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Numerical Methods versus Bjerksund and Stensland Approximations for American Options Pricing

Marasovic Branka, Aljinovic Zdravka, Poklepovic Tea

***Abstract*—**Numerical methods like binomial and trinomial trees and finite difference methods can be used to price a wide range of options contracts for which there are no known analytical solutions. American options are the most famous of that kind of options. Besides numerical methods, American options can be valued with the approximation formulas, like Bjerksund-Stensland formulas from 1993 and 2002. When the value of American option is approximated by Bjerksund-Stensland formulas, the computer time spent to carry out that calculation is very short. The computer time spent using numerical methods can vary from less than one second to several minutes or even hours. However to be able to conduct a comparative analysis of numerical methods and Bjerksund-Stensland formulas, we will limit computer calculation time of numerical method to less than one second. Therefore, we ask the question: Which method will be most accurate at nearly the same computer calculation time?

***Keywords*—**Bjerksund and Stensland approximations, Computational analysis, Finance, Options pricing, Numerical methods.

I. INTRODUCTION

OPTIONS are part of a larger class of financial instruments known as derivative products, or simply, derivatives. A derivative is an instrument whose value depends on values of other more basic underlying variables.

Option is a security that gives its owner the right, but not the obligation, to buy or sell another, underlying security, simply called underlying, at or before a future predetermined date for a predetermined price. The option that provides its owner the right to buy is called a call option. The option that provides its owner the right to sell is called a put option. If the owner of the option can buy or sell on a given date only, the option is called a European option. If the option gives the right to buy or sell up to (and including) a given date, it is called an American option. If the owner decides to buy or sell, we say that the owner exercises the option. The date on which the option can be exercised (or the last date on which it can be exercised for American options) is called maturity or the expiration date. The predetermined price at which the option can be exercised is called the strike price or the exercise price. Simple puts and calls written on basic assets such as stocks and bonds are common options, often called plain vanilla options. There are many other types of options payoffs, and they are usually referred to as exotic options.

B. Marasović is with the Faculty of Economics, University of Split, 21000 Split, Croatia (phone: 00385914430697; fax: 0038521430601; e-mail: branka.marasovic@ efst.hr).

Z. Aljinović and T. Poklepović are with the Faculty of Economics, University of Split, 21000 Split, Croatia (e-mail: zdravka.aljinovic@ efst.hr, tea.poklepovic@efst.hr).

One of main issue about option is how to determine the option price. The price of an option (like the price of a bond and the price of a stock) will depend on a number of factors. Some of these factors are the price of the underlying, the strike price, and the time left to maturity.

Because the values of option contracts depend on a number of different factors they are complex to value. There are many pricing models in use today. First and the most popular model for pricing European type of options is Black-Scholes-Merton model ([4], [11]). The American option can be exercised at any time up to its expiration date. This added freedom complicates the valuation of American options relative to their European counterparts. With a few exceptions, it is not possible to find an exact formula for the value of American options. Several researchers have, however, come up with excellent closed-form approximations [1]-[3]. These approximations have become especially popular because they execute more quickly on computers than the numerical techniques.

Numerical methods that can be used for evaluation of American options are binomial and trinomial trees and finite difference methods. These methods are more flexible then analytical solutions and can be used to price a wide range of options contracts for which there are no known analytical solutions including the American options.

The binomial method was first published by Cox, Ross and Rubinstein [7] and Rendleman and Bartter [12]. Trinomial trees were introduced in option pricing by Boyle [5] and are similar to binomial trees. The use of finite difference methods in finance was first described by Brennan and Schwartz [6]. Finite difference methods, also called grid models, are simply a numerical technique to solve partial differential equations. The main objection to these methods is that the computing time required for their algorithms is longer than for the analytical expressions. But with the development of computer technology computers become faster and the computation time is reduced significantly. The question arises of whether the price of American options obtained by numerical methods in a short time (less than one second) is closer to the correct value of the option than the price obtained by an approximation formula. This paper will try to give answers to this question by evaluating 280 American options by various numerical methods and Bjerksund and Stensland formulas for approximation values of American options.

The paper is organized as follows: following this introduction, in Section II, we describe the binomial and trinomial model for valuing options. Section III presents the applications of finite difference method in option pricing. In

Section IV we describe Bjerksund and Stensland formulas for approximation values of American options. In Section V we conduct a comparative analysis of specified numerical methods and approximation formulas. Section VI summarizes the paper and indicates the possible directions for further research.

