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Another Structure of Weakly Left C-wrpp Semigroups

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Abstract—It is known that a left C-wrpp semigroup can be described as curler structure of a left band and a C-wrpp semigroup. In this paper, we introduce the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass. We shall particularly show that the spined product of a left C-wrpp semigroup and a right normal band is a weakly left C-wrpp semifroup. Some equivalent characterizations of weakly left C-wrpp semigroups are obtained. Our results extend that of left C-wrpp semigroups.

Keywords—Left C-wrpp semigroup,left quasi normal regular band, weakly left C-wrpp semigroup.

I. Introduction

THROUGHOUT this paper, we adopt the notation and terminologies given by Howei[1] and Du[2].

Tang[3] considered a Green-like right congruence relation $\mathcal{L}^{**}$ on a semigroup S : for $a, b \in S, a\mathcal{L}^{**}b$ if and only if $ax\mathcal{R}ay \Leftrightarrow bx\mathcal{R}by$ for all $x, y \in S^1$. Moreover, Tang pointed out in [3] that a semigroup S is a wrpp semigroup if and only if each $\mathcal{L}^{**}$-class of S contains at least one idempotent.

Recall that a wrpp semigroup S is a C-wrpp semigroup if the idempotents of S are central. It is well known that a semigroup S is a C-wrpp semigroup if and only if S is a strong semilattice of left-$\mathcal{R}$ cancellative monoids(see[3]). Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids(see[4-9]), we see immediately that the concept of C-wrpp semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [2] first introduced the concept of left C-wrpp semigroups, that is, a left C-wrpp semigroup whose satisfy the following conditions: (i) for all $e \in E(L^{**}_a), a = ae$, where $E(L^{**}_a)$ is the set of idempotents in L^{**}_a; (ii) for all $a \in S$, there exists a unique idempotent a^+ satisfying $a\mathcal{L}^{**}a^+$ and $a = a^+a$; (iii) for all $a \in S, aS \subseteq L^{**}(a)$,where $L^{**}(a)$ is the smallest left **-ideal of S generated by a. For such semigroups, Du-Shum[2] gave a method of construction.

Zhang[10] showed that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semigroup by virtue of left C-full Ehremann cybergroups. In this paper, we first define the concept of weakly left C-wrpp semigroups. A equivalent descriptions of weakly left C-wrpp semigroups is therefore obtained and our results generalize that of Cao on weakly left C-rpp in[5]. In view of the theorems given in this paper, one can easily observe that the results of weakly left C-wrrp semigroups are a common generalizations of weakly left C-semigroups and left C-wrpp semigroups in range of wrpp semigroups.

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II. Preliminaries

We first recall some known results used in the sequel. To start with, we introduce the concept of simi-spined product.

Let $T = \cup_{\alpha\in Y}T_\alpha$ and $I = \cup_{\alpha\in Y}I_\alpha$ be the semilattice decomposition of the semigroups T and I with respect to semilattice Y respectively. For all $\alpha \in Y$, we denote the direct product $I_\alpha \times T_\alpha$ by S_α. Let $S = \cup_{\alpha\in Y}S_\alpha$. we define the mapping η by the following rules:
$\eta : S \to T_l(I), (i, a) \mapsto \eta(i, a), \eta(i, a) : I \to I, j \mapsto (i, a)^{\#}j$, where $T_l(I)$ is a left transformation semigroup on I. Suppose that the mapping η satisfies the following conditions:

(Q1)If $(i, a) \in S_\alpha, j \in I_\beta$, then $(i, a)^{\#}j \in I_{\alpha\beta}$;

(Q2)If $(i, a) \in S_\alpha, (j, b) \in S_\beta$ with $\alpha \le \beta$, then $(i, a)^{\#}j = ij$, where ij is the semigroup product in the semigroup $I = \cup_{\alpha\in Y}I_\alpha$;

(Q3)If $(i, a) \in S_\alpha, (j, b) \in S_\beta$, then $\eta(i, a)\eta(j, b) = \eta((i, a)^{\#}j, ab)$, where ab is the semigroup product in the semigroup $T = \cup_{\alpha\in Y}T_\alpha$.

Then we define a multiplication " $\circ$" on $S = \cup_{\alpha\in Y}S_\alpha$ by $(i, a) \circ (j, b) = ((i, a)^{\#}j, ab)$. By a straightforward verification, we can prove that the multiplication " $\circ$" satisfies the associative law and hence $(S, \circ)$ becomes a semigroup, denoted by $S = I \times_\eta T$. We call this semigroup the semi-spined product of I and T with respect to the structure mapping η.

