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Power Series Form for Solving Linear Fredholm Integral Equations of Second Order via Banach Fixed Point Theorem

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Abstract**—**In this paper, a new method for solution of second order linear Fredholm integral equation in power series form was studied. The result is obtained by using Banach fixed point theorem.

Keywords**—**Fredholm integral equation, power series, Banach fixed point theorem, Linear Systems.

I. Introduction

ONE of the outstanding practical problems met in the solution of Fredholm integral equation is the form of power series. Fredholm integral equation has wide application in physics and other science. For updating work, the electro elastic analysis of a piezoelectric strip with an anti-plane shear crack whose surfaces are parallel to the strip boundaries is made via solving a resulting Fredholm integral equation using the iterative method of power series [1]. Borzabadi and Fard used approximate solutions for nonlinear Fredholm integral equations of the first kind by converting it to moment problem [2]. The torsion problem of a finite elastic cylinder with a circumferential edge crack is studied in a Fredholm integral equation of the second kind [3]. Parandin and Araghi solved fuzzy Fredholm integral equation of the second kind numerically based on iterative interpolation [4]. In [5], the problem of the displacement of the surface of the plate outside the punch reduces to a Fredholm integral equation of the first kind in a function describing. These functions are sought in the form of a sum of a Schl-milch series and a power function with a root singularity. Maleknejad et al. solved first kind Fredholm integral equation numerically based on sinc function [6]. Fredholm integral equations of the second kind without explicit characteristic values and characteristic functions in the Neumann power series is studied [7]. Aruchunan and Sulaiman used the Half-sweep approximation equation based on central difference and repeated trapezoidal formulas to solve linear fredholm integro-differential equations of first order [8]. Hou et al. solved nonlinear Fredholm integral equation of the second kind numerically based on Chebyshev polynomials and hybrid blok-pulse functions [9].

The main purpose of this paper is to introduce and study the relation between Banach fixed point theorem and the solution of second order linear Fredholm integral equation in power series form.

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II. New Result for Solving Fredholm Integral Equations

A. Definition of Fredholm Integral Equation

A linear Fredholm integral equation of second order has the form

$$u(t) = f(t) + \lambda\int_a^b k(t,y)u(y)dy \tag{1}$$

Here, the functions $k:[a,b]\times[a,b]\to R$ and $f:[a,b]\to R$ are given, $\lambda\in R$ is an arbitrary parameter, and $u:[a,b]\to R$ is the unknown function [10]. The function k is called the kernel of the equation. The equation is said to be homogeneous if $f=0$ and non - homogeneous otherwise.

A metric space (X,d) is called complete if every Cauchy sequence in X converges to a point in X. A complete normed metric space (X,d) is called Banach [11].

Theorem 1 (Banach Fixed Point Theorem): Let T be a contraction on a complete metric space (X,d),then T has a unique fixed point $x\in X$.

Proof [10]:

Now we can suppose that k is continuous on the square $[0,1]\times[0,1]$ and that f is continuous on $[0,1]$. Then, the function $T(u)$ defined by

$$T(u(t)) = f(t) + \int_0^1 k(t,y)u(y)dy \tag{2}$$

is continuous on $[0,1]$ for each continuous function u on $[0,1]$.

In other words, the mapping $u\to T(u)$ is a transformation on $(C[0,1],D_2)$,

$$D_2^2(u,v) = \int_0^1 \left|u(t)-v(t)\right|^2 dt \cdot$$

Then the Fredholm equation (2) becomes

$$u = T(u) \tag{3}$$

and thus, solving (1) is equivalent to find the fixed points of the transformation T on Banach space $(C[0,1], D_2)$.

In order to apply the Banach fixed point theorem, all we need to suppose that T is a contraction (where $(C[0,1], D_2)$ is complete).

B. Second Order Power Series Fredholm Integral Equations

From above, one can study the following proposition.