II. Binomial and Trinomial Model for Valuing Options

A. Binomial Model

The procedure followed by binomial model is to assume that the stock price follows a discrete time process. The life of the option T – t is decomposed into n equal time steps of length (Δt = (T – t)/n). At each time interval (tj = j·Δt, j = 0, 1, ..., n), it is assumed that the underlying instrument will move up or down by a specific factor (u or d where, by definition $u \geq 1$, and $0 < d \leq 1$) per step of the tree with probability p, 1-p respectively . So, if S is the current price, then in the next period the price will either be $S_{up} = S \cdot u$ or $S_{down} = S \cdot d$. The binomial tree of stock's price is best illustrated in a Fig. 1.

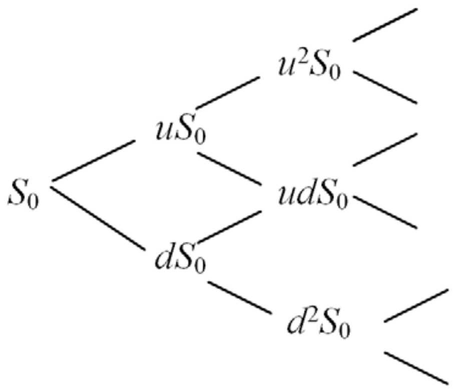

Fig. 1 Binomial tree

The up and down jump factors and corresponding probabilities are chosen to match the first two moments of the stock price distribution (mean and variance). There are, however, more unknowns than there are equations in this set of restrictions, implying that there are many ways of choosing the parameters and still satisfy the moment restrictions. Cox, Ross and Rubinstein [7] set the up and down parameters to

$$u = e^{\sigma\sqrt{\Delta t}}, \; d = e^{-\sigma\sqrt{\Delta t}},$$

where σ is volatility of the relative price change of the underlying stock price. The probability of the stock price increasing at the next time step is:

$$p = \frac{e^{r\Delta t} - d}{u - d},$$

where r is risk-free interest rate.

At each final node of the tree i.e. at expiration of the option the option value is simply its intrinsic, or exercise, value

Max [($S_n - K$), 0], for a call option

Max [($K - S_n$), 0], for a put option,

where K is the strike price and S_n is the spot price of the underlying asset at the n^{th} period.

Once the above step is complete, the option value is then found for each node, starting at the penultimate time step, and working back to the first node of the tree (the valuation date) where the calculated result is the value of the option.

Under the risk neutrality assumption, today's fair price of a derivative is equal to the expected value of its future payoff discounted by the risk free rate. Therefore, expected value is calculated using the option values from the later two nodes (Option up and Option down) weighted by their respective probabilities (probability p of an up move in the underlying, and probability 1-p of a down move). The expected value is then discounted at r, the risk free rate corresponding to the life of the option.

The following formula to compute the expectation value is applied at each node:

$$C_{t-\Delta t,i} = e^{-r\Delta t}\left(pC_{t,i+1} + (1-p)C_{t,i}\right)$$

where $C_{t,i}$ is the option's value for the i^{th} node at time t.

This result is the "Binomial Value". It represents the fair price of the derivative at a particular point in time (i.e. at each node), given the evolution in the price of the underlying asset to that point. It is the value of the option if it were to be held—as opposed to exercised at that point.

For an American option, since the option may either be held or exercised prior to expiry, the value at each node is: Max (Binomial Value, Exercise Value). The value of the initial node presents the required fair price of the option.

B. Trinomial Model

Under the trinomial model, in each period, the prices can go up, down or remain unchanged. The term "lattice" implies two or more branches protruding from the node of a tree. In the case of a binomial lattice there are two branches, three in the case of a trinomial, and so on. Where there are more than two branches, the lattice can be called a multinomial lattice.

A trinomial lattice works on the same principles as the binomial lattice, but assumes that the prices may also remain constant. So in the first step, the prices may go up, down or remain unchanged. For each of the three outcomes, there will be three outcomes each in the second time step, but the second outcome of the first node in the second step will be the same as the first outcome of the second node in the second step and so on.

The expected results are attained much faster, as the branches become intractable at a much earlier period of time. Trinomial trees can be used as an alternative to binomial trees, where there are numerous time steps. It is to be noted that the trinomial tree computation procedure is exactly the same as for the binomial model.

As the name suggests, trinomial model uses a similar approach to be binomial one. But the hedging and replication arguments do not take place in constructing trinomial trees. For a non - dividend paying stock, parameter values that match the mean and standard deviation of price changes are given below:

$$u = e^{\sigma\sqrt{3\Delta t}}, d = 1/u,$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6},$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{6},$$

$$p_m = 1 - p_u - p_d,$$

where u, d and r have the same meaning as in binomial model, σ is stock volatility, while p_u, p_d and p_m denote probabilities of the price going up, down or remaining unchanged, respectively.

