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Modified Hankel Matrix Approach for Model Order Reduction in Time Domain

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***Abstract*—**The author presented a method for model order reduction of large-scale time-invariant systems in time domain. In this approach, two modified Hankel matrices are suggested for getting reduced order models. The proposed method is simple, efficient and retains stability feature of the original high order system. The viability of the method is illustrated through the examples taken from literature.

***Keywords*—**Model Order Reduction, Stability, Hankel Matrix, Time-Domain, Integral Square Error.

I. Introduction

MODEL order reduction is a very attractive idea in CAD area. It replaces the original large scale systems model with a much smaller one, yet still retains the original behavior under investigation to high accuracy. Therefore, by simulating just the reduced small system one can still study the original system and thus make the design work much easier. With the ever increasing the scale of system models appearing in the engineering design practice, model order reduction has become an indispensible tool in numerous areas of science and technology. Model order reduction is also a very interesting and meaningful mathematical problem in its own right.

Several methods based on Hankel matrix have been used for deriving low order state models from a given complex system described by its transfer function matrix or state model. The problem of minimal realization of rational transfer function matrix based on Hankel matrix approach has drawn major attention of the several authors [1]-[7] from the last few decades. Rozsa et al. [2], Hickin & Sinha [5] and Shrikhande et al. [6], etc. has suggested reduction methods based on Hankel matrix approach in which Hankel matrix is converted into Hermite normal form by using outer products. The minimal realization can be achieved in fixed number of operations on the Hankel matrix. Shamash [3] has proposed a method based on Hankel matrix approach in which conversion of Hankel matrix into Hermite normal form is not required but using Silverman's algorithm [8] and the theory developed in reference [1], the reduced order state models are obtained by minimal realization. The method suggested by Shamash is applicable to linear SISO and MIMO dynamic systems. In this paper, the outer products algorithm is used to synthesize the reduced model which is equally applicable to linear multi-variable system as well.

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II. Problem Statement

Let the n^{th} order original high order linear SISO system be expressed as

$$\left.\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) \\ Y(t) &= CX(t) \end{aligned}\right\} \tag{1}$$

where $X(t)$, $U(t)$ and $Y(t)$ are state, input and output variable vectors and $[A]_{n\times n}$, $[B]_{n\times 1}$ and $[C]_{1\times n}$ are state, input, and output matrices of the original high order system.

The problem is to find the k^{th} $(k < n)$ order reduced model, which reflects the dominant properties of the original high order system (1) be expressed as

$$\left.\begin{aligned} \dot{X}_k(t) &= A_k X_k(t) + B_k U(t) \\ Y_k(t) &= C_k X_k(t) \end{aligned}\right\} \tag{2}$$

where $X_k(t)$, $Y_k(t)$ are reduced state and output vectors and $Y_k(t)$ is close approximation of $Y(t)$ and $[A]_{k\times k}$, $[B]_{k\times 1}$, and $[C]_{1\times k}$ are unknown matrices of the reduced order model.

III. Description of the Method

The transfer function $G(s)$, which can be expanded in power series of s^{-1} or S as

$$G(s) = C(sI - A)^{-1}B \tag{3}$$

$$= M_1 s^{-1} + M_2 s^{-2} + M_3 s^{-3} + ... \tag{4}$$

$$= T_1 + T_2 s + T_3 s^2 + ... \tag{5}$$

where

$$\left.\begin{aligned} M_i &= CA^{i-1}B \\ T_i &= CA^{-i}B \end{aligned}\right\} \quad i = 1, 2, 3, ... \tag{6}$$

and M_i & T_i are the i^{th} Markov parameter and time-moment respectively. The following two modified Hankel matrices $H_{ij}^{(0)}$ are defined in the following manner with $i = j = n$ as

