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Some Application of Random Fuzzy Queueing System Based On Fuzzy Simulation

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Abstract—This paper studies a random fuzzy queueing system that the interarrival times of customers arriving at the server and the service times are independent and identically distributed random fuzzy variables. We match the random fuzzy queueing system with the random fuzzy alternating renewal process and we do not use from α-pessimistic and α-optimistic values to estimate the average chance of the event "random fuzzy queueing system is busy at time *t*", we employ the fuzzy simulation method in practical applications. Some theorem is proved and finally we solve a numerical example with fuzzy simulation method.

Keywords—Random fuzzy variables, Fuzzy simulation, Queueing system, Interarrival times.

I. Introduction

QUEUEING system is the most applications of alternating renewal process and has been widely applied to many practical problems. In classic queueing systems, the interarrival times of customers and service times of servers are characterized as random variables. In such cases, Ning, Tang, and Zhao[1] proposed the concept of random fuzzy queueing systems with finite capacity, where the interarrival times and service times were characterized as random fuzzy variables. We proposes a theorem on the limit value of the average chance of the random fuzzy event "the queueing system is busy at time *t*". In this paper we use fuzzy simulation to estimate he average chance of the random fuzzy event "the queueing system is busy at time *t*" instead of the α_0-pessimistic value and the α_0-optimistic value method.
In Section II, we introduce some basic definition about credibility measure and random fuzzy variables. In Section III, we consider the random fuzzy queueing system and obtain the *Ch* {random fuzzy queueing system is busy at time *t*} and in Section IV we illustrate the fuzzy simulation and in Section V, an example is solved by the simulation method.

II. Definitions and preliminaries

Credibility theory, founded by Liu [2] in 2004 and refined by Liu [3] in 2007, is a branch of mathematics for studying the behavior of fuzzy phenomena. The emphasis in this section is mainly on credibility measure, credibility space, fuzzy variable, membership function, credibility distribution, expected value, random fuzzy variable and its expected value, independence, identical distribution,

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Let Θ be a nonempty set, and P the power set of Θ (i.e., the largest σ- algebra over Θ). Each element in P is called an *event*. In order to present an axiomatic definition of credibility, it is necessary to assign to each event A a number $Cr\{A\}$ which indicates the credibility that A will occur. In order to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect a credibility to have, we accept the following four axioms:
Axiom 1. (*Normality*) $Cr\{\Theta\} = 1$.
Axiom 2. (*Monotonicity*)$Cr\{A\} \leq Cr\{B\}$ for $A \subset B$.
Axiom 3. (*Self-Duality*) $Cr\{A\} + Cr\{A^c\} = 1$ for any event A.
Axiom 4. (*Maximality*) $C\{\cup_i A_i\} = \sup_i Cr\{A_i\}$ for any events $\{A_i\}$ with $\sup_i Cr\{A_i\} < 0.5$.

Definition 1 (Liu and Liu [4]) The set function Cr is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

Definition 2 Let Θ be a nonempty set, P the power set of Θ, and Cr a credibility measure. Then the triplet (Θ, P, Cr) is called a credibility space.

Product credibility measure may be defined in multiple ways. We accepts the following axiom.
Axiom 5. (*Product Credibility Axiom*) Let Θ_k be nonempty sets on which Cr_k are credibility measures, $k = 1, 2, ..., n$, respectively, and $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$. Then

$$Cr\{(\theta_1, \theta_2, ..., \theta_n)\} = Cr_1\{\theta_1\} \wedge Cr_2\{\theta_2\} \wedge ... \wedge Cr_n\{\theta_n\} \tag{1}$$

for each $(\theta_1, \theta_2, ..., \theta_n) \in \Theta$.
Let $(\theta_k, P_k, Cr_k), k = 1, 2, ..., n$ be credibility spaces, $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$ and $Cr_1 \wedge Cr_2 \wedge ... \wedge Cr_n$. Then (Θ, P, Cr) is called the product credibility space of $(\theta_k, P_k, Cr_k), k = 1, 2, ..., n$.

Definition 3 A fuzzy variable is a measurable function from a credibility space (Θ, P, Cr) to the set of real numbers.