Once the tree of prices has been calculated, the option price is found at each node largely as for the binomial model, by working backwards from the final nodes to today. The difference being that the option value at each non-final node is determined based on the three (as opposed to two) later nodes and their corresponding probabilities.

III. Finite Difference Method in Option Pricing

The finite difference method is basically a numerical approximation of the partial difference equation. Here we will give overview of the three most common finite difference techniques in option pricing: explicit finite difference, implicit finite difference, Crank-Nicolson finite difference. In all the finite difference models first we built a grid with time along one dimension/axis and price along the other dimension/axis. Time increases in increments of Δt, while the asset changes in amount of ΔS. These increments are then used to construct a grid of possible combinations of time and asset price levels. The finite difference technique is then used to approximately solve the relevant PDE on this grid. Just as in a tree model one starts at the end of the grid at time T, and rolls back through the grid. The finite difference models can be used to solve a large class of options. If we assume that the underlying asset follows a geometric Brownian motion, we get the following Black-Scholes-Merton PDE ([4], [11]) for any single asset derivatives:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (r - q) S \frac{\partial f}{\partial S} - rf = 0 \qquad (1)$$

where f is the value of a derivative security. We want to solve this PDE along the grid for the particular derivative instrument under consideration. How this is done will depend on the chosen finite difference techniques, as well as derivative's contractual details. Here we will give an overview of the three most common finite difference techniques in option pricing: explicit finite difference method, implicit finite difference method and Crank-Nicolson finite difference. The explicit finite difference method is more or less a generalization of the trinomial tree. The method approximates the PDE in (1) by using numerical differentiation (see [10]). The $\frac{\partial f}{\partial t}$ is approximated by using the forward difference (naturally, because time can only move forward):

$$\frac{\partial f}{\partial t} \approx \frac{f_{j+1,i} - f_{j,i}}{\Delta t}$$

where $f_{j,i}$ is the value of the derivative instrument at time step j and price level i.

The delta and the gamma approximated by central differences (the asset price can naturally move in both directions):

$$\frac{\partial f}{\partial S} \approx \frac{f_{j+1,i+1} - f_{j+1,i-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2} \approx \frac{f_{j+1,i+1} - 2f_{j+1,i} + f_{j+1,i-1}}{\Delta S^2}.$$

The implicit finite difference method is closely related to the explicit finite difference method. The main difference is that we approximate $\frac{\partial f}{\partial S}$ and $\frac{\partial^2 f}{\partial S^2}$ in PDE (1) by central differentiation at time step j instead at $j+1$ as in the explicit finite difference method.

In Crank-Nicolson method the approximation of the PDE is done by central differences at time step $j+\frac{1}{2}$ instead of at $j+1$ as in explicit finite difference method, or at point j as in the implicit finite difference method.

As we can see, the Crank-Nicolson method is a combination of the explicit and implicit methods. It is more efficient than the others. In combination with the same boundary conditions as in the implicit finite difference method, the Crank-Nicolson method will make up a tridiagonal system of equations. For an in-depth discussion of the Crank-Nicolson method applied to derivatives valuation, see [13].

IV. The Bjerksund and Stensland (1993) and (2002) Approximation

The Bjerksund and Stensland, 1993 approximation can be used to price American options on stocks, futures and currencies. Bjerksund and Stensland's approximation is based on an exercise strategy corresponding to a flat boundary I (trigger price).

Given this feasible but non-optimal strategy, the American call boils down to: (i) a European up-and-out call with knock-out barrier I, strike K, and maturity date T; and (ii) a rebate I-K that is received at the knock-out date if the option is knocked out prior to the maturity date.

Their American call approximation is

$$c=\alpha S^{\beta}-\alpha\phi(S,T,\beta,I,I)+\phi(S,T,1,I,I)-\phi(S,T,1,K,I)-$$
$$-K\phi(S,T,0,I,I)+K\phi(S,T,0,K,I),$$

where

$$\alpha=(I-K)I^{-\beta},$$
$$\beta=\left(\frac{1}{2}-\frac{b}{\sigma^2}\right)+\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+2\frac{r}{\sigma^2}}.$$