Type 1:

$$H_{nn}^{(0)} = \begin{bmatrix} T_1 & M_1 & \dots & \dots & M_{n-1} \\ M_{n-1} & M_{n-2} & \dots & \dots & M_n \\ \dots & \dots & \dots & \dots & \dots \\ M_{n-2} & M_{n-1} & \dots & \dots & M_{2n-3} \\ M_{n-1} & M_n & \dots & \dots & M_{2n-2} \end{bmatrix}$$

$$= \begin{bmatrix} CA^{-1}B & CB & \dots & \dots & CA^{n-2}B \\ CB & CAB & \dots & \dots & CA^{n-1}B \\ \dots & \dots & \dots & \dots & \dots \\ CA^{n-3}B & CA^{n-2}B & \dots & \dots & CA^{2n-4}B \\ CA^{n-2}B & CA^{n-1}B & \dots & \dots & CA^{2n-3}B \end{bmatrix} \tag{7}$$

which can also be written as

$$H_{nn}^{(0)} = \begin{bmatrix} CA^{-1} \\ C \\ \dots \\ CA^{n-3} \\ CA^{n-2} \end{bmatrix} \begin{bmatrix} B & AB & \dots & \dots & A^{n-1}B \end{bmatrix} = £\times\Omega \tag{8}$$

where £ and Ω are observability and controllability matrix.

By applying any of the available techniques [1], [2], a partial realization is then obtained as

$$\sum : B,\ A,\ CA^{-1} \tag{9}$$

from which the triple of matrices (A_k, B_k, C_k) can be obtained.

Type 2:

$$H_{nn}^{(0)} = \begin{bmatrix} M_1 & T_1 & \dots & \dots & T_{n-1} \\ T_1 & T_2 & \dots & \dots & T_n \\ \dots & \dots & \dots & \dots & \dots \\ T_{n-2} & T_{n-1} & \dots & \dots & T_{2n-3} \\ T_{n-1} & T_n & \dots & \dots & T_{2n-2} \end{bmatrix}$$

$$= \begin{bmatrix} CB & CA^{-1}B & \dots & \dots & CA^{-n+1}B \\ CA^{-1}B & CA^{-2}B & \dots & \dots & CA^{-n}B \\ \dots & \dots & \dots & \dots & \dots \\ CA^{-n+2}B & CA^{-n+1}B & \dots & \dots & CA^{-2n+3}B \\ CA^{-n+1}B & CA^{-n}B & \dots & \dots & CA^{-2n+2}B \end{bmatrix} \tag{10}$$

This matrix can also be written as

$$H_{nn}^{(0)} = \begin{bmatrix} C \\ CA^{-1} \\ \dots \\ CA^{-n+2} \\ CA^{-n+1} \end{bmatrix} \begin{bmatrix} B & A^{-1}B & \dots & \dots & A^{-n+1}B \end{bmatrix} = € \,.\, £ \tag{11}$$

where € and £ are observability and controllability matrix.

By applying any of the available techniques [1]-[2], a partial realization is then obtained as

$$\sum : B,\ A^{-1},\ C \tag{12}$$

from which the triple of matrices (A_k, B_k, C_k) can be obtained.

The algorithm of Rozsa and Sinha [2] is used to convert the Hankel matrix $H_{nn}^{(0)}$ into Hermite normal form using outer products. In this procedure the first $'k'$ columns of the Hankel matrix are transformed to unit vectors. The method requires exactly $'k'$ steps where $'k'$ is the order of minimal realization. The conversion algorithm of the both types of the Hankel matrix into Hermite normal form and realization in each steps, is illustrated as follows:

Let a Hankel matrix $H_{nn}^{(0)}$ be

$$H_{nn}^{(0)} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & \dots & e_{1n} \\ e_{21} & e_{22} & e_{23} & \dots & e_{2n} \\ e_{31} & e_{32} & e_{34} & \dots & e_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ e_{n1} & e_{n2} & e_{n3} & \dots & e_{nn} \end{bmatrix} \tag{13}$$

The Hankel matrix $H_{nn}^{(1)}$, in Hermite normal form is obtained from $H_{nn}^{(0)}$ using outer products method as follows:

$$H_{nn}^{(1)} = H_{nn}^{(0)} - \frac{1}{e_{11}} \begin{bmatrix} z \\ e_{21} \\ e_{31} \\ \dots \\ e_{n1} \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} & \dots & e_{1n} \end{bmatrix} \tag{14}$$

where $z = e_{11} - 1$

$$H_{nn}^{(1)} = \begin{bmatrix} 1 & e_{12}' & e_{13}' & \dots & e_{1n}' \\ 0 & e_{22}' & e_{23}' & \dots & e_{2n}' \\ 0 & e_{32}' & e_{33}' & \dots & e_{3n}' \\ \dots & \dots & \dots & \dots & \dots \\ 0 & e_{n2}' & e_{n3}' & \dots & e_{nn}' \end{bmatrix} \tag{15}$$