Lemma 1[2] Let I be a left regular band which is expressed as a semilattice of left zero bands I_α (that is, $I = \cup_{\alpha\in Y}I_\alpha$) and let $T = \cup_{\alpha\in Y}T_\alpha$ be a C-wrpp semigroup(that is, T is a strong semilattice of left-$\mathcal{R}$ cancellative monoids $[Y; T_\alpha, \phi_{\alpha,\beta}]$)(see[3]). If the structure mapping η satisfies the following condition:

(Q): $\ker\eta(i, a) = \ker\eta(j, b)$ for every $(i, a), (j, b) \in S_\alpha$. Then S is a left C-wrpp semigroup. Conversely, every left C-wrpp semigroup S can be constructed in terms of above method.

Lemma 2[5] A semigroup S is a weakly left C-semigroup, that is, S is a regular semigroup and

$$(\forall e \in E(S))\eta'_e : S \to eS, x \mapsto ex$$

is a homomorphism if and only if S is a completely regular and $E(S)$ is a left quasi-normal band.

Lemma 3[2] If S is a left C-wrpp, then Reg$S$ is a left C-semigroup.

Lemma 4[7] A band B is a left normal band (that is, a band satisfies identity $efeg = efg$) if and only if Green relation $\mathcal{L}$ and $\mathcal{R}$ are congruence on B and B/R is a right normal band.

Definition 1 A monoid T is called a left-$\mathcal{R}$ cancelattive monoid if for $a, b, c \in T, (ab, ac) \in \mathcal{R}$ implies $(b, c) \in \mathcal{R}$. We call the direct product of a left-$\mathcal{R}$ cancellative monoid T and a rectangular band I a left cancellative plank because the direct product looks like a two-dimensional plank. We denote the left-$\mathcal{R}$ cancellative plank by $I \times T$.

Lemma 5[2] Let $I = \cup_{\alpha\in Y} I_\alpha$ be a semilattice of left zero bands, and $T = [Y; T_\alpha, \phi_{\alpha,\beta}]$ a strong semilattice of left-$\mathcal{R}$ cancellative monoids T_α. Then $(i, a)\mathcal{R}(j, b)$ if and only if $a\mathcal{R}b$ and $i = j$ for any $(i, a), (j, b) \in S = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha)$.

III. The weakly left C-wrpp semigroups

In this section, the concept of weakly left C-wrpp semigroups is introduced. We shall give equivalent characterization for the structure of weakly left C-wrpp semigroups. First, we introduce the concept of weakly left C-wrpp semigroups.

Definition 2 A semigroup S is called a weakly left C-wrpp semigroup, if S is isomorphic to a semilattice of left-$\mathcal{R}$ cancellative planks, and

$$(\forall \in E(S))\eta'_e : S \to eS, x \mapsto ex$$

is a homomorphism.

We now characterize the weakly left C-wrpp semigroups.

Theorem 1 Let S be a semigroup. Then the following conditions are equivalent:

(1)S is a weakly left C-wrpp semigroup;

(2)S is a semilattice of left-$\mathcal{R}$ cancellative planks, and Reg$S$ is a weakly left C-semigroup;

(3)S is a semilattice of left-$\mathcal{R}$ cancellative planks, and $E(S)$ is a left quasi-normal band;

(4)S is a spined product of left C-wrpp semigroup and a right normal band.

Proof. (1)⇒(2). We only need show that Reg$S$ is a weakly left C-semigroup. Let $a, b \in$ Reg$S$. Then there exists $x, y \in S$ such that $a = axa, x = xax, b = byb$. So $e = xa \in E(S)$. According to (1), we know that η'_e is a semigroup homomorphism from S to eS. Thus

$$\begin{aligned} ab &= axabyb = a\eta'_e[(by)b] = a\eta'_e(by)\eta'_e(b) \\ &= axabyxab = (ab)(yx)(ab) \end{aligned}$$

So $ab \in$Reg$S$. Therefore, Reg$S$ is a subsemigroup of S. Again $E(\text{Reg}S) = E(S)$, according to Lemma 3, we obtain Reg$S$ is a weakly left C-semigroup.

(2)⇒ (3). Clearly, we omit it.