The function $\phi(S,T,\gamma,H,I)$ is given by:

$$\phi(S,T,\gamma,H,I)=e^{\lambda}S^{\gamma}\left[N(d)-\left(\frac{I}{S}\right)^{\kappa}N\left(d-\frac{2\ln\left(\frac{I}{S}\right)}{\sigma\sqrt{T}}\right)\right],$$
$$\lambda=\left[-r+\gamma b+\frac{1}{2}\gamma(\gamma-1)\sigma^2\right]T,$$
$$d=\frac{\ln\left(\frac{S}{H}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\quad \kappa=\frac{2b}{\sigma^2}+(2\gamma-1),$$

and the trigger price I is defined as

$$I=B_0+(B_\infty-B_0)\left(1-e^{h(T)}\right),$$
$$h(T)=-\left(bT+2\sigma\sqrt{T}\right)\left(\frac{B_0}{B_\infty-B_0}\right),B_0=\frac{\beta}{\beta-1}K,$$
$$B_\infty=\max\left\{K,\frac{r}{r-b}K\right\}.$$

If S>I, it is optimal to exercise the option immediately and the value must be equal to the intrinsic value of S-X. On the other hand, if $b\geq r$, it will never be optimal to exercise the American call option before expiration, and the value can be found using Black-Scholes formula [4]. The value of the American put is given by Bjerksund and Stensland put-call transformation:

$$p(S,K,T,r,b,\sigma)=c(S,K,T,r-b,-b,\sigma)$$

The Bjerksund and Stensland, 2002 approximation divides the time to maturity into two parts, each with a separate flat exercise boundary. They extend the flat boundary approximation above by allowing for one flat boundary I_1 that is valid from date 0 to date t, and another flat boundary I_2 that is valid from date t to date T, where $0<t<T$. Their American call approximation is:

$$c=\alpha_2 S^{\beta}-\alpha_2\phi(S,t_1,\beta,I_2,I_2)+\phi(S,t_1,1,I_2,I_2)-\phi(S,t_1,1,I_1,I_2)-$$
$$-K\phi(S,t_1,0,I_2,I_2)+K\phi(S,t_1,0,I_1,I_2)+\alpha_1\phi(S,t_1,\beta,I_1,I_2)-$$
$$-\alpha_1\Psi(S,T,\beta,I_1,I_2,I_1,t_1)+\Psi(S,T,1,I_1,I_2,I_1,t_1)-$$
$$-\Psi(S,T,1,K,I_2,I_1,t_1)-K\Psi(S,T,0,I_1,I_2,I_1,t_1)+$$
$$+\Psi(S,T,0,K,I_2,I_1,t_1),$$

where:

$$\alpha_1=(I_1-K)I_1^{-\beta},\ \alpha_2=(I_2-K)I_2^{-\beta},\beta=\left(\frac{1}{2}-\frac{b}{\sigma^2}\right)+\sqrt{\left(\frac{b}{\sigma^2}-\frac{1}{2}\right)^2+2\frac{r}{\sigma^2}}.$$

The function $\phi(S,T,\gamma,H,I)$ is given by:

$$\phi(S,T,\gamma,H,I)=e^{\lambda}S^{\gamma}\left[N(-d)-\left(\frac{I}{S}\right)^{\kappa}N(-d_2)\right],$$
$$\lambda=-r+\gamma b+\frac{1}{2}\gamma(\gamma-1)\sigma^2,$$
$$d=\frac{\ln\left(\frac{S}{H}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},$$
$$d_2=\frac{\ln\left(\frac{I^2}{SH}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\ \kappa=\frac{2b}{\sigma^2}+(2\gamma-1),$$

The trigger price I is defined as:

$$I_1=B_0+(B_\infty-B_0)\left(1-e^{h_1}\right),$$
$$I_2=B_0+(B_\infty-B_0)\left(1-e^{h_2}\right),$$
$$h_1=-\left(bt_1+2\sigma\sqrt{t_1}\right)\left(\frac{K^2}{(B_\infty-B_0)B_0}\right),$$
$$h_2=-\left(bT+2\sigma\sqrt{T}\right)\left(\frac{K^2}{(B_\infty-B_0)B_0}\right),$$
$$t_1=\frac{1}{2}\left(\sqrt{5}-1\right)T,$$
$$B_0=\frac{\beta}{\beta-1}K,$$
$$B_\infty=\max\left\{K,\frac{r}{r-b}K\right\}.$$