Thus, the 1st order partial realization is obtained as follows:

For type 1 Hankel matrix: $B_1 = [1]$, $A_1 = [e_{12}']$ and $C_1 A_1^{-1} = [e_{11}]$

For type 2 Hankel matrix: $B_1 = [1]$, $A^{-1}{}_1 = [e_{12}']$ and $C_1 = [e_{11}]$

In the second step, $H_{nn}^{(2)}$ is obtained from the $H_{nn}^{(1)}$ as follows

$$H_{nn}^{(2)} = H_{nn}^{(1)} - \frac{1}{e_{22}'} \begin{bmatrix} e_{12}' \\ z \\ e_{32}' \\ \dots \\ e_{n2}' \end{bmatrix} \begin{bmatrix} 0 & e_{22}' & e_{23}' & \dots & e_{2n}' \end{bmatrix} \tag{16}$$

where $z = e_{22}' - 1$

$$H_{nn}^{(2)} = \begin{bmatrix} 1 & 0 & e_{13}'' & \dots & e_{1n}'' \\ 0 & 1 & e_{23}'' & \dots & e_{2n}'' \\ 0 & 0 & e_{33}'' & \dots & e_{3n}'' \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & e_{n3}'' & \dots & e_{nn}'' \end{bmatrix} \quad (17)$$

2nd order partial realization is thus obtained as

For type-1 Hankel matrix: $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & e_{13}'' \\ 1 & e_{23}'' \end{bmatrix}$, and $C_2 A_2^{-1} = \begin{bmatrix} e_{11} & e_{12} \end{bmatrix}$

For type 2 Hankel matrix: $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A^{-1}{}_2 = \begin{bmatrix} 0 & e_{13}'' \\ 1 & e_{23}'' \end{bmatrix}$, and $C_2 = \begin{bmatrix} e_{11} & e_{12} \end{bmatrix}$

Similarly, in the third step, $H_{nn}^{(3)}$ is obtained from the Hankel matrix $H_{nn}^{(2)}$ and is written as

$$H_{nn}^{(3)} = H_{nn}^{(2)} - \frac{1}{e_{33}''} \begin{bmatrix} e_{13}'' \\ e_{23}'' \\ z \\ \dots \\ e_{n3}'' \end{bmatrix} \begin{bmatrix} 0 & 0 & e_{33}'' & \dots & e_{3n}'' \end{bmatrix} \quad (18)$$

where $z = e_{33}'' - 1$

$$H_{nn}^{(3)} = \begin{bmatrix} 1 & 0 & 0 & e_{14}''' & \dots & e_{1n}''' \\ 0 & 1 & 0 & e_{24}''' & \dots & e_{2n}''' \\ 0 & 0 & 1 & e_{34}''' & \dots & e_{3n}''' \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & e_{n3}''' & \dots & e_{nn}''' \end{bmatrix} \quad (19)$$

The 3rd order partial realization is then obtained as

For type 1 Hankel matrix: $B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 & e_{14}''' \\ 1 & 0 & e_{24}''' \\ 0 & 1 & e_{34}''' \end{bmatrix}$, and $C_3 A_3^{-1} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \end{bmatrix}$

For type-2 Hankel matrix: $B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

$A_3^{-1} = \begin{bmatrix} 0 & 0 & e_{14}''' \\ 1 & 0 & e_{24}''' \\ 0 & 1 & e_{34}''' \end{bmatrix}$, and $C_3 = \begin{bmatrix} e_{11} & e_{12} & e_{13} \end{bmatrix}$

Extension to MIMO Systems:

The proposed method is also applicable to linear MIMO systems. The algorithms of the proposed method for the MIMO systems are almost same as discussed for the reduction of linear SISO systems. But for the sake of convenience, a brief procedure for the reduction of MIMO systems is discussed as follows:

Let the n^{th} order linear MIMO system with $'p'$ inputs and $'q'$ outputs be taken as

$$\left.\begin{aligned} \dot{X}(t) &= AX(t) + BU(t) \\ Y(t) &= CX(t) \end{aligned}\right\} \quad (20)$$

where $X(t)$, $U(t)$ and $Y(t)$ are state, input and output variable vectors and $[A]_{n\times n}$, $[B]_{n\times p}$ and $[C]_{q\times n}$ are state, input, and output matrices of the original high order system.