(3)⇒(4). Let $S = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha \times \Lambda_\alpha)$ is a semilattice decomposition Y of left-$\mathcal{R}$ cancellative planks, and $E(S)$ is a left quasi-normal band, and put $S_l = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha), \Lambda = \cup_{\alpha\in Y}\Lambda_\alpha, S_\alpha = I_\alpha \times T_\alpha \times \Lambda_\alpha$, where I_α, T_α and Λ_α are a left zero band, a left-$\mathcal{R}$ cancellative monoid and a right zero band, respectively. Next, we verify that $S_l = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha)$ is a left C-wrpp semigroup, and $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$ is a right normal band.

Step 1 Let $T = \cup_{\alpha\in Y} T_\alpha$. we shall show that T is a C-wrpp semigroup, and $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$ is a right normal band. For this purpose, we only need to show that T is a strong semilattice of left-$\mathcal{R}$ cancellative monoids T_α, and a strong semilattice of rihgt zero bands Λ_α, respectively.

Identity in T_α denoted by I_α, obviously, we have $E(S) = \{(i, 1_\alpha, \lambda)|(i, \lambda) \in I_\alpha \times \lambda_\alpha, \alpha \in Y\}$, and

$$(i, 1_\alpha, \lambda)\mathcal{L}^E(j, 1_\beta, \mu) \Leftrightarrow \alpha = \beta, \lambda = \mu, \quad (1)$$

$$(i, 1_\alpha, \lambda)\mathcal{R}^E(j, 1_\beta, \mu) \Leftrightarrow \alpha = \beta, i = j \quad (2)$$

where $\mathcal{L}^E$ and $\mathcal{R}^E$ are Green's relations on semigroup $E(S)$.

For all $\alpha \geq \beta$, let $a = (i, g, \lambda) \in S_\alpha$, if $(j, \mu) \in I_\beta \times \Lambda_\beta$, then there exists $(j_1, h_1, \mu_1) \in S_\beta$ such that $(j, 1_\beta, \mu)a = (j_1, h_1, \mu_1)$. Since $(j_1, h_1, \mu_1) = (j, 1_\beta, \mu)[(j, 1_\beta, \mu)a] = (j, h_1, \mu_1)$, we obtain $j_1 = j$. On the other hand, for all $j\prime \in I_\beta$, we have $(j\prime, 1_\beta, \mu)a = (j\prime, 1_\beta, \mu) = [(j, 1_\beta, \mu)a] = (j\prime, h_1, \mu_1)$. So h_1, μ_1 do not depend on the choice of j in I_β. Let $h_1 = \mu(i, g, \lambda)\chi_{\alpha,\beta}, \mu_1 = \mu(i, g, \mu)\psi_{\alpha,\beta}$. Then we have

$$(j, 1_\beta, \mu)(i, g, \lambda) = (j, \mu(i, g, \lambda)\chi_{\alpha,\beta}, \mu(i, g, \mu)\psi_{\alpha,\beta}). \quad (3)$$

Similarly, we show that there exists $\phi_{\beta,\alpha}(i, g, \lambda)j \in I_\beta$, $\varphi_{\beta,\alpha}(i, g, \lambda)j \in T_\beta$ such that

$$(i, g, \lambda)(j, 1_\beta, \mu) = (\phi_{\beta,\alpha}(i, g, \lambda)j, \varphi_{\beta,\alpha}(i, g, \lambda)j, \mu). \quad (4)$$

For all $\lambda' \in \Lambda_\alpha$, we have obtain $(i, 1_\alpha, \lambda)\mathcal{R}^E(i, 1_\alpha, \lambda')$ by (2). According to lemma 4, we know that $\mathcal{R}^E$ is a congruence on $E(S)$, it follows that $(i, 1_\alpha, \lambda)(j, 1_\beta, \mu)\mathcal{R}^E(i, 1_\alpha, \lambda')(j, 1_\beta, \mu)$. Since $E(S)$ is a band, Referring to (2) and (4), we can follow that $(i, 1_\alpha, \lambda)(j, 1_\beta, \mu) = (i, 1_\alpha, \lambda')(j, 1_\beta, \mu)$, multiplied with from left side of above formula's both sides, by (4), we obtain $\phi_{\beta,\alpha}(i, g, \lambda)j = \phi_{\beta,\alpha}(i, g, \lambda')j, \varphi_{\beta,\alpha}(i, g, \lambda)j = \varphi_{\beta,\alpha}(i, g, \lambda')j$. Therefore, $\phi_{\beta,\alpha}(i, g, \lambda)j$ and $\varphi_{\beta,\alpha}(i, g, \lambda)$ do not depend on the choice of λ, let