Moreover, the function $\Psi(S,T,\gamma,H,I_2,I_1,t_1,r,b,\sigma)$ is given by:

$$\Psi\left(S,T,\gamma,H,I_2,I_1,t_1,r,b,\sigma\right)=e^{\lambda T}S^{\gamma}\cdot$$
$$\cdot\left[M\left(-e_1,-f_1,\sqrt{\frac{t_1}{T}}\right)-\left(\frac{I_2}{S}\right)^{\kappa}M\left(-e_2,-f_2,\sqrt{\frac{t_1}{T}}\right)-\right.$$
$$\left.-\left(\frac{I_1}{S}\right)^{\kappa}M\left(-e_3,-f_3,\sqrt{\frac{t_1}{T}}\right)+\left(\frac{I_1}{I_2}\right)^{\kappa}M\left(-e_4,-f_4,\sqrt{\frac{t_1}{T}}\right)\right],$$

where $M(\cdot,\cdot,\cdot)$ cumulative bivariate normal distribution and

$$e_1=\frac{\ln\left(\frac{S}{I_1}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},\quad e_2=\frac{\ln\left(\frac{I_2^2}{SI_1}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},$$
$$e_3=\frac{\ln\left(\frac{S}{I_1}\right)-\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},\quad e_4=\frac{\ln\left(\frac{I_2^2}{SI_1}\right)-\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]t_1}{\sigma\sqrt{t_1}},$$
$$f_1=\frac{\ln\left(\frac{S}{H}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\quad f_2=\frac{\ln\left(\frac{I_2^2}{SH}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},$$
$$f_3=\frac{\ln\left(\frac{I_1^2}{SH}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}},\quad f_4=\frac{\ln\left(\frac{SI_1^2}{HI_2^2}\right)+\left[b+\left(\gamma-\frac{1}{2}\right)\sigma^2\right]T}{\sigma\sqrt{T}}.$$

The computer code for Bjerksund and Stensland, 1993 and 2002 American option approximation is taken from [8].

V. Comparison of Numerical Methods and Bjerksund - Stensland Approximations

Comparative analysis of observed models will be carried out by their application to pricing American put options on nondividend-paying stocks. We will compare the Bjerksund and Stensland 1993 and 2002 approximation with binomial model, trinomial model and Crank-Nicolson finite difference method. We will limit computer calculation time of numerical method to less than one second, which nearly corresponds to the calculation time of the Bjerksund and Stensland approximation.

Since there is no formula that can calculate the exact value of American options offer, for the calculation of reference value, we will use trinomial model with a very large number of steps (5000 steps) that achieves high precision and the resulting value can be considered accurate. The calculation of the reference value using trinomial model in this analysis required over 100 hours of computer processing. The values obtained by the observed models are compared with the reference values. Errors of each particular model will be represented by the absolute value of the difference between the values obtained by the observed model and the reference value (on the same way as in [9]).

The survey is conducted by evaluating 280 American options with the exercise price of 150, and the volatility of 25%, with a risk-free interest rate of 6%. Time to maturity takes values of the interval $\left[0.05,1\right]$, and the current price of the stock values are taken from the interval $\left[50,180\right]$.

The option values obtained by the analysis are given in Tables I-VI. In applying the binomial and trinominal model, as well as the Crank-Nicolson finite difference method, the biggest number (rounded to the tens) was taken for the number of periods, for which computer computation is less than one second.

The main aim is to find out whether the errors in the observed methods differ significantly.

For this purpose, we will apply the Friedman non-parametric test.

This test is used for more than two dependent variable samples measured using the sequence scale. The following hypotheses are set:

H0. There is no difference in the rank of model errors,

H1. There is a difference in the rank of model errors.

Fig. 2 indicates the results of the conducted Friedman test. Friedman test was used to test the differences in the error ranks for all five models based on the results obtained for the option offer. The obtained results show that in both cases there is a difference in ranks of error for the observed models, i.e. the initial hypothesis H_0 is rejected.

The binomial model has shown to be the best, followed by the trinomial, Crank-Nicolson finite difference method, and Bjerksund-Stensland model 2002, with the Bjerksund-Stensland model 1993 taking the last position.

TABLE I
EVALUATING THE AMERICAN PUT OPTIONS FROM THE SAMPLE USING THE TRINOMINAL MODEL (N=5000)