Let a reduced model of the order $'k'$ $(k < n)$, which reflects the dominant properties of the original high order system be expressed as

$$\left.\begin{aligned} \dot{X}_k(t) &= A_k X_k(t) + B_k U(t) \\ Y_k(t) &= C_k X_k(t) \end{aligned}\right\} \quad (21)$$

where $X_k(t)$, $Y_k(t)$ are reduced state and output vectors and $Y_k(t)$ is close approximation of $Y(t)$ and $[A]_{k\times k}$, $[B]_{k\times p}$, and $[C]_{q\times k}$ are unknown matrices of the reduced order model.

In case of MIMO system, the size of the Hankel matrix $H_{ij}^{(0)}$ is calculated as follows:

$$i = \left[\frac{n}{q}\right] \text{ and } j = \left[\frac{n+p}{p}\right] \quad (22)$$

where $[\alpha]$ is the largest integer greater than α and p, q, and n are inputs, outputs and order of the original system respectively.

The following two types of Hankel matrices with $'i'$ rows and $'j'$ columns are taken as

Type 1:

$$H_{ij}^{(0)} = \begin{bmatrix} T_1 & M_1 & M_2 & \dots & \dots & M_{j-1} \\ M_1 & M_2 & M_3 & \dots & \dots & M_j \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ M_{i-2} & M_{i-1} & M_i & \dots & \dots & M_{i+j-3} \\ M_{i-1} & M_i & M_{i+1} & \dots & \dots & M_{i+j-2} \end{bmatrix} \quad (23)$$

This matrix can be written as:

$$H_{ij}^{(0)} = \begin{bmatrix} CA^{-1} \\ C \\ \dots \\ \dots \\ CA^{i-3} \\ CA^{i-2} \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \dots & \dots & A^{j-1}B \end{bmatrix} \quad (24)$$

A partial realization is then obtained as:

$$\sum : B,\ A,\ CA^{-1} \tag{25}$$

Type 2:

$$H_{ij}^{(0)} = \begin{bmatrix} M_1 & T_1 & T_2 & \dots & \dots & T_{j-1} \\ T_1 & T_2 & T_3 & \dots & \dots & T_j \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ T_{i-2} & T_{i-1} & T_i & \dots & \dots & T_{i+j-3} \\ T_{i-1} & T_i & T_{i+1} & \dots & \dots & T_{i+j-2} \end{bmatrix} \tag{26}$$

The above matrix can be also written as:

$$H_{ij}^{(0)} = \begin{bmatrix} C \\ CA^{-1} \\ \dots \\ \dots \\ CA^{-i+2} \\ CA^{-i+1} \end{bmatrix} \begin{bmatrix} B & A^{-1}B & A^{-2}B & \dots & \dots & A^{-j+1}B \end{bmatrix} \tag{27}$$

The partial realization is obtained from above Hankel matrix as:

$$\sum : B,\ A^{-1},\ C \tag{28}$$

Let the general Hankel matrix $H_{44}^{(0)}$ of the type-1 be written as:

$$H_{ij}^{(0)} = \begin{bmatrix} [T_1]_{q\times p} & [M_1]_{q\times p} & [M_2]_{q\times p} & \dots & \dots & [M_{j-1}]_{q\times p} \\ [M_1]_{q\times p} & [M_2]_{q\times p} & [M_3]_{q\times p} & \dots & \dots & [M_j]_{q\times p} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ [M_{i-2}]_{q\times p} & [M_{i-1}]_{q\times p} & [M_i]_{q\times p} & \dots & \dots & [M_{i+j-3}]_{q\times p} \\ [M_{i-1}]_{q\times p} & [M_i]_{q\times p} & [M_{i+1}]_{q\times p} & \dots & \dots & [M_{i+j-2}]_{q\times p} \end{bmatrix} \tag{29}$$