$$\phi_{\beta,\alpha}(i, g)j = \phi_{\beta,\alpha}(i, g, \lambda)j, \varphi_{\beta,\alpha}(i, g)j = \varphi_{\beta,\alpha}(i, g, \lambda)j, \quad (5)$$

where $\lambda \in \Lambda_\alpha, \alpha \geq \beta$. Similarly, by $\mathcal{L}^E$ is a congruence on $E(S)$, we follow that $\mu(i, g, \lambda)\chi_{\alpha,\beta}$ and $\mu(i, g, \lambda)\psi_{\alpha,\beta}$ do not depend on the choice of i in I_α, let

$$\mu(g, \lambda)\chi_{\alpha,\beta} = \mu(i, g, \lambda)\chi_{\alpha,\beta}, \mu(g, \lambda) = \mu(i, g, \lambda)\psi_{\alpha,\beta} \quad (6)$$

where $i \in I_\alpha, \alpha \geq \beta$. It follows that $(j, \mu(g, \lambda)\chi_{\alpha,\beta}, \mu) = [(j, 1_\beta, \mu)(i, g, \lambda)](j, 1_\beta, \mu) = (j, 1_\beta, \mu)[(i, g, \lambda)(j, 1_\beta, \mu)] = (j, \varphi_{\beta,\alpha}(i, g)j, \mu)$. So $\mu(g, \lambda)\chi_{\alpha,\beta} = \varphi_{\beta,\alpha}(i, g)j$, write as c. Clearly, c is determined by g but does not depend on the choice of i, j, λ and μ. Let

$$g\sigma_{\alpha,\beta} = \mu(g, \lambda)\chi_{\alpha,\beta} = \varphi_{\beta,\alpha}(i, g)j, \quad (7)$$

where $i \in I_\alpha, j \in I_\beta, \lambda \in \Lambda_\alpha$ and $\mu \in \Lambda_\beta$. According to $\mathcal{L}^E$ being a right normal band congruence on $E(S)$, for all $\mu, \mu' \in \Lambda_\beta$, we have $(j, 1_\beta, \mu')(j, 1_\beta, \mu)(i, 1_\alpha, \lambda)\mathcal{L}^E(j, 1_\beta, \mu)(j, 1_\beta, \mu')(i, 1_\alpha, \lambda)$, that is, $(j, 1_\beta, \mu)(i, 1_\alpha, \lambda)\mathcal{L}^E(j, 1_\beta, \mu')(i, 1_\alpha, \lambda)$. we can follow that $(j, 1_\beta, \mu)(i, 1_\alpha, \lambda) = (j, 1_\beta, \mu')(i, 1_\alpha, \lambda)$ in view of (1) and (3), multiplied with (i, g, λ) from right side of above formula's both sides, referring to (3)and (6), we obtain $\mu(g,h)\psi_{\alpha,\beta} = \mu'(g,h)\psi_{\alpha,\beta}$. Therefore, $\mu(g,h)\psi_{\alpha,\beta}$ does not depend on the choice of μ in Λ_β, let

$$(g,\lambda)\psi_{\alpha,\beta} = \mu(g,\lambda)\psi_{\alpha,\beta} \quad (8)$$

where $\mu \in \Lambda_\beta, \alpha \geq \beta$, In view of (3)-(8), we have

$$\begin{aligned}
&(j, g\sigma_{\alpha,\beta}, (g,\lambda)\psi_{\alpha,\beta})\\
&= (j, 1_\beta, \mu)(i, g, \lambda)\\
&= [(j, 1_\beta, \mu)(i, g, \lambda)](i, 1_\alpha, \lambda)\\
&= (j, g\sigma_{\alpha,\beta}, (g,\lambda)\psi_{\alpha,\beta})(i, 1_\alpha, \lambda)\\
&= (j, g\sigma_{\alpha,\beta}, (g,\lambda)\psi_{\alpha,\beta})[(j, 1_\beta, (g,\lambda)\psi_{\alpha,\beta})(i, 1_\alpha, \lambda)]\\
&= (j, g\sigma_{\alpha,\beta}, (g,\lambda)\psi_{\alpha,\beta})(j, 1_\alpha\sigma_{\alpha,\beta}, (1_\alpha,\lambda)\psi_{\alpha,\beta})\\
&= (j, (g\sigma_{\alpha,\beta})(1_\alpha\sigma_{\alpha,\beta}), (1_\alpha,\lambda)\psi_{\alpha,\beta}).
\end{aligned}$$