		Asset price						
		120	130	140	150	160	170	180
Time to maturity	0,05	30,00000	20,00000	10,19847	3,15076	0,47987	0,03346	0,00110
	0,1	30,00000	20,00000	10,73557	4,34832	1,25706	0,25606	0,03739
	0,15	30,00000	20,00815	11,26361	5,22684	1,95552	0,58906	0,14462
	0,2	30,00000	20,07938	11,74576	5,94098	2,57269	0,95582	0,30776
	0,25	30,00000	20,19474	12,18433	6,55056	3,12455	1,32659	0,50549
	0,3	30,00000	20,33329	12,58522	7,08619	3,62410	1,68963	0,72317
	0,35	30,00000	20,48377	12,95450	7,56603	4,08074	2,04012	0,95110
	0,4	30,00000	20,63969	13,29662	8,00188	4,50215	2,37668	1,18346
	0,45	30,00000	20,79738	13,61572	8,40195	4,89336	2,69902	1,41655
	0,5	30,00000	20,95455	13,91454	8,77218	5,25892	3,00771	1,64802
	0,55	30,00252	21,10980	14,19563	9,11706	5,60216	3,30326	1,87626
	0,6	30,01404	21,26219	14,46096	9,44008	5,92563	3,58663	2,10051
	0,65	30,03380	21,41166	14,71236	9,74400	6,23132	3,85824	2,32038
	0,7	30,05967	21,55743	14,95144	10,03107	6,52212	4,11939	2,53533
	0,75	30,09046	21,69955	15,17871	10,30311	6,79823	4,37049	2,74496
	0,8	30,12525	21,83806	15,39616	10,56167	7,06211	4,61209	2,94982
	0,85	30,16341	21,97301	15,60362	10,80805	7,31371	4,84520	3,14959
	0,9	30,20373	22,10444	15,80301	11,04336	7,55549	5,07012	3,34429
	0,95	30,24636	22,23236	15,99396	11,26855	7,78698	5,28714	3,53410
	1	30,29034	22,35679	16,17769	11,48444	8,00939	5,49662	3,71918

TABLE II
EVALUATING THE AMERICAN PUT OPTIONS FROM THE SAMPLE USING THE BJERKSUND-STENDSLAND (1993) MODEL

		Asset price						
		120	130	140	150	160	170	180
Time to maturity	0,05	30,00000	20,00000	10,17537	3,13141	0,47725	0,03334	0,00110
	0,1	30,00000	20,00000	10,68140	4,30999	1,24605	0,25417	0,03717
	0,15	30,00000	20,00110	11,18646	5,17112	1,93365	0,58305	0,14337
	0,2	30,00000	20,04965	11,65041	5,86954	2,53929	0,94409	0,30431
	0,25	30,00000	20,14402	12,07379	6,46493	3,07945	1,30789	0,49888
	0,3	30,00000	20,26491	12,46203	6,98779	3,56752	1,66314	0,71235
	0,35	30,00000	20,40066	12,82041	7,45614	4,01341	2,00560	0,93546
	0,4	30,00000	20,54434	13,15332	7,88169	4,42440	2,33399	1,16250
	0,45	30,00000	20,69174	13,46431	8,27250	4,80604	2,64827	1,38990
	0,5	30,00000	20,84025	13,75623	8,63444	5,16257	2,94899	1,61546
	0,55	30,00000	20,98821	14,03140	8,97191	5,49736	3,23691	1,83781
	0,6	30,00276	21,13457	14,29174	9,28830	5,81309	3,51284	2,05612
	0,65	30,01390	21,27867	14,53882	9,58632	6,11196	3,77760	2,26989
	0,7	30,03210	21,42009	14,77398	9,86813	6,39579	4,03198	2,47887
	0,75	30,05606	21,55859	14,99835	10,13552	6,66610	4,27668	2,68295
	0,8	30,08470	21,69403	15,21289	10,38996	6,92418	4,51238	2,88210
	0,85	30,11716	21,82635	15,41844	10,63273	7,17113	4,73967	3,07639
	0,9	30,15273	21,95554	15,61573	10,86486	7,40789	4,95910	3,26589
	0,95	30,19080	22,08162	15,80538	11,08730	7,63529	5,17117	3,45073
	1	30,23091	22,20465	15,98797	11,30081	7,85406	5,37634	3,63104

TABLE III
EVALUATING THE AMERICAN PUT OPTIONS FROM THE SAMPLE USING THE BJERKSUND-STENSLAND (2002) MODEL

		Asset price						
		120	130	140	150	160	170	180
Time to maturity	0,05	30,00000	20,00000	10,18504	3,13842	0,47776	0,03335	0,00110
	0,1	30,00000	20,00000	10,70275	4,32352	1,24882	0,25446	0,03719
	0,15	30,00000	20,00192	11,21543	5,19049	1,93969	0,58423	0,14353
	0,2	30,00000	20,05988	11,68517	5,89411	2,54901	0,94675	0,30488
	0,25	30,00000	20,16173	12,11320	6,49413	3,09293	1,31248	0,50015
	0,3	30,00000	20,28857	12,50527	7,02110	3,58470	1,66997	0,71462
	0,35	30,00000	20,42911	12,86686	7,49311	4,03413	2,01486	0,93899
	0,4	30,00000	20,57670	13,20248	7,92190	4,44849	2,34576	1,16748
	0,45	30,00000	20,72732	13,51576	8,31560	4,83327	2,66257	1,39649
	0,5	30,00000	20,87850	13,80963	8,68009	5,19272	2,96580	1,62377
	0,55	30,00000	21,02868	14,08646	9,01982	5,53021	3,25616	1,84790
	0,6	30,00535	21,17689	14,34819	9,33821	5,84842	3,53445	2,06803
	0,65	30,01939	21,32254	14,59644	9,63798	6,14958	3,80147	2,28363
	0,7	30,04016	21,46525	14,83258	9,92134	6,43550	4,05800	2,49442
	0,75	30,06638	21,60482	15,05775	10,19008	6,70771	4,30474	2,70028
	0,8	30,09703	21,74114	15,27295	10,44570	6,96752	4,54235	2,90117
	0,85	30,13127	21,87418	15,47901	10,68948	7,21604	4,77144	3,09714
	0,9	30,16841	22,00394	15,67670	10,92249	7,45421	4,99255	3,28826
	0,95	30,20787	22,13047	15,86664	11,14566	7,68289	5,20618	3,47466
	1	30,24920	22,25384	16,04942	11,35980	7,90280	5,41280	3,65646