The same procedure as discussed earlier for SISO systems for getting $H_{ij}^{(1)}$, $H_{ij}^{(2)}$,……., $H_{ij}^{(k)}$ using outer products is adopted here to get partial realizations in each case. The order of the reduced model $'k'$ is $p \le k < n$ in case of MIMO system reduction, hence the process for conversion of the Hankel matrix $H_{ij}^{(0)}$ into Hermite normal forms (i.e., $H_{ij}^{(1)}$, $H_{ij}^{(2)}$……..) can be stopped just after the k^{th} step. For getting the k^{th} order partial realization from the matrix $H_{ij}^{(k)}$, the first $'k'$ rows and first $'p'$ columns of the matrix $H_{ij}^{(k)}$ are selected to realize the matrix B_k, the first $'k'$ rows and next $'k'$ columns are selected to obtain the matrix A_k and the first $'q'$ rows and first $'k'$ columns of the Hankel matrix $H_{ij}^{(0)}$ are selected to obtain the matrix $C_k A_k^{-1}$. The same algorithm can be applied on the type-2 Hankel matrix to realize the reduced order models.

IV. Numerical Examples

Two numerical examples are taken from the literature to illustrate the algorithm of the proposed method and solved in details by using the Hankel matrices of the type 1 and type 2. The reduced order models are graphically compared with the original high order system with the help of its unit step responses. To check the goodness of the reduced order models, the ISE [9] and RISE [10] are calculated between the transient parts of the original and reduced systems and are defined as follows:

$$\text{ISE} = \int_0^\infty [y(t) - y_k(t)]^2\, dt \tag{30}$$

$$\text{RISE} = \int_0^\infty [y(t) - y_k(t)]^2\, dt \Big/ \int_0^\infty [y(t) - y(\infty)]^2\, dt \tag{31}$$

where $y(t)$ and $y_k(t)$ are the unit step responses of original and reduced order system respectively and $y(\infty)$ is the steady-state value of the original high order system.

Example 1: Consider the 4th order state model of a fuel control system of an actual boiler, which integrates a real power plant from Aguirre [7].

$$A = \begin{bmatrix} -2.06 & -0.4558 & -0.1524 & -0.05683 \\ 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \end{bmatrix},\quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0.9704 & 0.2901 & 0.1127 & 0.05437 \end{bmatrix}$$

(a) Let the type 1 Hankel matrix be used to reduce the system.

The type 1 Hankel matrix $H_{44}^{(0)}$ is obtained as

$$H_{44}^{(0)} = \begin{bmatrix} T_1 & M_1 & M_2 & M_3 \\ M_1 & M_2 & M_3 & M_4 \\ M_2 & M_3 & M_4 & M_5 \\ M_3 & M_4 & M_5 & M_6 \end{bmatrix}$$

$$H_{44}^{(0)} = \begin{bmatrix} -0.9567 & 0.9704 & -0.8386 & 0.8599 \\ 0.9704 & -0.8386 & 0.8599 & -1.2081 \\ -0.8396 & 0.8599 & -1.2081 & 1.7227 \\ 0.8599 & -1.2081 & 1.7227 & -2.2040 \end{bmatrix}$$

using the outer products, the following Hankel matrices in the Hermite normal form and corresponding reduced order models are obtained as:

$$H_{44}^{(1)} = H_{44}^{(0)} + \frac{1}{0.9567}\begin{bmatrix} -1.9567 \\ 0.9704 \\ -0.8396 \\ 0.8599 \end{bmatrix}\begin{bmatrix} -0.9567 & 0.9704 & -0.8386 & 0.8599 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1.0143 & 0.8766 & -0.8988 \\ 0 & 0.1457 & 0.0093 & -0.3359 \\ 0 & 0.0093 & -0.4730 & 0.9690 \\ 0 & -0.3359 & 0.9690 & -1.4311 \end{bmatrix}$$

hence, 1st order partial realization gives the following

$$B_1 = [1], \quad A_1 = \begin{bmatrix} -1.0143 \end{bmatrix} \text{ and } C_1 A_1^{-1} = \begin{bmatrix} -0.9567 \end{bmatrix}$$

