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# A New Proof on the Growth Factor in Gaussian Elimination for Generalized Higham Matrices

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*Abstract*—The generalized Higham matrix is a complex symmetric matrix $A = B + iC$, where both $B \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{n\times n}$ are Hermitian positive definite, and $i = \sqrt{-1}$ is the imaginary unit. The growth factor in Gaussian elimination is less than $3\sqrt{2}$ for this kind of matrices. In this paper, we give a new brief proof on this result by different techniques, which can be understood very easily, and obtain some new findings.

*Keywords*—CSPD matrix, positive definite, Schur complement, Higham matrix, Gaussian elimination, Growth factor.

## I. Introduction

COMPLEX symmetric matrices arise frequently, specially in algebraic eigenvalue problems see [2,3] and in the computational electrodynamics see [4] etc. The Higham matrix is a complex symmetric matrix $A = B + iC$, where both $B$ and $C$ are real, symmetric and positive definite, which was firstly presented by Higham in [3] (It was called by a CSPD matrix.). In order to research the accuracy and stability of their LU factorizations, the growth factor (denoted by $\rho_n(A)$) in Gaussian elimination was conjectured in [2] that

$$\rho_n(A) \le 2$$

for any Higham matrix $A$. Subsequently, the paper proved the following result

$$\rho_n(A) < 3, \tag{1}$$

for such matrix $A$, and so LU factorization without pivoting is perfectly normwise backward stable see [3]. Moreover, they pointed out that if the Higham matrix is extended by allowing $B$ and $C$ to be arbitrary Hermitian positive definite matrices, i.e., $A = B + iC$ is a generalized Higham matrix, then

$$\rho_n(A) < 3\sqrt{2}, \tag{2}$$

whose proof was quite lengthy in [1]. In addition, authors in [5] also noted that the above bound (2) remains true when $B$ or $C$ or both are negative (rather than positive) definite.

In this paper, we mainly give a new brief proof of the results (1) and (2) by different techniques, which can be understood more easily than the proof of [1]. Next, for convenience, we use the same notations as in [1].

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## II. Auxiliary results

In this section, we mainly list some results and lemmas which will be essential to prove our results.

**Lemma 1** ([6]). Let $A$ be a CSPD matrix, then $A$ is nonsingular, and any principal submatrix of $A$ and any schur complement in $A$ are also CSPD matrices. Obviously, Lemma 1 shows that, being a CSPD matrix is a hereditary property of active submatrices in Gaussian elimination.

**Lemma 2** ([6]). The largest element of a CSPD matrix $A$ lies on its main diagonal.

The above property also holds for generalized Higham matrices in the following slightly weakened form.

**Lemma 3** ([1]). If $A$ is a generalized Higham matrix, then

$$\sqrt{2}\max_{l}|a_{ll}| \ge \max_{l\neq j}|a_{lj}|, \tag{3}$$

Thus, for a CSPD matrix $A$, the growth factor

$$\rho_n(A) = \frac{\max_{i,j,k}|a_{ij}^{(k)}|}{\max_{i,j}|a_{ij}|} \tag{4}$$

can be replaced by

$$\rho_n(A) = \frac{\max_{j,k}|a_{jj}^{(k)}|}{\max_{j}|a_{jj}|}. \tag{5}$$

By Lemma 1 and 2, one can obtain broader bounds for the growth factor of a CSPD matrix $A$.

**Lemma 4** ([7]). Let $Z_1$ and $Z_2$ be $m \times n$ matrices and

$$H = Z_1^* Z_2 + Z_2^* Z_1, \tag{6}$$

then

$$H \le Z_1^* Z_1 + Z_2^* Z_2. \tag{7}$$

**Lemma 5** ([8]). If $B_1$ and $B_2$ are $n \times n$ Hermitian positive definite, then inequalities $B_1 \ge B_2$ if and only if $B_2^{-1} \ge B_1^{-1}$.

In addition, according to the Theorem 2.1 in [1] and its proof, we easily obtain the following corollary.

**Corollary 1**. Let $A = B+iC$, where $B$ and $C$ are Hermitian and positive definite matrices, then $A$ is nonsingular, and $A^{-1} = X + iY$, $X$ is a positive (semi)definite matrix when $B$ is positive (semi)definite and $Y$ is a negative (semi)definite matrix when $C$ is positive (semi)definite.