TABLE IV
EVALUATING THE AMERICAN PUT OPTIONS FROM THE SAMPLE USING THE BINOMINAL MODEL (N=350)

		Asset price						
		120	130	140	150	160	170	180
Time to maturity	0,05	30,00000	20,00000	10,19905	3,14915	0,47988	0,03310	0,00108
	0,1	30,00000	20,00000	10,73589	4,34628	1,25886	0,25574	0,03726
	0,15	30,00000	20,00773	11,26556	5,22452	1,95659	0,58774	0,14459
	0,2	30,00000	20,07851	11,74720	5,93846	2,57603	0,95751	0,30741
	0,25	30,00000	20,19513	12,18542	6,54787	3,12583	1,32843	0,50610
	0,3	30,00000	20,33354	12,58790	7,08335	3,62837	1,69249	0,72422
	0,35	30,00000	20,48266	12,95861	7,56305	4,07923	2,03889	0,95270
	0,4	30,00000	20,64152	13,29634	7,99880	4,50425	2,38003	1,18397
	0,45	30,00000	20,79536	13,61625	8,39877	4,89911	2,69582	1,41589
	0,5	30,00000	20,95667	13,91870	8,76892	5,26463	3,01202	1,65048
	0,55	30,00000	21,11104	14,20110	9,11371	5,60535	3,30652	1,87639
	0,6	30,01409	21,26105	14,46596	9,43666	5,92498	3,58333	2,10171
	0,65	30,03241	21,41378	14,71539	9,74051	6,22770	3,86069	2,32419
	0,7	30,05792	21,56030	14,95113	10,02750	6,52274	4,12527	2,53458
	0,75	30,09084	21,70058	15,17562	10,29946	6,80229	4,37606	2,74487
	0,8	30,12435	21,83530	15,39544	10,55796	7,06798	4,61475	2,95430
	0,85	30,16074	21,97351	15,60558	10,80426	7,32124	4,84284	3,15380
	0,9	30,20302	22,10717	15,80654	11,03948	7,56311	5,06833	3,34467
	0,95	30,24708	22,23600	15,99896	11,26460	7,79462	5,29013	3,52932
	1	30,29083	22,36017	16,18358	11,48044	8,01666	5,50280	3,72087

TABLE V
EVALUATING THE AMERICAN PUT OPTIONS FROM THE SAMPLE USING THE TRINOMINAL MODEL (N=100)

		Asset price						
		120	130	140	150	160	170	180
Time to maturity	0,05	30,00000	20,00000	10,19825	3,14746	0,47911	0,03304	0,00103
	0,1	30,00000	20,00000	10,73506	4,34383	1,25827	0,25513	0,03699
	0,15	30,00000	20,00346	11,25985	5,22140	1,95855	0,58726	0,14413
	0,2	30,00000	20,07530	11,74857	5,93480	2,56940	0,95335	0,30810
	0,25	30,00000	20,19105	12,18049	6,54370	3,13005	1,32292	0,50548
	0,3	30,00000	20,33235	12,58898	7,07865	3,62876	1,69367	0,72438
	0,35	30,00000	20,47753	12,95696	7,55781	4,07649	2,03364	0,95214
	0,4	30,00000	20,63658	13,29196	7,99310	4,50051	2,38195	1,18100
	0,45	30,00000	20,78916	13,60944	8,39265	4,89876	2,70119	1,41891
	0,5	30,00000	20,95150	13,91357	8,76237	5,26711	2,99895	1,64913
	0,55	30,00000	21,10956	14,19801	9,10670	5,61026	3,30600	1,87051
	0,6	30,00000	21,25993	14,46479	9,42915	5,93174	3,59367	2,10467
	0,65	30,01188	21,40273	14,71600	9,73249	6,23434	3,86434	2,32448
	0,7	30,04478	21,55168	14,95357	10,01905	6,52075	4,12035	2,53260
	0,75	30,08104	21,69705	15,17882	10,29062	6,79247	4,36384	2,73814
	0,8	30,11786	21,83681	15,39303	10,54880	7,05161	4,60438	2,95212
	0,85	30,15865	21,97092	15,59722	10,79485	7,30321	4,84506	3,15566
	0,9	30,20119	22,09973	15,79234	11,02985	7,54903	5,07466	3,34992
	0,95	30,24326	22,22363	15,97915	11,25474	7,78412	5,29440	3,53604
	1	30,28451	22,34286	16,16452	11,47026	8,00937	5,50504	3,71516

