International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:8, No:3, 2014

Stress Variation around a Circular Hole in Functionally Graded Plate under Bending

Parveen K. Saini, Mayank Kushwaha

***Abstract*—**The influence of material property variation on stress concentration factor (SCF) due to the presence of a circular hole in a functionally graded material (FGM) plate is studied in this paper. A numerical method based on complex variable theory of elasticity is used to investigate the problem. To achieve the material property, variation plate is decomposed into a number of rings. In this research work, Young's modulus is assumed to be varying exponentially and it is found that stress concentration factor can be reduced by increasing Young's modulus progressively away from the hole.

***Keywords*—**Stress Concentration, Circular Hole, FGM Plate, Bending.

I. Introduction

APPLICATION of pure materials in engineering processes is very restricted as they provide a small amount of desirable properties. This issue can be overcome with the use of composite materials. FGMs are one of the different types of composite materials where the properties vary as a function of spatial coordinates. The concept of FGM was first examined in Japan in 1984 during a space plane project. Since then a great attention has been paid to the study of the behavior FGM under various loading and application conditions. FGMs can be used for thermal barrier coating in gas turbine engines and rocket nozzles, sensors, wear resistant coating, Nano-composites and aircraft construction.

In recent years, researchers have investigated the problem to reduce SCF around the hole in FGM concept. Birman et al. [1] proposed a review of development and possible application of functionally graded materials. Dryden and Batra [2] obtained the optimum Young's modulus of a homogeneous cylinder equivalent to FG cylinder. Effect of different geometries on stress concentration factor has also been paid attention by many authors as Hosseini – Hasemi [3], Batra [4]. Buskirk et al. [5] & Nagpal et al. [6] found that the radial increase in modulus of elasticity can cause SCF to be reduced in comparison to homogeneous plate. Kim et al. [7] presented an interaction integral method for the evaluation of the stress in an orthotropic FGM plate under thermal loading. Zhao et al. [8] studied the natural vibration analysis of FG plate using Kp-Ritz method. Ying et al. [9] presented two-dimensional elastic solution of FG beam resting on elastic foundation. Allam et al. [10] studied the stress distribution in a fiber reinforced composite plate under uniaxial tension. Dai et al. [11] investigated SCF in transversely isotropic piezoelectric solid due to circular hole. Kosmodamianskii et al. [12] studied the bending of anisotropic plate with arbitrary hole shape. To compute the complex potentials of Lekhnitskii, they used discrete method of least squares at the boundary. Bending of an elliptical anisotropic plate with two elliptical holes was investigated by Meglinskii [13]. Santhanam et al. [14] examined a finite plate weakened by a crack due to the presence of a hole under far-field uniform bending. Analytical and FEM methods were used to investigate the cases of on-center and off-center hole. Yang et al. [15] have shown that Poisson's ratio has a lesser influence on SCF than Young's modulus. The distribution of stress, in an FGM plate having circular hole, was investigated under the application of tensile loading. The solution was obtained by the complex variable function using the piecewise homogeneous layer method. Kubair and Bhanuchandar [16] worked on the influence of material inhomogeneity on SCF due to the circular hole in FGM plate using power law of material property variation. They concluded that with the increase in Young's Modulus away from the hole, the SCF decreases in FGM plate. Mohammadi et al. [17] assumed Young's modulus to be a function of two adjustable parameters. A plate was modeled where the functionally graded response is bounded in a region around the hole. SCF in FGM plate was investigated considering elastic properties to be exponentially variable in radial direction. Sburlati [18] decomposed a homogeneous plate into functionally graded rings around the round hole. The solution was obtained in closed form for uniaxial tensile loading. Young's modulus was considered to be varied with a monotonic power law.

This work is related to stress concentration reduction in FGM plate by varying the material properties as a known function of the spatial position.

II. Problem Description

In a fixed coordinate system (x, y, z) the state of stress of an isotropic solid can be formulated as [19], [20],

$$\sigma_y + \sigma_x = 4Re[\varphi'(z)] \tag{1}$$

$$\sigma_y + \sigma_x - 2i\sigma_{xy} = 2[\bar{z}\varphi''(z) + \psi'(z)] \tag{2}$$

where $z = x + iy$ and σ_x,σ_y, σ_{xy} are the stress components. $\varphi(z)$and $\psi(z)$ are some potential functions which can be evaluated using following boundary conditions

P. K. Saini is an Associate Professor of Mechanical Engineering at National Institute of Technology, Kurukshetra, 132119, India (phone: 0091-1744-228880; fax: 0091-1744-238050; e-mail: pksaini@ nitkkr.ac.in).

