, in this paper is to investigate the solutions, stability character and asymptotic behavior of the system of difference equations,

$$\begin{cases} x_{n+1} = \frac{x_n + x_{n-1}}{q + y_n y_{n-1}}, \\ y_{n+1} = \frac{y_n + y_{n-1}}{p + x_n x_{n-1}}, \end{cases} \quad n = 0, 1, \dots \quad (1)$$

where $p, q \in (0, \infty)$ and initial conditions $x_i, y_i \in (0, \infty), i = -1, 0$. Finally, we give some numerical examples to illustrate results obtained.

II. PRELIMINARIES

Let I_x, I_y be some intervals of real number and $f : I_x^2 \times I_y^2 \rightarrow I_x, g : I_x^2 \times I_y^2 \rightarrow I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \end{cases} \quad n = 0, 1, 2, \dots, \quad (2)$$

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has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$.

Definition 1. A point $(\bar{x}, \bar{y}) \in I_x \times I_y$ is called an equilibrium point of (2) if $\bar{x} = f(\bar{x}, \bar{y}, \bar{y})$, $\bar{y} = g(\bar{x}, \bar{y}, \bar{y})$, i. e., $(x_n, y_n) = (\bar{x}, \bar{y})$ for all $n \geq 0$.

Definition 2. Let I_x and I_y are some intervals of real numbers, interval $I_x \times I_y$ is called invariant for (2) if, for all $n > 0$,

$$x_{-1}, x_0 \in I_x, \quad y_{-1}, y_0 \in I_y \Rightarrow x_n \in I_x, \quad y_n \in I_y.$$

Definition 3. Assume that (\bar{x}, \bar{y}) be a fixed point of (2). Then
(i) (\bar{x}, \bar{y}) is said to be stable relative to $I_x \times I_y$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial conditions $(x_i, y_i) \in I_x \times I_y$ ($i = -1, 0$), with $\sum_{i=-1}^0 |x_i - \bar{x}| < \delta$, $\sum_{i=-1}^0 |y_i - \bar{y}| < \delta$, implies $|x_n - \bar{x}| < \varepsilon$, $|y_n - \bar{y}| < \varepsilon$.
(ii) (\bar{x}, \bar{y}) is called an attractor relative to $I_x \times I_y$ if for all $(x_i, y_i) \in I_x \times I_y$ ($i = -1, 0$), $\lim_{n \rightarrow \infty} x_n = \bar{x}$, $\lim_{n \rightarrow \infty} y_n = \bar{y}$.
(iii) (\bar{x}, \bar{y}) is called asymptotically stable relative to $I_x \times I_y$ if it is stable and an attractor.
(iv) Unstable if it is not stable.

Proposition 1.[16] Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \dots$, is a system of difference equations and \bar{X} is the equilibrium point of this system i.e., $F(\bar{X}) = \bar{X}$. If all eigenvalues of the Jacobian matrix J_F , evaluated at \bar{X} lie inside the open unit disk $|\lambda| < 1$, then \bar{X} is locally asymptotically stable. If one of them has modulus greater than one then \bar{X} is unstable.

Proposition 2.[17] Assume that $X(n+1) = F(X(n))$, $n = 0, 1, \dots$, is a system of difference equations and \bar{X} is the equilibrium point of this system, the characteristic polynomial of this system about the equilibrium point \bar{X} is $P(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_{n-1} \lambda + a_n = 0$, with real coefficients and $a_0 > 0$. Then all roots of the polynomial $p(\lambda)$ lie inside the open unit disk $|\lambda| < 1$ if and only if

$$\Delta_k > 0 \text{ for } k = 1, 2, \dots, n, \quad (3)$$

where Δ_k is the principal minor of order k of the $n \times n$ matrix

$$\Delta_n = \begin{bmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{bmatrix}.$$

III. MAIN RESULTS

System (1) has equilibrium points $(0, 0)$ and $(\bar{x}, \bar{y}) = (\sqrt{2-p}, \sqrt{2-q})$, for $p < 2, q < 2$. In addition, if $q = 2$, then every point on the x -axis is an equilibrium point, If $p = 2$, then every point on the y -axis is an equilibrium point, Finally, if $p > 2, q > 2$, then $(0, 0)$ is unique equilibrium points.

Lemma 1.[16] Consider second order difference equation

$$x_{n+2} + ax_{n+1} + bx_n = 0, \quad n = 0, 1, 2, \dots,$$

where $a, b \in R$. Then a necessary and sufficient condition for the asymptotic stability of this equation, is that

$$|a| < 1 + b < 2.$$

Lemma 2.[17] Suppose $c_j(i) \in R^+, i \in Z, j \in \{0, 1, \dots, h\}$ and $\sup_{i \in Z} \{c_j(i)\} = \eta < 1$. Let $\{u_i\}$ be a sequence of real numbers satisfying the following difference inequality:

$$u_{i+1} \leq \sum_{j=1}^h c_j(i) u_{i-j}.$$

Then

$$u_i \leq d e^{-\lambda i}.$$

where $d, \lambda \in R^+$.