Example 1. Take (Θ, P, Cr) to be $\{\theta_1, \theta_2\}$ with $Cr\{\theta_1\} = Cr\{\theta_2\} = 0.5$. Then the function $\xi(\theta) = \begin{cases} 0, & \theta = \theta_1 \\ 1, & \theta = \theta_2 \end{cases}$ is a fuzzy variable.

Definition 4 A fuzzy variable ξ is said to be
(a) *Nonnegative* if $Cr\{\xi < 0\} = 0$;
(b) *Positive* if $Cr\{\xi \leq 0\} = 0$;
(c) *Continuous* if $Cr\{\xi = x\}$ is a continuous function of x.

Definition 5 Let ξ be a fuzzy variable defined on the credibility space (Θ, P, Cr). Then its membership function is derived from the credibility measure by

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1, x \in \Re. \tag{2}$$

Membership function represents the degree that the fuzzy variable ξ takes some prescribed value. There are several methods reported in the past literature. Anyway, the membership degree $\mu(x) = 0$ if x is an impossible point, and $\mu(x) = 1$ if x is the most possible point that ξ takes.

Definition 6 Let ξ be a fuzzy variable defined on the credibility space $(\Theta, P(\Theta), Pos)$, and $\alpha \in (0, 1]$. Then

$$\xi'_\alpha = \inf\{r|\mu_\xi(r) \geq \alpha\}, \xi''_\alpha = \sup\{r|\mu_\xi(r) \geq \alpha\}, \tag{3}$$

are called α-pessimistic value and α-optimistic value of ξ, respectively.

Definition 7(Liu [5]) The credibility distribution $\Phi : \Re \to [0, 1]$ of a fuzzy variable ξ is defined by

$$\Phi(x) = Cr\{\theta \in \Theta|\xi(\theta) \leq x\}. \tag{4}$$

That is, $F(x)$ is the credibility that the fuzzy variable ξ takes a value less than or equal to x. Generally speaking, the credibility distribution F is neither left-continuous nor right-continuous.

There are many ways to define an expected value operator for fuzzy variables. The most general definition of expected value operator of fuzzy variable was given by Liu and Liu [5]. This definition is applicable to not only continuous fuzzy variables but also discrete ones.

Definition 8 Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr, \tag{5}$$

provided that at least one of the two integrals is finite.
In particular, if the fuzzy variable ξ is positive (i.e. $Cr\{\xi \leq 0\} = 0$), then

$$E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr. \tag{6}$$

Proposition 1. Let ξ be a fuzzy variable defined on the credibility space $(\Theta, P(\Theta), Pos)$. Then we have

$$E[\xi] = \frac{1}{2}\int_0^1 [\xi'_\alpha + \xi''_\alpha]d\alpha. \tag{7}$$

Proof. Let ξ is normalized, i.e., there exists a real number r_0 such that $\mu_\xi(r_0) = 1$ and if $r_0 > 0$, then the equation (5) can be rewritten as

$$E[\xi] = \frac{1}{2}[r_0 + \int_{r_0}^{+\infty} Cr(\xi \geq r)dr + r_0 - \int_{-\infty}^{r_0} Cr(\xi \leq r)dr]$$
$$= \frac{1}{2}\int_0^1 (\xi'_\alpha + \xi''_\alpha)d\alpha,$$

The same result can be obtained when $r_0 > 0$.•

Definition 9 A random fuzzy variable is a function from the credibility space (Θ, P, Cr) to the set of random variables.

Definition 10 The expected value of a random fuzzy variable ξ is defined by

$$E[\xi] = \int_0^\infty Cr\{\theta \in \Theta|E[\xi(\theta)] \geq r\}dr$$
$$- \int_{-\infty}^0 Cr\{\xi \leq r\}Cr\{\theta \in \Theta|E[\xi(\theta)] \leq r\}dr, \tag{8}$$

Proposition 2. Let ξ be a random fuzzy variable defined on (Θ, P, Cr). Then, for any $\theta \in \Theta$, $E[\xi(\theta)]$ is a fuzzy variable provided that $E[\xi(\theta)]$ is finite for fixed $\theta \in \Theta$.

Definition 11 The random fuzzy variables ξ and η are said to be identically distributed if

$$\sup_{Cr\{A\}\geq\alpha} \inf_{\theta\in A}\{Pr\{\xi(\theta) \in B\}\} \sup_{Cr\{A\}\geq\alpha} = \inf_{\theta\in A}\{Pr\{\eta(\theta) \in B\}\} \tag{9}$$

for any $\alpha \in (0, 1]$ and Borel set B of real numbers.