M. Kushwaha is with the Mechanical Engineering Department, National Institute of Technology, Kurukshetra, 132119, India (phone: 0091-1744-233469; fax: 0091-1744-238050; e-mail: kush.048@gmail.com).

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = \pm i \int_0^s (X_n + Y_n) ds \quad (3)$$

$$\varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} = 2G(u + iv) \quad (4)$$

X_n, Y_n, u and v in the above equations stand for stress and displacement components at the boundary of hole, respectively. $\varkappa$ and G are the elastic constants which have different values in different cases, i.e. plane stress or plane strain.

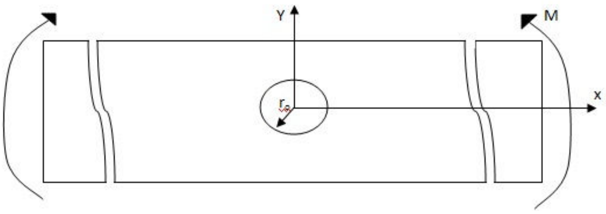

Fig. 1 Infinite FGM plate having a circular hole subjected to bending moment

Fig. 1 shows an FGM plate subjected to a far field uniform bending moment at the boundary. A round hole is considered at the center of the plate and assumed to be free from all the external forces and displacements. FGM plate is decomposed of N number of annular rings around the hole as in Fig. 2. Material properties are assumed to vary exponentially with radius. The outer boundary of the plate is considered to be (N+1) th ring.

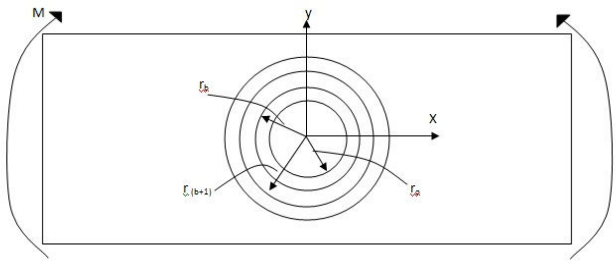

Fig. 2 FGM plate decomposed into rings

Inside of any of the N rings the complex potential functions $\varphi_b(z)$ and $\psi_b(z)$ can be taken as

$$\varphi_b(z) = \sum_{-\infty}^{\infty} \alpha^b_{-a} \left(\frac{R_0}{z}\right)^a \quad (5)$$

$$\psi_b(z) = \sum_{-\infty}^{\infty} \beta^b_{-a} \left(\frac{R_0}{z}\right)^a \quad (6)$$

(b = 1, 2 ………… N) and R_0 is some arbitrary radius. α^b_{-a} and β^b_{-a} are the constants to be determined.

In the outermost ring, (N+1) th ring, potential functions can take the form

$$\varphi_{N+1}(z) = \Gamma z^2 + \varphi_0(z) \quad (7)$$

$$\psi_{N+1}(z) = \Gamma' z^2 + \psi_0(z) \quad (8)$$

where Γ and Γ' are constants. $\varphi_0(z)$ and $\psi_0(z)$ are the holomorphic complex functions of the state of stress in plate without hole and can be expressed as

$$\varphi_0(z) = \sum_{a=0}^{\infty} \alpha^{(N+1)}_{-a} \left(\frac{R_0}{z}\right)^a \quad (9)$$

$$\psi_0(z) = \sum_{a=0}^{\infty} \beta^{(N+1)}_{-a} \left(\frac{R_0}{z}\right)^a \quad (10)$$

where $\alpha^{(N+1)}_{-a}$ and $\beta^{(N+1)}_{-a}$ are unknown constants.

From (7)-(10) it can be formulated that

$$\varphi_{(N+1)}(z) = \sum_{-\infty}^{\infty} \alpha^{(N+1)}_{-a} \left(\frac{R_0}{z}\right)^a \quad (11)$$

$$\psi_{(N+1)}(z) = \sum_{-\infty}^{\infty} \beta^{(N+1)}_{-a} \left(\frac{R_0}{z}\right)^a \quad (12)$$

For any of the rings the analytical functions can be written in following form by combining (5), (6), (11), and (12).

$$\varphi_b(z) = \sum_{-\infty}^{\infty} \alpha^b_{-a} \left(\frac{R_0}{z}\right)^a \quad (13)$$

$$\psi_b(z) = \sum_{-\infty}^{\infty} \beta^b_{-a} \left(\frac{R_0}{z}\right)^a \quad (14)$$

where b = 1, 2, 3………….. (N+1)