Therefore

$$g\sigma_{\alpha,\beta} = (g\sigma_{\alpha,\beta})(1_\alpha\sigma_{\alpha,\beta}), (g,\lambda)\psi_{\alpha,\beta} = (1_\alpha,\lambda)\psi_{\alpha,\beta}.$$

Since T_β is a left-$\mathcal{R}$ cancellative monoid,

$$1_\alpha\sigma_{\alpha,\beta}\mathcal{R}1_\beta(\alpha \geq \beta), \quad (9)$$

let

$$\lambda\theta_{\alpha,\beta} = (1_\alpha,\lambda)\psi_{\alpha,\beta} = (g,\lambda)\psi_{\alpha,\beta}, (g \in T_\alpha, \alpha \geq \beta). \quad (10)$$

Thus, summing up the above cases, we conclude that there exists the mapping:$\phi_{\beta,\alpha} : I_\alpha \times T_\alpha \to T_l(I_\beta), (i,g) \mapsto \phi_{\beta,\alpha}(i,g); \sigma_{\alpha,\beta} : T_\alpha \to T_\beta, g \mapsto g\sigma_{\alpha,\beta}; \theta_{\alpha,\beta} : \Lambda_\alpha \to \Lambda_\beta, \lambda \mapsto \lambda\theta_{\alpha,\beta}$ such that

$$(j, 1_\beta, \mu)(i, g, \lambda) = (i, g\sigma_{\alpha,\beta}, \lambda\theta_{\alpha,\beta}) \quad (11)$$

$$(i, g, \lambda)(j, 1_\beta, \mu) = (\phi_{\beta,\alpha}(i,g)j, g\sigma_{\alpha,\beta}, \mu) \quad (12)$$

for all $(i, g, \lambda) \in S_\alpha, (j, \mu) \in I_\beta \times \Lambda_\beta$.

The following we verify that $\sigma_{\alpha,\beta}$ and $\theta_{\alpha,\beta}$ are the structure homomorphism of strong semilattice on semigroups T_α and Λ_α, respectively. For all $\alpha, \beta \in Y, (i, g, \lambda) \in S_\alpha, (j, h, \mu) \in S_\beta$, let $(k, m, n) = (i, g, \lambda)(j, h, \mu) \in S_{\alpha\beta}$. Then for $\gamma \leq \alpha\beta$ and $(I, v) \in I_\gamma \times \Lambda_\gamma$, according to (11), we have

$$\begin{aligned}
(l, m\sigma_{\alpha,\beta}, n\theta_{\alpha\beta,\gamma}) &= (l, 1_\gamma, v)(k, m, n)\\
&= (l, 1_\lambda, v)(i, g, \lambda)(j, h, \mu)\\
&= (l, g\sigma_{\alpha,\gamma}, \lambda\theta_{\alpha,\gamma})(j, h, \mu)\\
&= (l, g\sigma_{\alpha,\gamma}, \lambda\theta_{\alpha,\gamma})(l, 1_\gamma, \lambda\theta_{\alpha,\gamma})(j, h, \mu)\\
&= (l, g\sigma_{\alpha,\gamma}, \lambda\theta_{\alpha,\gamma})(l, h\sigma_{\beta,\gamma}, \mu\theta_{\beta,\gamma})\\
&= (l, (g\sigma_{\alpha,\gamma})(h\sigma_{\beta,\gamma}), (\lambda\theta_{\alpha,\gamma})(\mu\theta_{\beta,\gamma}))
\end{aligned}$$

Therefore,

$$m\sigma_{\alpha\beta,\gamma} = (g\sigma_{\alpha,\gamma})(h\sigma_{\beta,\gamma}), n\theta_{\alpha\beta,\gamma} = (\lambda\theta_{\alpha,\gamma})(\mu\theta_{\beta,\gamma}), \quad (13)$$

$$(l, 1_\gamma, v)(i, g, \lambda)(j, h, \mu) = (l, (g\sigma_{\alpha,\gamma})(h\sigma_{\beta,\gamma}), (\lambda\theta_{\alpha,\gamma})(\mu\theta_{\beta,\gamma})). \quad (14)$$