Proposition 1: Suppose that f and k are continuous in Banach space $(C[0,1], D_2)$. Then there is a unique power series solution

$$u(t) = \sum_{i=0}^{N} c_i t^i$$

For Fredholm integral equation

$$u(t) = f(t) + \int_0^1 k(t,y)u(y)dy \cdot$$

Proof:

One can suppose that u is represented in a power series form:

$$u(t) = \sum_{i=0}^{N} c_i t^i \tag{4}$$

Then

$$\Delta = D_2^2(u, T(u))$$

$$\Delta = \int_0^1 \left| u(t) - T(u(t)) \right|^2 dt$$

$$\Delta = \int_0^1 \left| \sum_{i=0}^{N} c_i t^i - f(t) - \int_0^1 k(t,y) \sum_{i=0}^{N} c_i y^i dy \right|^2 dt \tag{5}$$

By letting

$$g_i = \int_0^1 y^i k(t,y)dy, i = 0,1,\ldots,N \tag{6}$$

Then

$$\Delta = \int_0^1 \left(\sum_{i=0}^{N} c_i (t^i - g_i) - f(t)\right)^2 dt \tag{7}$$

By using least square approximation, the values c's are calculated from the following linear system

$$\begin{bmatrix} <(1-g_0)(1-g_0)> & <(t-g_1)(1-g_0)> & \ldots & <(t^N-g_N)(1-g_0)> \\ <(1-g_0)(t-g_1)> & <(t-g_1)(t-g_1)> & \ldots & <(t^N-g_N)(t-g_1)> \\ \vdots & \vdots & \vdots & \vdots \\ <(1-g_0)(t^N-g_N)> & <(t-g_1)(t^N-g_N)> & \ldots & <(t^N-g_N)(t^N-g_N)> \end{bmatrix} . \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix} = \begin{bmatrix} <f.(1-g_0)> \\ <f.(t-g_1)> \\ \vdots \\ <f.(t^N-g_N)> \end{bmatrix} \tag{8}$$

where

$$<z> = \int_0^1 z(t)dt$$

In matrix notation, the linear system (8) can be presented as

$$GC = b \tag{9}$$

C. Examples

For explaining the introduced propositions we can use the following examples in Tables I, II, and III for N=2 where f and k are known to solve $u(t) = \sum_{i=0}^{3} c_i t^i$ such that $\bar{u}(t)$ is approximated to u and e represents the error.

TABLE I
POWER SERIES SOLUTION

NO	1	2	3
$f(t)$	t	t	t^2
k	ty	t^2y^2+ty	ty
G	$\begin{bmatrix} \frac{7}{12} & \frac{2}{9} & \frac{1}{8} \\ \frac{2}{9} & \frac{4}{27} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{9} & \frac{23}{240} \end{bmatrix}$	$\begin{bmatrix} \frac{7}{15} & \frac{21}{160} & \frac{61}{1200} \\ \frac{21}{160} & \frac{167}{2160} & \frac{769}{14400} \\ \frac{61}{1200} & \frac{769}{14400} & \frac{239}{6000} \end{bmatrix}$	$\begin{bmatrix} \frac{7}{12} & \frac{2}{9} & \frac{1}{8} \\ \frac{2}{9} & \frac{4}{27} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{9} & \frac{23}{240} \end{bmatrix}$
B	$\begin{bmatrix} 1/3 \\ 2/9 \\ 1/6 \end{bmatrix}$	$\begin{bmatrix} 1/4 \\ 23/144 \\ 7/60 \end{bmatrix}$	$\begin{bmatrix} 5/24 \\ 1/6 \\ 11/80 \end{bmatrix}$
$u(t)$	$\begin{bmatrix} 0 \\ 3/2 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \frac{192}{113} \\ \frac{60}{113} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 3/8 \\ 1 \end{bmatrix}$
$\bar{u}(t)$	$u(t)$	$u(t)$	$u(t)$
e	0	0	0

TABLE II
POWER SERIES SOLUTION

NO	4	5	6
$f(t)$	t^2+t	t^4+1	e^t
k	ty	ty	ty
G	$\begin{bmatrix} \frac{7}{12} & \frac{2}{9} & \frac{1}{8} \\ \frac{2}{9} & \frac{4}{27} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{9} & \frac{23}{240} \end{bmatrix}$	$\begin{bmatrix} \frac{7}{12} & \frac{2}{9} & \frac{1}{8} \\ \frac{2}{9} & \frac{4}{27} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{9} & \frac{23}{240} \end{bmatrix}$	$\begin{bmatrix} 0.50 & 0.14 & 0.05 \\ 0.14 & 0.08 & 0.05 \\ 0.05 & 0.05 & 0.05 \end{bmatrix}$
B	$\begin{bmatrix} 13/24 \\ 7/18 \\ 73/240 \end{bmatrix}$	$\begin{bmatrix} 13/15 \\ 4/9 \\ 13/42 \end{bmatrix}$	$\begin{bmatrix} 1.2183 \\ 0.6667 \\ 0.4683 \end{bmatrix}$
$u(t)$	$\begin{bmatrix} 0 \\ 15/8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 38/35 \\ 3/35 \\ 12/7 \end{bmatrix}$	$\begin{bmatrix} 1.013 \\ 2.3511 \\ 0.8392 \end{bmatrix}$
$\bar{u}(t)$	U	t^4+t+1	$e^t+\frac{2}{3}t$
E	0	0.0381	0.0053