Since the surface of the hole is considered free from forces, i.e. $X_n = 0$ and $Y_n = 0$, then from (3)

$$\varphi_1(z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)} = 0 \quad (15)$$

where $z = r_0\gamma$ and $\gamma = e^{i\theta}$

Equating the coefficients of γ^a on the both sides of (15) after substituting $\varphi_1(z)$ and $\psi_1(z)$ from (13) and (14), we get

$$\left(\frac{R_0}{r_0}\right)^a \alpha^1_{-a} + (a+2)\left(\frac{r_0}{R_0}\right)^{(a+2)} \overline{\alpha}^{(1)}_{a+2} + \left(\frac{r_0}{R_0}\right)^a \overline{\beta}^{(1)}_a = 0 \quad (16)$$

$$\left(\frac{r_0}{R_0}\right)^a \alpha^1_a - (a-2)\left(\frac{R_0}{r_0}\right)^{(a-2)} \overline{\alpha}^{(1)}_{-(a-2)} + \left(\frac{R_0}{r_0}\right)^a \overline{\beta}^{(1)}_{-a} = 0 \quad (17)$$

where r_o is the hole radius, a = 1, 2…… M

For the study of stress distribution in the plate it is reasonable to assume that there are no residual stresses at the interface of rings. This condition can be expressed as

$$X_n^b = -X_n^{(b+1)}, Y_n^b = -Y_n^{(b+1)} \quad (18)$$

$$u^b = u^{(b+1)}, \; v^b = v^{(b+1)} \quad (19)$$

On substituting (3) into (18) and (4) into (19),

$$\varphi_b(z) + z\overline{\varphi_b'(z)} + \overline{\psi_b(z)} = \varphi_{(b+1)}(z) + z\overline{\varphi_{(b+1)}'(z)} + \overline{\psi_{(b+1)}(z)} \quad (20)$$

$$\frac{1}{G_b}\left[\varkappa_b\varphi_b(z) - z\overline{\varphi_b'(z)} - \overline{\psi_b(z)}\right] = \frac{1}{G_{(b+1)}}\left[\varkappa_{(b+1)}\varphi_{(b+1)}(z) - z\overline{\varphi_{(b+1)}'(z)} - \overline{\psi_{(b+1)}(z)}\right] \quad (21)$$

where $z = r_b e^{i\theta} = r_b \gamma$, b = 1, 2, 3 ……… N

From (13), (14), (20), (21) and by following the same procedure as for (16) and (17), we obtain

$$\left(\frac{R_0}{r_b}\right)^a \alpha^b_{-a} + (a+2)\left(\frac{r_b}{R_0}\right)^{(a+2)} \overline{\alpha}^{(b)}_{a+2} + \left(\frac{r_b}{R_0}\right)^a \overline{\beta}^{(b)}_a = \left(\frac{R_0}{r_b}\right)^a \alpha^{(b+1)}_{-a} + (a+2)\left(\frac{r_b}{R_0}\right)^{(a+2)} \overline{\alpha}^{(b+1)}_{a+2} + \left(\frac{r_b}{R_0}\right)^a \overline{\beta}^{(b+1)}_a \quad (22)$$

$$\left(\frac{r_b}{R_0}\right)^a \alpha^b_a - (a-2)\left(\frac{R_0}{r_b}\right)^{(a-2)} \overline{\alpha}^{(b)}_{-(a-2)} + \left(\frac{R_0}{r_b}\right)^a \overline{\beta}^{(b)}_{-a} = \left(\frac{r_b}{R_0}\right)^a \alpha^{(b+1)}_a - (a-2)\left(\frac{R_0}{r_b}\right)^{(a-2)} \overline{\alpha}^{(b+1)}_{-(a-2)} + \left(\frac{R_0}{r_b}\right)^a \overline{\beta}^{(b+1)}_{-a} \quad (23)$$

$$\frac{1}{G_b}\left[\varkappa_b \left(\frac{R_0}{r_b}\right)^a \alpha^b_{-a} - (a+2)\left(\frac{r_b}{R_0}\right)^{(a+2)} \overline{\alpha}^{(b)}_{a+2} - \left(\frac{r_b}{R_0}\right)^a \overline{\beta}^{(b)}_a\right] = \frac{1}{G_{(b+1)}}\left[\varkappa_{(b+1)} \left(\frac{R_0}{r_b}\right)^a \alpha^{(b+1)}_{-a} - (a+2)\left(\frac{r_b}{R_0}\right)^{(a+2)} \overline{\alpha}^{(b+1)}_{a+2} - \left(\frac{r_b}{R_0}\right)^a \overline{\beta}^{(b+1)}_a\right] \quad (24)$$