Thus, the 1st order reduced model is obtained as :

$$\dot{X}_1 = \begin{bmatrix} -1.0143 \end{bmatrix} X_1 + [1]U$$
$$Y_1 = [0.9704] X_1$$

or ,

$$R_1(s) = \frac{0.9704}{\text{s} + 1.014}$$

now in the second step, $H_{44}^{(2)}$ can be obtained as:

$$H_{44}^{(2)} = H_{44}^{(1)} - \frac{1}{0.1457}\begin{bmatrix} -1.0143 \\ -0.8543 \\ 0.0093 \\ -0.3359 \end{bmatrix}\begin{bmatrix} 0 & 0.1457 & 0.0093 & -0.3359 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0.9413 & -3.2372 \\ 0 & 1 & 0.0638 & -2.3054 \\ 0 & 0 & -0.4736 & 0.9904 \\ 0 & 0 & 0.9904 & -2.2055 \end{bmatrix}$$

Similarly, for the 2nd order partial realization, we have

$$B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.9413 \\ 1 & 0.0638 \end{bmatrix} \text{ and}$$
$$C_2 A_2^{-1} = \begin{bmatrix} -0.9567 & 0.9704 \end{bmatrix}$$

2nd order reduced model is unstable in this case.
In the third step, $H_{44}^{(3)}$ can be obtained similarly

$$H_{44}^{(3)} = H_{44}^{(2)} + \frac{1}{0.4736}\begin{bmatrix} 0.9413 \\ 0.0638 \\ -1.4736 \\ 0.9904 \end{bmatrix}\begin{bmatrix} 0 & 0 & -0.4736 & 0.9904 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & -1.2687 \\ 0 & 1 & 0 & -2.1720 \\ 0 & 0 & 1 & -2.0912 \\ 0 & 0 & 0 & -0.1344 \end{bmatrix}$$

The 3rd order partial realization is given by

$$B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & -1.2687 \\ 1 & 0 & -2.1720 \\ 0 & 1 & -2.0912 \end{bmatrix} \text{ and}$$
$$C_3 A_3^{-1} = \begin{bmatrix} -0.9567 & 0.9704 & -0.8386 \end{bmatrix}$$

The 3rd order reduced model in time-domain is thus obtained as

$$\dot{X}_3 = \begin{bmatrix} 0 & 0 & -1.2687 \\ 1 & 0 & -2.1720 \\ 0 & 1 & -2.0912 \end{bmatrix} X_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$
$$Y_3 = [0.9704 \quad -0.8386 \quad 0.8598] X_3$$

or

$$R_3(s) = \frac{0.9704s^2 + 1.191s + 1.214}{s^3 + 2.091s^2 + 2.172s + 1.269}$$

The step responses of the first, second and third order reduced models are compared with the original system and shown in Fig. 1 and the error indices ISE and RISE are calculated between the original and reduced systems and shown in Table I.

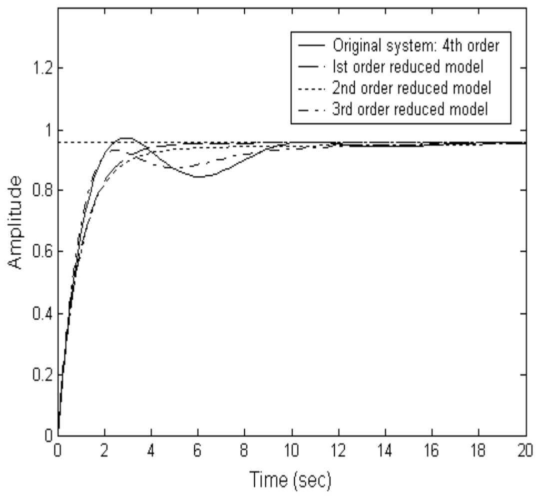

Fig. 1 Comparison of the step responses for the example 1

TABLE I
COMPARISON OF REDUCED ORDER MODELS FOR EXAMPLE I

Reduced Models	Type-1		Type-2	
	ISE	RISE	ISE	RISE
First Order	0.04180	0.10140	0.044180	0.10140
Second Order			0.039960	0.09170
Third Order	0.02943	0.06755	0.009103	0.02089

Example 2: A simplified dynamic model of a power system *i.e.* single machine connected to infinite bus power system is considered and symbols used have usual meanings as defined in paper [11].