(i)If $\beta = \alpha$, then $m = gh, n = \lambda\mu$. By (13), we have $(gh)\sigma_{\alpha,\gamma} = (g\sigma_{\alpha,\gamma})(h\sigma_{\alpha,\gamma}), (\lambda\mu)\theta_{\alpha,\gamma} = (\lambda\theta_{\alpha,\gamma})(\mu\theta_{\alpha,\gamma})$, where $g, h \in T_\alpha, \lambda, \mu \in \Lambda_\alpha$. So $\sigma_{\alpha,\gamma}$ and $\theta_{\alpha,\gamma}$ are semigroup homomorphism of from T_α to T_β and from Λ_α to Λ_β, respectively, where $\alpha \geq \gamma$. Similarly, it follows that $\sigma_{\alpha,\beta}$ is also a semigroup homomorphism, by (9), we have

$$1_\alpha\sigma_{\alpha,\beta} = 1_\beta, (\alpha \geq \beta). \quad (15)$$

(ii)If $\beta = \alpha$, let $\gamma = \alpha, h = 1_\alpha, \mu = \lambda$. In view of (14) and (15), it follows that $g = g\sigma_{\alpha,\alpha}, \chi = \lambda\theta_{\alpha,\alpha}$ for any $g \in T_\alpha, \lambda \in \Lambda_\alpha$. So $\sigma_{\alpha,\alpha}$ and $\theta_{\alpha,\alpha}$ are identical mapping on T_α and T_γ, respectively.

(iii)Let $\gamma = \alpha\beta, l = k$. According to (13), (14) and the results above (ii), we have

$$m = (g\sigma_{\alpha,\alpha\beta})(h\sigma_{\beta,\alpha\beta}), n = (\lambda\theta_{\alpha,\alpha\beta})(\mu\theta_{\beta,\alpha\beta}), \quad (16)$$

$$(i, g, \lambda)(j, h, \mu) = (k, (g\sigma_{\alpha,\alpha\beta})(h\sigma_{\beta,\alpha\beta}), (\lambda\theta_{\alpha,\alpha\beta})(\mu\theta_{\beta,\alpha\beta})). \quad (17)$$

(iv)If $\alpha \geq \beta \geq \gamma$, then $\alpha\beta = \beta$. Referring to (13),(16) and (17), we have $(g\sigma_{\alpha,\beta})\sigma_{\beta,\alpha} = [(g\sigma_{\alpha,\beta})(1_\beta)\sigma_{\beta,\beta}]\sigma_{\beta,\gamma} = (g\sigma_{\alpha,\gamma})(1_\beta\sigma_{\beta,\gamma}) = (g\sigma_{\alpha,\gamma})1_\gamma = g\sigma_{\alpha,\gamma}, (\lambda\theta_{\alpha,\beta})\theta_{\beta,\gamma} = [(\lambda\theta_{\beta,\beta})(\lambda\theta_{\alpha,\beta})]\theta_{\beta,\gamma} = (\lambda\theta_{\beta,\gamma})(\lambda_{\alpha,\gamma}) = \lambda\theta_{\alpha,\gamma}$. This leads to $\sigma_{\alpha,\beta}\sigma_{\beta,\gamma} = \sigma_{\alpha,\gamma}, \theta_{\alpha,\beta}\theta_{\beta,\gamma} = \theta_{\alpha,\gamma}$.

Define multiplication operations on $T = \cup_{\alpha\in Y}T_\alpha$ and $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$, as follows respectively:

$$g \circ h = (g\sigma_{\alpha,\alpha\beta})(h\sigma_{\beta,\alpha\beta})(g \in T_\alpha, h \in T_\beta), \quad (18)$$
$$\lambda \circ \mu = (\lambda\theta_{\alpha,\alpha\beta})(\mu\theta_{\beta,\alpha\beta})(\lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta). \quad (19)$$

According to (i),(ii) and (iv), we know that $T = [Y; T_\alpha, \sigma_{\alpha,\beta}]$ is a strong semilattice of left-$\mathcal{R}$ cancellative monoid T_α and $\Lambda = [Y; \Lambda_\alpha, \theta_{\alpha,\beta}]$ is a strong semilattice of right zero band Λ_α, that is, $(T, \circ)$ is a C-wrpp semigroup and $(\Lambda, \circ)$ is a right normal band. It follows that

$$(i, g, \lambda)(j, h, \mu) = (k, g \circ h, \lambda \circ \mu) \quad (20)$$

by (18)-(20).