$$\frac{1}{G_b}\left[\varkappa_b \left(\frac{r_b}{R_0}\right)^a \alpha^b_a + (a-2)\left(\frac{R_0}{r_b}\right)^{(a-2)} \overline{\alpha}^{(b)}_{-(a-2)} - \left(\frac{R_0}{r_b}\right)^a \overline{\beta}^{(b)}_{-a}\right] = \frac{1}{G_{(b+1)}}\left[\varkappa_{(b+1)} \left(\frac{r_b}{R_0}\right)^a \alpha^{(b+1)}_a + (a-2)\left(\frac{R_0}{r_b}\right)^{(a-2)} \overline{\alpha}^{(b+1)}_{-(a-2)} - \left(\frac{R_0}{r_b}\right)^a \overline{\beta}^{(b+1)}_{-a}\right] \quad (25)$$

Equations (16), (17) and (21)–(25) constitute a set of linear equations which can be solved for the unknown variables. Complex analytical functions can be evaluated by using these variables and by the use of these functions the value of stress can easily be calculated.

III. Results and Discussions

The results of the comprehensive two dimensional finite analysis that has carried out to determine the stress distribution around circular hole in FGM plate for various values of Young's modulus is presented in this section.

A. Influence of the Variation of Young's Modulus on SCF

To illustrate the effect of variation of Young's modulus on distribution, three different models are used while the Poisson's ratio is assumed to be constant.

$$E_{dec}(r) = E_{\propto}\, e^{(1.0\rho^{-1})}$$

$$E_0(r) = E_{\propto}$$

$$E_{inc}(r) = E_{\propto}\, e^{(-1.0\rho^{-1})}$$

The stress variation around the hole is shown in Fig. 3

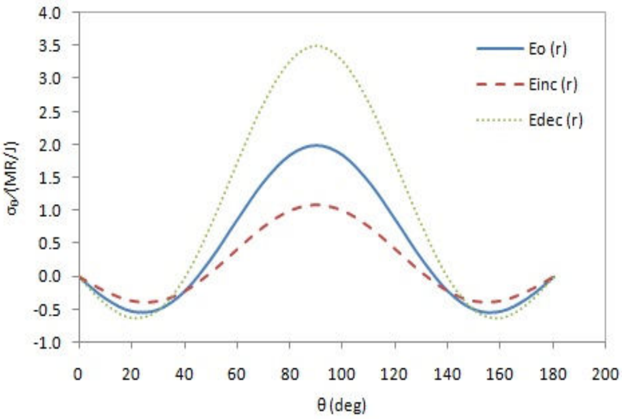

Fig. 3 Variations of stress on the circumference of the hole under bending

From Fig. 3 it is clear that SCF tends to decrease when Young's modulus increases radially away from the hole. On the other hand, decrease in Young's modulus in radial direction results in the increase of SCF around the hole.

B. Effect of Different Young's Modulus Variation on SCF

In this section SCF is determined by using different exponential functions for the increase of Young's modulus. Fig. 4 depicts the stress distribution for the case where the plate has three different increase functions of the Young's modulus.

$$E_1(r) = E_{\propto}\, e^{(-1.0\rho^{-1})}$$

$$E_2(r) = E_{\propto}\, e^{(-2.0\rho^{-1})}$$

$$E_3(r) = E_{\propto}\, e^{(-3.0\rho^{-1})}$$

Again three different functions for the decrement of Young's modulus are taken here and SCF is calculated as shown in Fig. 5.