Definition 12 The random fuzzy variables $\xi_i, i = 1, ..., n$ are said to be independent if
(1) $\xi_i(\theta), i = 1, ..., n$ are independent random variables for each $\theta \in \Theta$.
(2) $E[\xi_i(.)], i = 1, ..., n$ are independent fuzzy variables.

Definition 13 Let ξ be a random fuzzy variable on the possibility space $(\Theta, P(\Theta), Cr)$. Then the average chance of random fuzzy event $\xi \leq 0$ is defined as

$$Ch\{\xi \leq 0\} = \int_0^1 Cr\{\theta \in \Theta|Pr\{\xi(\theta) \leq 0\} \geq p\}dp. \tag{10}$$

III. Random fuzzy queueing system

Consider a random fuzzy queueing system with single server, that the interarrival times of customers arriving at the server are independent and identically distributed random fuzzy variables, $\xi_i \sim EXP(\lambda_i)$, where λ_i are fuzzy variables defined on the credibility space $(\Theta_i, P(\Theta_i), Cr_i), i = 1, 2, ...,$ and the service times are independent and identically distributed random fuzzy variables, $\eta_i \sim EXP(\mu_i)$, where μ_i are fuzzy variables defined on credibility space $(\Gamma_i, P(\Gamma_i), Cr'_i), i = 1, 2, ...$ ξ_i and η_i are independent. We denote this random fuzzy queueing system with $RF/RF/1/FCFS/K/\infty$, where *RF* denotes the interarrival times and service times are random fuzzy variables and the queue discipline is first come, first served(*FCFS*).
In classic stochastic queueing systems, the event $\{system\ \ is\ \ busy\ \ at\ \ time\ \ t\}$ is one of the most important applications of queueing systems (see [6,7]). It is clear that the busy period and idle period in the random fuzzy queueing systems follow a random fuzzy alternating renewal process. In random fuzzy alternating renewal processes, the event $\{system\ is\ on\ at\ time\ t\}$, is a random fuzzy event. Therefore, the event

$\{random\ fuzzy\ queueing\ system\ is\ busy\ at\ time\ t\}$ is a random fuzzy event. In what follows, we will concentrate on the average chance of the event random fuzzy queueing system is busy at time t. By definition 8, we have

$$Ch\{random\ fuzzy\ queueing\ system\ is\ busy\ at\ time\ t\}$$

$$= \int_0^1 Cr\{\theta \in \Theta | Pr\{queueing\ system\ is\ busy\ at\ time\ t\} \geq p\}dp$$

Set ***P(t)***=*Pr*$\{queueing\ system\ is\ busy\ at\ time\ t\}$, then ***P(t)*** is a fuzzy variable and if P'_{α_0} and P''_{α_0} are the α_0-pessimistic values and the α_0-optimistic values $P(t)$ and $E[\frac{\lambda}{\mu}] < 1$.

Theorem 1. Assume that, in a random fuzzy queueing system $RF/RF/1/FCFS/K/\infty$, the fuzzy variables λ has the same α_0-pessimistic values and the α_0-optimistic values λ_i and the fuzzy variables μ has the same α_0-pessimistic values and the α_0-optimistic values μ_i and are continuous at the point $\alpha_0, \alpha_0 \in [0, 1]$, then we have

$$\lim_{t\to\infty} Ch\{the\ fuzzy\ queueing\ system\ is\ busy\ at\ time\ t\}$$

$$= E[\frac{\lambda}{\mu}]. \tag{11}$$

Proof. From definition 13 and proposition 1, it follows that

$$Ch\{the\ fuzzy\ queueing\ system\ is\ busy\ at\ time\ t\}$$

$$= \int_0^1 Cr\{\theta \in \Theta | P(t)(\theta) \geq p\}dp$$

$$= \int_0^\infty Cr\{\theta \in \Theta | P(t)(\theta) \geq p\}dp$$

$$E[P(t)] = \frac{1}{2}\int_0^1 (P'_\alpha(t) + P''_\alpha(t))dp$$

The following results have been proved in [8]