Step 2 We shall show that $S_l = \cup_{\alpha\in Y}(I_\alpha \times T_\alpha)$ forms a left C-wrpp semigroup. Let $I = \cup_{\alpha\in Y}I_\alpha$. We wish to define a mapping $\eta : S_l \to T_l(I)$ so that S_l can be made into a semi-spined product. For all $k' \in I_{\alpha\beta}$, we have

$$\begin{aligned}
(k, m, n) = (k, m, n)(k', 1_{\alpha\beta}, n) &= (i, g, \lambda)(j, h, \mu)(k', 1_{\alpha\beta}, n)\\
&= (i, g, \lambda)(\phi_{\alpha\beta}(j,h)k', \ldots, \ldots)\\
&= (\phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j,h)k', \ldots, \ldots).
\end{aligned}$$

So $k = \phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j,h)k'$. Therefore, $\phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j,h)$ is a constant mapping on $I_{\alpha\beta}$, write as $k =< \phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j,h) >$, we have

$$\begin{aligned}
(k, m, n) &= (k, m, n)(j, 1_\beta, \mu)(j, h, \mu)(k', 1_{\alpha\beta}, n)\\
&= (i, g, \lambda)(j, 1_\beta, \mu)(\phi_{\alpha\beta,\beta}(j,h)k', \cdots, \cdots)\\
&= (\phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j, 1_\beta)[\phi_{\alpha\beta,\beta}(j,h)k'], \ldots, \ldots)\\
&= (< \phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j, 1_\beta) >, \ldots, \ldots).
\end{aligned}$$

Thus $k =< \phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j, 1_\beta) >$ does not depend on the choice of h, let $k = \eta(i,g)j$. We define the mapping η by the following rules:

$$\eta(i,g) : S_l \to T_l(I), (i,g) \mapsto \eta(j,g);$$
$$\eta(i,g) : I \to I, j \mapsto \eta(i,g)j,$$

and such that

$$(i,g,\lambda)(j,h,\mu) = (\eta(i,g), g\circ h, \lambda\circ\mu)$$

for $(i,g,\lambda),(j,h,\mu)\in S$.

To see that η is a structure mapping defining a semi-spined product $I\times_\eta T$, we need to verify that η satisfies the required conditions (Q1)-(Q3). If $(i,g)\in I_\alpha\times T_\alpha, j\in I_\beta, \alpha\le\beta$, then $\eta(i,g)j = <\phi_{\alpha\beta,\alpha}(i,g)\phi_{\alpha\beta,\beta}(j,1_\beta)>\in I_{\alpha\beta}$, (Q1) holds. To verify that (Q2) holds, we let $(i,g)\in I_\alpha\times T_\alpha, j\in I_\beta, \alpha\le\beta$, then we obtain

$$\begin{aligned}(\eta(i,g)j, g\circ h, \lambda\circ\mu) &= (i,g,\lambda)[(i,1_\alpha,\lambda)(j,h,\mu)]\\ &= (i,g,\lambda)(i, h\sigma_{\beta,\alpha}, \mu\theta_{\beta,\alpha})\\ &= (i,\ldots,\ldots)\end{aligned}$$

by (11) and (20). Consequently, we have $\eta(i,g)j=i$. Thus, (Q2) holds. Finally, we let $(i,g)\in I_\alpha\times T_\alpha, (j,h)\in I_\alpha\times T_\beta$. For all $\gamma\in Y, l\in I_\gamma, v\in\Lambda_\alpha$, according to (20), we have

$$\begin{aligned}&(\eta(\eta(i,g)j, g\circ h)l, (g\circ h)\circ 1_\gamma, \lambda\circ\mu)\\ &= (i,g,\lambda)(j,h,\mu)(l,1_\gamma,\nu)\\ &= (i,g,\lambda)(\eta(j,h)l,\ldots,\ldots)\\ &= (\eta(i,g)\eta(j,h)l,\ldots,\ldots).\end{aligned}$$

This leads to $\eta(\eta(i,g)j, g\circ h)l = \eta(i,g)\eta(j,h)l$, so $\eta(\eta(i,g)j, g\circ h) = \eta(i,g)\eta(j,h)$. In fact, we have shown that (Q3) holds. Thus, η satisfies (Q1)-(Q3) and we do have a semi-spined product $I\times_\eta T$.