TABLE VI
EVALUATING THE AMERICAN PUT OPTIONS FROM THE SAMPLE USING THE CRANK-NICOLSON MODEL (N=150; M=150)

		Asset price						
		120	130	140	150	160	170	180
Time to maturity	0,05	30,00000	20,00000	10,19673	3,14138	0,47926	0,03422	0,00120
	0,1	30,00000	20,00000	10,72998	4,33476	1,25531	0,25698	0,03847
	0,15	30,00000	20,00122	11,25180	5,21001	1,95298	0,58814	0,14649
	0,2	30,00000	20,07238	11,73842	5,92137	2,56189	0,95233	0,31046
	0,25	30,00000	20,18465	12,16851	6,52851	3,12039	1,32056	0,50739
	0,3	30,00000	20,32522	12,57519	7,06189	3,61778	1,68956	0,72539
	0,35	30,00000	20,46933	12,94124	7,53969	4,06389	2,02739	0,95191
	0,4	30,00000	20,62565	13,27441	7,97367	4,48572	2,37461	1,17998
	0,45	30,00000	20,77601	13,59137	8,37199	4,88287	2,69297	1,41632
	0,5	30,00000	20,93751	13,89462	8,74057	5,25039	2,98835	1,64589
	0,55	30,00000	21,09411	14,17784	9,08385	5,59274	3,29428	1,86526
	0,6	30,00000	21,24302	14,44345	9,40527	5,91339	3,58133	2,09867
	0,65	30,00188	21,38453	14,69358	9,70765	6,21515	3,85124	2,31784
	0,7	30,02923	21,53359	14,92995	9,99327	6,50055	4,10633	2,52499
	0,75	30,06405	21,67833	15,15414	10,26399	6,77126	4,34870	2,72824
	0,8	30,10029	21,81724	15,36731	10,52135	7,02897	4,58666	2,94170
	0,85	30,14155	21,95049	15,57050	10,76664	7,27844	4,82683	3,14474
	0,9	30,18461	22,07841	15,76461	11,00088	7,52367	5,05602	3,33846
	0,95	30,22723	22,20131	15,95042	11,22495	7,75824	5,27525	3,52395
	1	30,26906	22,31957	16,13626	11,43970	7,98308	5,48541	3,70209

```
» [p,table,stats] = friedman(Y)

p =

     0


table =

    'Source'     'SS'                       'df'     'MS'                               'Chi-sq'                       'Prob>Chi-sq'
    'Columns'    [1.026192307692308e+003]   [  4]    [2.565480769230769e+002]           [4.152684824902723e+002]       [          0]
    'Error'      [2.588076923076924e+002]   [516]    [       0.50156529516995]                                 []                  []
    'Total'      [                   1285]  [649]                          []                                  []                  []


stats =

       source: 'friedman'
            n: 130
    meanranks: [4.84230769230769 3.85000000000000 1.32692307692308 2.01153846153846 2.96923076923077]
        sigma: 1.57199040905275
```

Fig. 2 Results of the Friedman test for the error sample obtained by the Bjerksund-Stensland 1993 model, Bjerksund-Stensland 2002 model, binominal, trinominal and Crank-Nicolson model in evaluating American options

VI. Conclusion

Taking into account the development of computer technology, i.e. architecture improvements and the increased speed of the new computer models, it is clear that the calculation accuracy of numerical methods in the same time period will be significantly higher on the modern computers than it was at the time when Bjerksund-Stensland models were published. The results of this study confirmed our assumptions and proved that the numerical methods provide a greater precision of calculations when compared to the Bjerksund-Stensland model if the computation time is limited to one second. Out of the set of numerical methods presented for the evaluation of plain vanilla American options, it was the binomial model that proved to be the most precise, followed by the trinomial model and the Crank-Nicolson finite difference method.

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