$$\lim_{t\to\infty} P'_\alpha(t) = \frac{\lambda'_\alpha}{\mu''_\alpha},$$

and

$$\lim_{t\to\infty} P''_\alpha(t) = \frac{\lambda''_\alpha}{\mu'_\alpha}.$$

It follows from the definition of the limit that there exist two real numbers t_1 and t_2 with $t_1 \geq 0$ and $t_2 \geq 0$ such that for any $t \geq t_1$

$$0 \leq P'_\alpha(t) \leq \frac{\lambda'_\alpha}{\mu''_\alpha},$$

$$0 \leq P''_\alpha(t) \leq \frac{\lambda''_\alpha}{\mu'_\alpha}.$$

Therefor, we have for any $t \geq \max(t_1, t_2)$

$$0 \leq P'_\alpha(t) + P''_\alpha(t) \leq 2 + \frac{\lambda'_\alpha}{\mu''_\alpha} + \frac{\lambda''_\alpha}{\mu'_\alpha}.$$

Since, $E[\frac{\lambda}{\mu}]$ is finite, than $2+\frac{\lambda'_\alpha}{\mu''_\alpha}+\frac{\lambda''_\alpha}{\mu'_\alpha}$ is an integrable function of α. It follows from Fatou's lemma that

$$\liminf_{t\to\infty} \int_0^1 (P'_\alpha(t) + P''_\alpha(t))d\alpha$$

$$\geq \int_0^1 \liminf_{t\to\infty} (P'_\alpha(t) + P''_\alpha(t))d\alpha,$$

and

$$\limsup_{t\to\infty} \int_0^1 (P'_\alpha(t) + P''_\alpha(t))d\alpha$$

$$\leq \int_0^1 \limsup_{t\to\infty} (P'_\alpha(t) + P''_\alpha(t))d\alpha.$$

Since $\lambda'_\alpha, \lambda''_\alpha, \mu'_\alpha, \mu''_\alpha$ are almost surely continuous at point α, then

$$\lim_{t\to\infty} Ch\{the\ fuzzy\ queueing\ system\ is\ busy\ at\ time\ t\}$$

$$= \frac{1}{2}\lim_{t\to\infty}\int_0^1 (P'_\alpha(t)+P''_\alpha(t))dp = \frac{1}{2}\int_0^1 \lim_{t\to\infty}(P'_\alpha(t)+P''_\alpha(t))dp$$

$$= \frac{1}{2}\int_0^1 (\frac{\lambda'_\alpha}{\mu''_\alpha} + \frac{\lambda''_\alpha}{\mu'_\alpha})d\alpha$$

$$= E[\frac{\lambda}{\mu}].$$

The proof is completed.●

Remark 1. It is obvious that

$$\lim_{t\to\infty} Ch\{the\ fuzzy\ queueing\ system\ is\ idle\ at\ time\ t\}$$

$$= 1 - E[\frac{\lambda}{\mu}]. \tag{12}$$

IV. Fuzzy simulation method

In order to evaluate the expected value of a fuzzy variable, Liu and Liu [2] designed a fuzzy simulation for both discrete and continuous cases.

(a) Discrete fuzzy vector: assume that f is a function, and $\xi = (\xi_1, ..., \xi_m)$ is discrete fuzzy vector whose joint possibility distribution function is defined by

$$\mu_\xi(u) = \begin{cases} \mu_1, & u = u_1 \\ \mu_2, & u = u_2 \\ \dots & \\ \mu_n, & u = u_n \end{cases} \tag{13}$$

where $\mu_u = \min_{1\leq i\leq m} \mu^{(i)}(u_i)$ and $u = (u_1, ..., u_m) \in \Re^m$ and $\mu^{(i)}$ are the possibility distribution function of ξ_i for $i = 1, 2, ..., m$.

Let $a_i = f(u_i)$. Without loss of generality ,we assume that $a_1 \leq a_2 \leq ... \leq a_n$,then the expected value is given by

$$E[f(\xi)] = \sum_{i=1}^n a_i p_i, \tag{14}$$

where

$$p_i = \frac{1}{2}(\vee_{j=i}^n \mu_j - \vee_{j=i+1}^{n+1}\mu_j) + \frac{1}{2}(\vee_{j=1}^i \mu_j - \vee_{j=0}^{i-1}\mu_j) \tag{15}$$

where $(\mu_0 = \mu_{n+1} = 0)$ for $i = 1, 2, ..., n$.