We summarize the local stability of the equilibria of (1) as follows.

Theorem 1. If

$$p > 2, \quad q > 2 \quad (4)$$

Then the unique equilibrium $(0, 0)$ is globally asymptotically stable.

Proof. We can easily obtain that the linearized system of (1) about the equilibrium $(0, 0)$ is

$$\Phi_{n+1} = D \Phi_n, \quad (5)$$

where

$$\Phi_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}, \quad D = \begin{pmatrix} \frac{1}{q} & \frac{1}{q} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{p} & \frac{1}{p} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

whose characteristic equation is

$$\left(\lambda^2 - \frac{1}{p}\lambda - \frac{1}{p}\right) \left(\lambda^2 - \frac{1}{q}\lambda - \frac{1}{q}\right) = 0 \quad (6)$$

Noting condition (4), we have

$$\frac{1}{p} < 1 + \frac{1}{p} < 2, \quad \frac{1}{q} < 1 + \frac{1}{q} < 2.$$

Applying Lemma 1, it follows that all the roots of characteristic equation lie inside unit disk. So the unique equilibrium $(0, 0)$ is locally asymptotically stable.

On the other hand, (1) gives

$$x_{n+1} \leq \frac{1}{q} x_n + \frac{1}{q} x_{n-1}, \quad y_{n+1} \leq \frac{1}{p} y_n + \frac{1}{p} y_{n-1}.$$

From condition (4), Lemma 2, we have

$$x_n \leq d_1 e^{-\lambda_1 n}, \quad y_n \leq d_2 e^{-\lambda_2 n}, \quad d_i, \lambda_i \in R^+, \quad i = 1, 2.$$

Which implies $\lim_{n \rightarrow \infty} x_n = 0$, $\lim_{n \rightarrow \infty} y_n = 0$. Hence the unique equilibrium $(0, 0)$ is globally asymptotically stable.

Theorem 2. Assume that

$$p < 2, \quad q < 2.$$

Then the equilibrium $(0, 0)$ and positive equilibrium $(\bar{x}, \bar{y}) = (\sqrt{2-p}, \sqrt{2-q})$ are locally unstable.

Proof. Its characteristic equation (5) about equilibrium $(0, 0)$ of (4) is

$$\begin{aligned} f(\lambda) &= f_1(\lambda)f_2(\lambda) \\ &= \left(\lambda^2 - \frac{1}{p}\lambda - \frac{1}{p}\right)\left(\lambda^2 - \frac{1}{q}\lambda - \frac{1}{q}\right) = 0. \end{aligned} \quad (7)$$

Since

$$f_1(-1) = 1 > 0, \quad f_1(0) = -\frac{1}{p} < 0, \quad f_1(1) = 1 - \frac{2}{p} < 0.$$

$$f_2(-1) = 1 > 0, \quad f_2(0) = -\frac{1}{q} < 0, \quad f_2(1) = 1 - \frac{2}{q} < 0.$$

It is clear that there are two roots of (7) modulus larger than 1 and other roots of (7) less than in absolute value. Therefore the equilibrium $(0, 0)$ is unstable.

The linearized equation of (1) about equilibrium $(\sqrt{2-p}, \sqrt{2-q})$ is

$$\Phi_{n+1} = B\Phi_n, \quad (8)$$

where

$$B = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \alpha & \alpha \\ 1 & 0 & 0 & 0 \\ \alpha & \alpha & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\alpha = -\frac{1}{2}\sqrt{(2-p)(2-q)}$.

Its characteristic equation is

$$g(\lambda) = \lambda^4 - \lambda^3 - \left(\frac{3}{4} + \alpha^2\right)\lambda^2 + \left(\frac{1}{2} - 2\alpha^2\right)\lambda + \frac{1}{4} - \alpha^2 \quad (9)$$

From (9), we have

$$\Delta_6 = \begin{bmatrix} -1 & \frac{1}{2} - 2\alpha^2 & 0 & 0 \\ 1 & -\frac{3}{4} - \alpha^2 & \frac{1}{4} - \alpha^2 & 0 \\ 0 & -1 & \frac{1}{2} - 2\alpha^2 & 0 \\ 0 & 1 & -\frac{3}{4} - \alpha^2 & \frac{1}{4} - \alpha^2 \end{bmatrix}$$

It is clear that not all of $\Delta_k > 0, k = 1, 2, 3, 4$. Therefore by Proposition 2 the positive equilibrium $(\sqrt{2-p}, \sqrt{2-q})$ is locally unstable.

Theorem 3. Consider system (1), suppose that (4) hold. Then the following statements are true, for $i = -1, 0$.