## III. Main results

The following theorem has been proved in [1], but the proof of [1] is lengthy. Based on the ideas in [1] and [7], we next give its a new proof, which can be understood more easily than the proof of [1].

**Theorem 1**. Let $A$ be a generalized Higham matrix, then

$$\frac{|a_{jj}^{(k)}|}{|a_{jj}|} < 3, \quad j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, n-1. \quad (8)$$

**Proof.** Similarly to [1], fix the number $k \in \{1, 2, \cdots, n-1\}$ and $j$, where $j \geq k+1$. Denote $A_k$, $B_k$ and $C_k$ by the leading principal order $k$ submatrices in $A$, $B$ and $C$, respectively. We split the matrix $A_{kj}$, a principal order $(k+1) \times (k+1)$ submatrix in $A$, into

$$A_{kj} = \begin{pmatrix} A_k & \alpha \\ \beta^T & a_{jj} \end{pmatrix},$$

where

$$\alpha^T = \begin{pmatrix} a_{1j}, a_{2j}, \cdots, a_{kj} \end{pmatrix},$$

and

$$\beta^T = \begin{pmatrix} a_{j1}, a_{j2}, \cdots, a_{jk} \end{pmatrix}.$$

Defining the vectors

$$b^T = \begin{pmatrix} b_{1j}, b_{2j}, \cdots, b_{kj} \end{pmatrix}$$

and

$$c^T = \begin{pmatrix} c_{1j}, c_{2j}, \cdots, c_{kj} \end{pmatrix},$$

we can rewrite $A_{kj}$ as

$$A_{kj} = \begin{pmatrix} B_k + iC_k & b + ic \\ b^* + ic^* & b_{jj} + ic_{jj} \end{pmatrix}. \quad (9)$$

It is easy to see that $a_{jj}^{(k)}$ can be obtained by performing block Gaussian elimination in $A_{kj}$, namely,

$$a_{jj}^{(k)} = a_{jj} - \beta^T A_k^{-1} \alpha.$$

Similarly to [1], setting $a_{jj}^{(k)} = \beta + i\gamma,\ \beta,\ \gamma \in \mathbb{R}$ and using (9), we have

$$\beta + i\gamma = b_{jj} + ic_{jj} - (b^* + ic^*)(B_k + iC_k)^{-1}(b + ic). \quad (10)$$

Next, we use the same method as in [1] to deal with $(B_k + iC_k)^{-1}$. By [1], we know that $(B_k + iC_k)^{-1}$ can be written as

$$(B_k + iC_k)^{-1} = X_k + iY_k, \quad (11)$$

where $X_k$ is positive definite and $Y_k$ is negative definite by Corollary 1. Substituting (11) into (10) yields

$$\beta + i\gamma = b_{jj} + ic_{jj} - (b^* + ic^*)(X_k + iY_k)(b + ic),$$

we have

$$\beta = b_{jj} - b^* X_k b + c^* X_k c + c^* Y_k b + b^* Y_k c, \quad (12)$$

and

$$\gamma = c_{jj} - b^* Y_k b + c^* Y_k c - c^* X_k b - b^* X_k c. \quad (13)$$

Now, we use the other technique, which is different from [1], to obtain the upper bounds on $\beta$ and $\gamma$.

Since $X_k$ is a positive definite matrix, $Y_k$ is negative definite. It is obvious that $-b^* X_k b$ in (12) and $c^* Y_k c$ in (13) are negative semidefinite, so (12) and (13) can rewrite

$$\beta \leq b_{jj} + c^* X_k c + c^* Y_k b + b^* Y_k c, \quad (14)$$

and

$$\gamma \leq c_{jj} - b^* Y_k b - c^* X_k b - b^* X_k c. \quad (15)$$

Now we mainly consider the last two summands on the right hand side for the above two inequalities (14) and (15). First, for (14), we apply the Lemma 4 with

$$Z_1 = Gb \text{ and } Z_2 = Gc,$$

where $G$ is the Hermitian positive definite square root of the matrix $-Y_k$, we get

$$c^* Y_k b + b^* Y_k c \leq -b^* Y_k b - c^* Y_k c,$$

thus

$$\beta \leq b_{jj} + c^* X_k c - b^* Y_k b - c^* Y_k c. \quad (16)$$

The last summand on the right-hand side of (15) may be proved in the same way. Thus we have the following inequality