Next we need to prove that the structure mapping η on this semispined product satisfies the condition (Q) in lemma 1. For this purpose, we let (i,a) and $(j,b)\in I_\alpha\times T_\alpha$. Take $k\in I_\tau$ and $l\in I_\delta$ for some τ and δ, and suppose that $\eta(i,a)k=\eta(i,a)l$, that is, $(i,a)^{\#}k=(i,a)^{\#}l$. By condition (Q1), we have $\delta\alpha=\tau\alpha$. Denote the identity elements of the monoids T_δ and T_τ by 1_δ and 1_τ, respectively. Since T is a strong semilattice of T_α, we have $a1_\delta = a1_\tau$. By invoking Lemma 5, we have $(i,a)(k,1_\tau)\mathcal{R}(i,a)(l,1_\delta)$. Since $i\mathcal{L}j$, we have $(i,a)\mathcal{L}^{**}(j,b)$ so that $(j,b)(k,1_\tau)\mathcal{R}(j,b)(l,1_\delta)$. Hence we have

$$((j,b)^{\#}k, b1_\tau)\mathcal{R}((j,b)^{\#}l, b1_\delta)\Rightarrow (j,b)^{\#}k=(j,b)^{\#}l.$$

This shows that ker$\eta(i,a)\subseteq$ ker$\eta(j,b)$. Analogously, we can also prove that ker$\eta(j,b)\subseteq$ ker$\eta(i,a)$. Thus ker$\eta(i,a)$ = ker$\eta(j,b)$ and so condition (Q) is satisfied. This shows that $S_l=\cup_{\alpha\in Y}(I_\alpha\times T_\alpha)$ is indeed a left C-wrpp semigroup.

Summing up step1 and step2, we conclude that S is the spined product of a left C-wrpp semigroup S_l and a right normal band Λ.

(4)⇒(1). Let S be the spined product of a left C-wrpp semigroup $S_l = I\times_{Y,\eta} T$ and a right normal band $\Lambda = [Y;\Lambda_\alpha,\theta_{\alpha,\beta}]$. Clearly, S is a semilattice of left-$\mathcal{R}$ cancellative planks, and for all $e=(i,1_\alpha,\lambda)\in E(S)\cap(I_\alpha\times T_\alpha\times\Lambda_\alpha), x=(j,h,\mu)\in I_\beta\times T_\beta\times\Lambda_\beta, y=(k,m,n)\in I_\gamma\times T_\gamma\times\Lambda_\gamma$, let $(l,q)=(i,1_\alpha)(j,h)\in I_{\alpha\beta}\times T_{\alpha\beta}$. According to S_l is a left C-wrpp semigroup and Lemma 1, we have $(i,q)(i,1_\alpha) = (\eta(l,g)i, (q\sigma_{\alpha\beta,\alpha\beta})(1_\alpha\sigma_{\alpha,\alpha\beta})) = (l,q) = (i,1_\alpha)(j,h)\in I_{\alpha\beta}\times T_{\alpha\beta}$, so

$$\begin{aligned}\eta'_e(xy) = exy &= ((i,1_\alpha)(j,h)(k,m),\lambda\mu\nu)\\ &= ((l,q)(i,1_\alpha)(i,1_\alpha)(k,m),\lambda\mu\nu)\\ &= ((i,1_\alpha)(j,h)(i,1_\alpha)(k,m),\lambda\mu\nu)\\ &= exey = \eta'_e(x)\eta'_e(y).\end{aligned}$$

Consequently, η'_e is a semigroup homomorphism from S to eS, thus S is a weakly left C-wrpp semigroup.

Corollary 1 Let S be a semigroup. Then the following conditions are equivalent:

(1) S is a weakly left C-rpp semigroup;

(2) S is a semilattice of left cancellative monoids, and Reg$S$ is a weakly left C-semigroup;

(3) S is a semilattice of left cancellative monoids, and S is a left quasi-normal band;

(4) S is a spined product of left C-rpp semigroup and a right normal band.

Corollary 2 A weakly left C-wrpp semigroup is a wrpp semigroup.

Proof. According to theorem 1, a weakly left C-wrpp semigroup is a spined product of a left C-wrpp semigroup and right normal band, but a left C-wrpp semigroup and a right normal band are wrpp semigroups, it follows that a weakly left C-wrpp emigroup is a wrpp semigroup.

By above corollary, we have the following results:

Corollary 3 A weakly left C-rpp semigroup is a rpp semigroup.

Corollary 4 A semigroup S is a weakly left C-semigroup if and only if S is a spined product of left C-semigroup and a right normal band.

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