- (i) $(x_i, y_i) \in (0, \sqrt{2-p}) \times (\sqrt{2-q}, +\infty) \Rightarrow (x_n, y_n) \in (0, \sqrt{2-p}) \times (\sqrt{2-q}, +\infty)$;
- (ii) $(x_i, y_i) \in (\sqrt{2-p}, +\infty) \times (0, \sqrt{2-q}) \Rightarrow (x_n, y_n) \in (\sqrt{2-p}, +\infty) \times (0, \sqrt{2-q})$.

Proof. Let $(x_i, y_i) \in (0, \sqrt{2-p}) \times (\sqrt{2-q}, +\infty), (i = -1, 0)$, from (1), we have

$$\begin{cases} x_1 = \frac{x_0 + x_{-1}}{q + y_0 y_{-1}} < \frac{2\bar{x}}{q + \bar{y}^2} = \bar{x} = \sqrt{2-p}, \\ y_1 = \frac{y_0 + y_{-1}}{p + x_0 x_{-1}} > \frac{2\bar{y}}{p + \bar{x}^2} = \bar{y} = \sqrt{2-q}. \end{cases} \quad (10)$$

We prove by induction that

$$(x_n, y_n) \in (0, \sqrt{2-p}) \times (\sqrt{2-q}, +\infty) \quad (11)$$

Suppose that (11) is true for $n = k > 1$. Then from (1) we have

$$\begin{cases} x_{k+1} = \frac{x_k + x_{k-1}}{q + y_k y_{k-1}} < \frac{2\bar{x}}{q + \bar{y}^2} = \bar{x} = \sqrt{2-p}, \\ y_{k+1} = \frac{y_k + y_{k-1}}{p + x_k x_{k-1}} > \frac{2\bar{y}}{p + \bar{x}^2} = \bar{y} = \sqrt{2-q}. \end{cases}$$

Therefore (11) is true. This completes the proof of (i). Similarly we can obtain the proof of (ii). Hence it is omitted.

IV. NUMERICAL EXAMPLES

In order to illustrate the results of the previous sections and to support our theoretical discussions, we consider several interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions system of nonlinear difference equations.

Example 1. If the initial conditions $x_0 = 0.8, x_{-1} = 1.2, y_0 = 1.5, y_{-1} = 0.3$ and $p = 3, q = 4$, then the solution of (1) converge to $(0, 0)$ (see Fig. 1, Theorem 1).

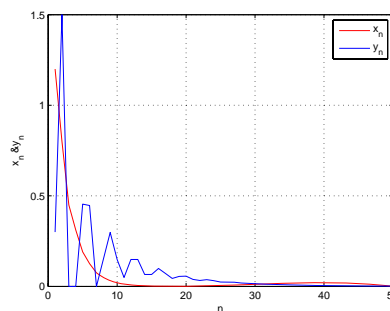


Fig. 1. The fixed point $(0, 0)$ is globally asymptotically stable

Example 2. If the initial conditions $x_0 = 0.8, x_{-1} = 1.2, y_0 = 1.5, y_{-1} = 0.3$ and $p = 1.8, q = 1.9$, then the equilibrium $(0, 0)$ is unstable (see Fig. 2, Theorem 2).

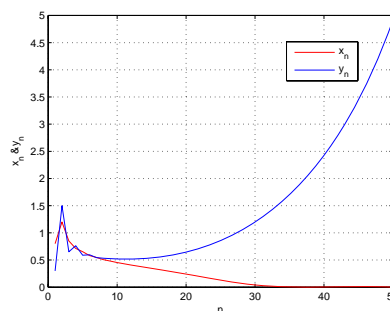


Fig. 2. The fixed point $(0, 0)$ is unstable

Example 3. If the initial conditions $x_0 = 1.3, x_{-1} = 1.2, y_0 = 0.3, y_{-1} = 0.2$ and $p = 1.4, q = 1.6$, then the equilibrium $(\sqrt{2-p}, \sqrt{2-q})$ is unstable (see Fig. 3, Theorem 2).

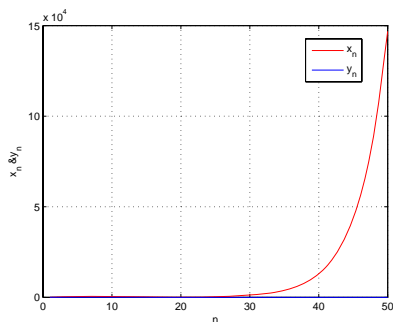


Fig. 3. The fixed point $((\sqrt{2-p}, \sqrt{2-q}))$ is unstable

V. CONCLUSION AND FUTURE WORK

In this paper, we have studied the behavior of positive solution to system (1) under some conditions. If $p > 2$ and $q > 2$, the system (1) has an unique equilibrium $(0, 0)$ which is globally asymptotically stable. If $A < 2$ and $B < 2$, the system (1) has equilibrium $(0, 0)$ and $(\sqrt{2-p}, \sqrt{2-q})$, and these equilibriums are unstable. We will study the behavior of positive solution to system under the conditions $p > 2, q < 2$ or $p = q = 2$ in the future.

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