$$-c^* X_k b - b^* X_k c \leq b^* X_k b + c^* X_k c.$$

So

$$\gamma \leq c_{jj} - b^* Y_k b + b^* X_k b + c^* X_k c. \quad (17)$$

In addition, by [1], we see that

$$X_k = (B_k + C_k B_k^{-1} C_k)^{-1} \leq \frac{1}{2} C_k^{-1} \quad (18)$$

and

$$-Y_k = (C_k + B_k C_k^{-1} B_k)^{-1} \leq \frac{1}{2} B_k^{-1}. \quad (19)$$

Note that $\begin{pmatrix} C_k & c \\ c^* & c_{jj} \end{pmatrix}$ and $\begin{pmatrix} B_k & b \\ b^* & b_{jj} \end{pmatrix}$ are positive definite, by [1], the Schur complement $C_{kj}/C_k$ and $B_{kj}/B_k$ are also positive definite, i.e.,

$$c^* {C_k}^{-1} c < c_{jj} \text{ and } - b^* {B_k}^{-1} b < b_{jj},$$

which implies that

$$c^* X_k c < \frac{1}{2} c_{jj}, \text{ and } - b^* Y_k b < \frac{1}{2} b_{jj},$$

Coming back to (18), from the trivial inequality

$$B_k + C_k B_k^{-1} C_k \geq B_k,$$

we can deduce the bound $X_k \leq B_k^{-1}$ by Lemma 5. In addition, note that $\begin{pmatrix} B_k & b \\ b^* & b_{jj} \end{pmatrix}$ is positive definite, we have

$$b^* X_k b \leq b^* B_k^{-1} b < b_{jj}.$$

Similarly, (19) implies the bound

$$-Y_k < C_k^{-1}$$

and

$$-c^* Y_k c \leq c^* C_k^{-1} c < c_{jj}.$$

Summarizing the above results, we conclude that

$$\begin{aligned}\beta &\le b_{jj} + c^*X_kc - b^*Y_kb - c^*Y_kc\\ &< b_{jj} + \tfrac{1}{2}c_{jj} + \tfrac{1}{2}b_{jj} + c_{jj}\\ &= \tfrac{3}{2}(b_{jj} + c_{jj})\end{aligned}$$

and

$$\begin{aligned}\gamma &\le c_{jj} - b^*Y_kb + b^*X_kb + c^*X_kc\\ &< c_{jj} + \tfrac{1}{2}b_{jj} + b_{jj} + \tfrac{1}{2}c_{jj}\\ &= \tfrac{3}{2}(b_{jj} + c_{jj}).\end{aligned}$$

So both the matrix $\beta$ and matrix $\gamma$ are bounded above by the same matrix $\frac{3}{2}(b_{jj} + c_{jj})$.

It follows that

$$\begin{aligned}\beta^2 + \gamma^2 &< [\tfrac{3}{2}(b_{jj} + c_{jj})]^2 + [\tfrac{3}{2}(b_{jj} + c_{jj})]^2\\ &= \tfrac{9}{2}(b_{jj} + c_{jj})^2\\ &= \tfrac{9}{2}(b_{jj}^2 + c_{jj}^2) + 9b_{jj}c_{jj}\\ &\le \tfrac{9}{2}(b_{jj}^2 + c_{jj}^2) + \tfrac{9}{2}(b_{jj}^2 + c_{jj}^2)\\ &= 9(b_{jj}^2 + c_{jj}^2).\end{aligned}$$

which is equivalent to (8). ■

**Remark 1**. Here, we obtain the same result as the paper [1] by Lemma 4, but our proof may be easily understood. In addition, according to the above analysis, we know that both the matrix $\beta$ and matrix $\gamma$ are bounded by the same matrix $\frac{3}{2}(b_{jj} + c_{jj})$, while the paper [1] indicated that

$$\beta < 2b_{jj} + c_{jj} \quad \text{and} \quad \gamma < b_{jj} + 2c_{jj}.$$

This seems to be interesting, and we will continue to study them in the future.

Finally, by (5), the following results are obvious.

**Corollary 2** ([1]). Let $A$ be a Higham matrix, then

$$\rho_n(A) < 3. \tag{20}$$

**Corollary 3** ([1]). Let $A$ be a generalized Higham matrix, then

$$\rho_n(A) < 3\sqrt{2}. \tag{21}$$

## IV. Conclusion

The main result of the paper has been proved in [1], but the proof of [1] is lengthy and is not to be understood easily. Based on the ideas in [1] and [7], we give its a new proof, which can be understood more easily than the proof of [1], and obtain some new findings.

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