TABLE III
POWER SERIES SOLUTION

NO	7	8
$f(t)$	$\sin t$	$\cos t$
k	$t^2y^2+ty^2$	ty
G	$\begin{bmatrix} 0.50 & 0.14 & 0.05 \\ 0.14 & 0.08 & 0.05 \\ 0.05 & 0.05 & 0.05 \end{bmatrix}$	$\begin{bmatrix} \frac{7}{12} & \frac{2}{9} & \frac{1}{8} \\ \frac{2}{9} & \frac{4}{27} & \frac{1}{9} \\ \frac{1}{8} & \frac{1}{9} & \frac{23}{240} \end{bmatrix}$
B	$\begin{bmatrix} 0.2477 \\ 0.1515 \\ 0.1072 \end{bmatrix}$	$\begin{bmatrix} 0.6506 \\ 0.2545 \\ 0.1437 \end{bmatrix}$
$u(t)$	$\begin{bmatrix} -0.007 \\ 1.55 \\ 0.3702 \end{bmatrix}$	$\begin{bmatrix} 1.003 \\ 0.536 \\ -0.431 \end{bmatrix}$
$\bar{u}(t)$	$\sin t+0.45t+0.6t^2$	$\cos t+0.75t$
e	0.0027	0.0015

III. CONCLUSION

A new method for solution of Fredholm integral equation was studied in condition that it presented as in power series form. In the proof of the main result, well known Banach fixed point theorem has been used. For test the efficiency of introduced result, many programmed examples were presented. The calculations in Table I showed that the error is very small and tends to zero speedy.

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REFERENCES

[1] X. F. Li, “Electro Elastic Analysis of an Anti-Plane Shear Crack in a Piezoelectric Ceramic Strip”, International, Journal of Solids and Structures 39, 2002, pp1097–1117.
[2] A. Borzabadi and O. Fard, “Approximate Solution of Nonlinear Fredholm Integral Equations of the First Kind via Converting to Optimization Problems”, World Academy of Science, Engineering and Technology, Vol: 9 2007, pp802-805.
[3] P. Malits, “Torsion of a Cylinder with a Shallow External Crack”, International Journal of Solids and Structures 46, 2009, pp 3061– 3067.
[4] N. Parandin, and M. Araghi, “The Approximate Solution of Linear Fuzzy Fredholm Integral Equations of the Second Kind by Using Iterative Interpolation”, World Academy of Science, Engineering and Technology, Vol: 25 2009, pp.907-913.
[5] N.A. Bazarenko, “The Contact Problem for a Circular Plate with a Stress-Free End Face”, Journal of Applied Mathematics and Mechanics, 74, 2010, pp. 699–709.
[6] K. Maleknejad, R. Mollapourasl,P. Torabi and M. Alizadeh, “Solution of First Kind Fredholm Integral Equation by Sinc Function”, World Academy of Science, Engineering and Technology, Vol:42 2010, pp.1539-1543.
[7] C. Gu & Y. Tao, “Function-Valued Padé-Type Approximant via E-Algorithm and Its Applications in Solving Integral Equations”, Applied Mathematics and Computation 217, 2011, pp 7975–7984.
[8] E. Aruchunan and J. Sulaiman, “Application of the Central-Difference with Half Sweep Gauss-Seidel Method for Solving First Order Linear Fredholm Integro-Differential Equations”, World Academy of Science, Engineering and Technology, International Journal of Mathematical Science and Engineering Vol:68, 2012,pp.335-339.
[9] J. Hou, C. Yang, B. Qin, “Hybrid Function Method for Solving Non Linear Fredholm Integral Equations of The Second Kind”, World Academy of Science, Engineering and Technology, International Journal of Mathematical Science and Engineering Vol: 7 No: 2, 2013, pp.729-732.
[10] J. Jerry, “Introduction to Integral Equations with Applications”, Marcel Dekker, INC, 1985.
[11] J. Hutchinson, “Fractals and Self-Similarity”, Journal of Indiana University Mathematics, 30, 1981, pp.713-740.

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