$$\begin{bmatrix} \Delta \dot{E}_q \\ \Delta \dot{\delta} \\ \dot{\omega} \\ \dot{p} \\ \dot{p}_1 \end{bmatrix} = \begin{bmatrix} -0.188 & 0 & 0.227 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1.815 & -0.570 & -0.50 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -20 & -12 \end{bmatrix} \begin{bmatrix} \Delta E_q \\ \Delta \delta \\ \omega \\ p \\ p_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} U$$

$$\begin{bmatrix} \Delta V_t \\ \Delta \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta E_q \\ \Delta \delta \\ \omega \\ p \\ p_1 \end{bmatrix}$$

The proposed method is applied to the above power system model and the following 2nd and 3rd order reduced models are obtained which are given as

(1) Using type-1 Hankel matrix , the following 2nd and 3rd order models are obtained as

$$\dot{X}_2 = \begin{bmatrix} 0 & 0 \\ 1 & -0.1880 \end{bmatrix} X_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U$$
$$Y_2 = \begin{bmatrix} 1 & -0.1880 \\ 0 & 0 \end{bmatrix} X_2$$

$$\dot{X}_3 = \begin{bmatrix} 0 & 0 & -0.1072 \\ 1 & 0 & -1.0761 \\ 0 & 1 & -0.6881 \end{bmatrix} X_3 + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$
$$Y_3 = \begin{bmatrix} 1 & -0.1880 & -0.3765 \\ 0 & 0 & -1.8157 \end{bmatrix} X_3$$

(2) Using type-2 Hankel matrix , the 2nd order reduced model is obtained as

$$\dot{X}_2 = \begin{bmatrix} -1.9738 & 1 \\ -0.1949 & 0 \end{bmatrix} X_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} U$$
$$Y_2 = \begin{bmatrix} 1 & -5.3191 \\ 0 & 16.937 \end{bmatrix} X_2$$

The step responses of the third order model from type-1 Hankel matrix and 2nd order model from type-2 Hankel are shown in Fig. 2.

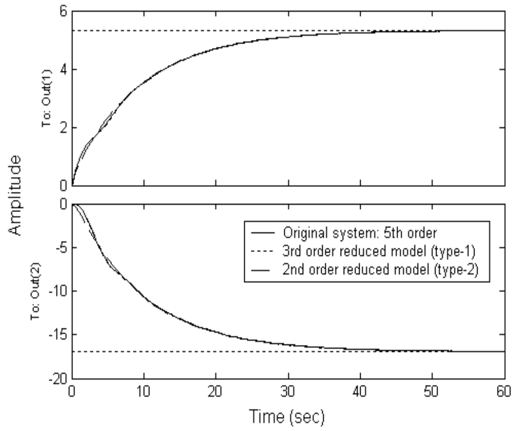

Fig. 2 Step response comparison of the reduced models

TABLE II
COMPARISON OF THE ORDER REDUCED METHODS FOR EXAMPLE 2

Method of reduction	Reduced order model		ISE
Proposed method	r_{11}	$\frac{s+1.037}{s^2+1.974s+0.1949}$	0.0819
	r_{21}	$\frac{-3.301}{s^2+1.974s+0.1949}$	1.976
Hankel Norm Approximation [11]	r_{11}	$\frac{2.734s+1.297}{s^2+2.753s+0.2175}$	0.781
	r_{21}	$\frac{5.266s-3.86}{s^2+2s+0.2495}$	10.24
Balanced Realization [11]	r_{11}	$\frac{1.04s+0.2183}{s^2+0.6535s+0.03662}$	0.2265
	r_{21}	$\frac{0.2663s-1.265}{s^2+0.5944s+0.08173}$	3.12
Routh approximation [11]	r_{11}	$\frac{12.172s+11.4}{17.006s^2+19.42s+2.086}$	0.5236
	r_{21}	$\frac{-21.78s-36.3}{17.006s^2+19.42s+2.086}$	12.45

V. CONCLUSION

In this paper, modified hankel matrices are used to synthesize the reduced order models .The proposed method is based on outer product and it has been concluded that the algorithm is simple and efficient. Two numerical examples are taken from literature and solved by using the proposed method to get lower order models. In the example 1, 2nd and 3rd order model are synthesized which matched the step response of original system very closely. In example 2, multivariable system is reduced to 2nd and 3rd order models which are very closely matching the features of original system. The proposed method is also compared with the existing well-known reduction methods and found better in quality.

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