(b) Continuous fuzzy vector: assume that ξ is a continuous fuzzy vector with a possibility distribution function μ. In this case, we can estimate the expected value by the formula (16).

TABLE I
THE AVERAGE CHANCE OF RANDOM FUZZY QUEUEING SYSTEM IS BUSY AT TIME *t* WITH FUZZY SIMULATION METHOD

$Number\ of\ iteratins$	$The\ average\ chance$
500	0.6355
1000	0.6370
5000	0.6393
10000	0.6422
20000	0.6423
50000	0.6432
70000	0.6432

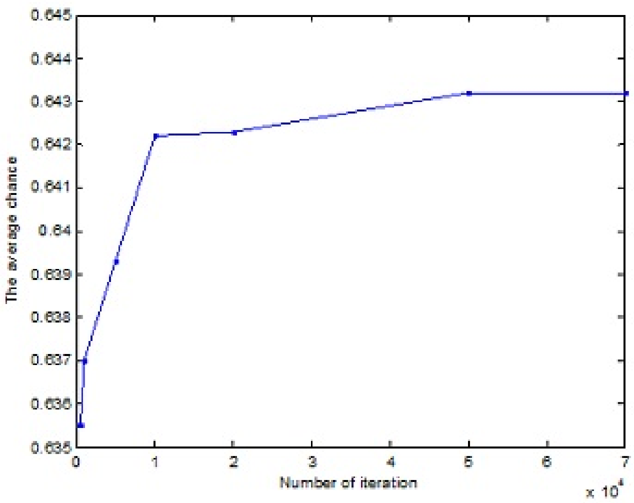

Fig. 1. The convergence of the fuzzy simulation method

V. APPLICATION EXAMPLE

Now, for illustrating the method, we consider an example. We present an application practical of this model how using fuzzy simulation method to estimate the average chance.

Example 1. Let us consider a random fuzzy queueing system. λ and μ are trapezoidal fuzzy variables, $\lambda = (1/2/3/4)$ and $\mu = (3/4/5/6)$. We want to calculate the average chance of "random fuzzy queueing system is busy at time *t*". From Theorem 1, for estimating the $E[\frac{\lambda}{\mu}]$, we use the simulation method in Section IV. The simulation results are shown in Table I and Fig 1.

Table I and Fig 1 show that the average chance of random fuzzy queueing system is busy at time *t* after 70000 times is equal 0.6432, in fact it remains at 0.6432, level. Also, from Remark 1,

$$\lim_{t\to\infty} Ch\{the\ fuzzy\ queueing\ system\ is\ idle\ at\ time\ t\}$$

$$= 1 - E[\frac{\lambda}{\mu}] = 1 - 0.6432 = 0.3568$$

VI. CONCLUSION

In this paper, we considered a fuzzy queueing system that the interarrival times of customers and service times of servers are fuzzy random variables and we simulate the average chance of the random fuzzy event "the queueing system is busy at time *t*" by fuzzy simulation. In this method we simulate the expected value of $E[\frac{\lambda}{\mu}]$ (or when it is difficult to compute the expected value) and obtain the real value. If we increase the number of iterations or improve the random numbers in simulation algorithm, we approach to real solution.

REFERENCES

[1] Y. Ning, W. Tang, R. Zhao, *Analysis on random fuzzy queueing systems with finite capacity*, Technical Report, 2006.
[2] B. Liu, Y. Liu, *Expected value of fuzzy variable and fuzzy expected value model*, IEEE Transactions on Fuzzy Systems 10 (2002) 445-450.
[3] B. Liu, *A survey of credibility theory*, Fuzzy Optimization and Decision Making, Vol 5, No. 4, pp. 387-408, 2006.
[4] Y.K. Liu, B.D. Liu, *Expected value operator of random fuzzy variable and random fuzzy expected value models*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 11 (2003) 195-215.
[5] B. Liu, *A survey of credibility theory*, Fuzzy Optimization and Decision Making, Vol 5, No. 4, pp. 387-408, 2006.
[6] S. Li. *Some properties of fuzzy alternating renewal processes*, Mathematical and Computer Modelling 54 (2011) 1886-1896.
[7] D. Gross, C. M. Harris, *Fundamentals of Queueing Theory*, Wiley, New York, 1998.
[8] Y. F. Ning, Busy period of random fuzzy queueing systems, International conference, Hong Kong, 19-22, 2007.