$$E_1(r) = E_{\propto}\, e^{(1.0\rho^{-1})}$$

$$E_2(r) = E_{\propto}\, e^{(2.0\rho^{-1})}$$

$$E_3(r) = E_{\propto}\, e^{(3.0\rho^{-1})}$$

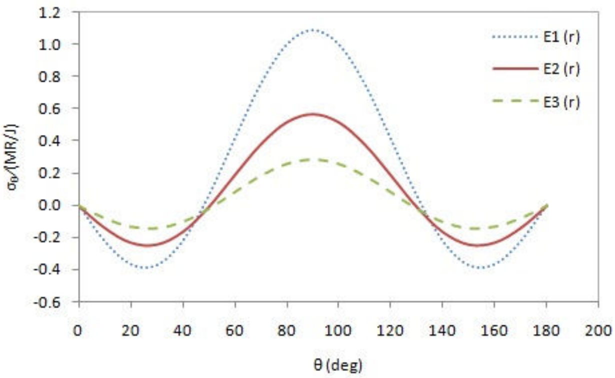

Fig. 4 Variation of stress around hole for different ways of E increase

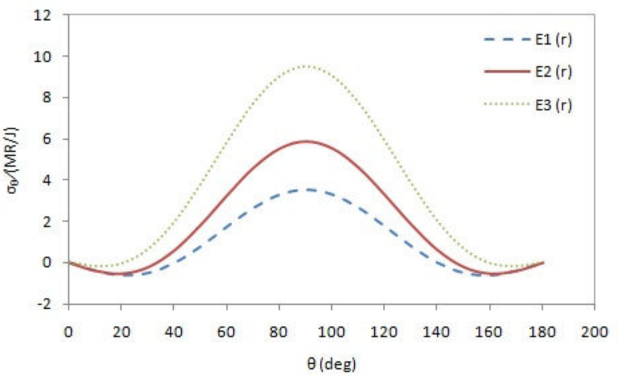

Fig. 5 Variation of stress around hole for different ways of E decrease

Above study of effect of different ways of increasing and decreasing Young's modulus, was conducted for hole radius, $r_o = 4$

It is clear form from Figs. 4 & 5 that stress concentration around hole is greatly affected by the way material properties vary. So for the reduction of stress concentration, selection of this variation should be done carefully.

C. Influence of Hole Size on Stress Concentration

In this section two different FGM plate having different variation of Young's modulus are considered.

$$E_{dec}\,(r) = E_{\propto}\, e^{(1.0\rho^{-1})}$$

$$E_{inc}\,(r) = E_{\propto}\, e^{(-1.0\rho^{-1})}$$

Figs. 6 and 7 shows the stress distribution around the hole for different hole sizes. The stress concentration factor always increases with the increase in hole size for an FGM plate under bending.

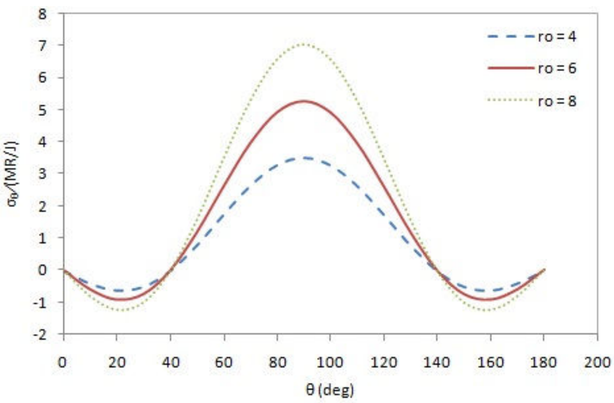

Fig. 6 Variation of stress around hole for different hole size when E decreases

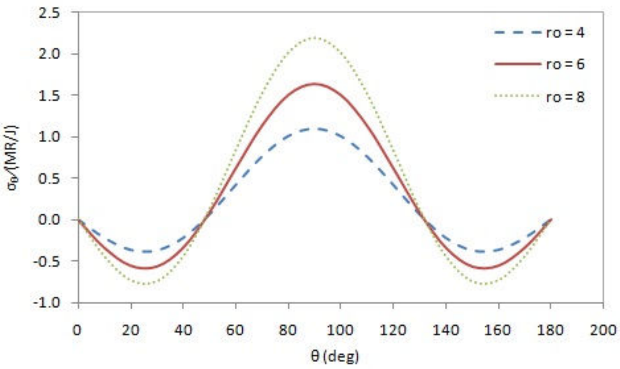

Fig. 7 Variation of stress around hole for different hole size when E increases

IV. Conclusion

Here we have considered an infinite functionally graded plate with circular hole subjected to bending moment. For the study of stress distribution around hole, the plate is decomposed into number of rings concentric with the hole. Complex variable theory of elasticity, proposed by Muskhelishvili [19], is used. The results obtained here are in coordination with those given by Savin [20] for homogeneous plate. From the graphs, shown above, it can be concluded that stress concentration factor can be reduced if Young's modulus increases radially away from the hole.

Additionally, it is found that bending stress concentration factor increases on increasing the hole size for FGM plate.

References

[1] V. Birman, and L. W. Byrd, "Modeling and Analysis of Functionally Graded Materials and Structures," *Applied Mechanics Review*, vol. 60, pp. 195-216, 2007.

[2] J. R. Dryden, and R. C. Batra, "Optimum Young's Modulus of a Homogeneous Cylinder Energetically Equivalent to a Functionally Graded Cylinder," *Journal of Elasticity*, vol. 110, pp. 95-110, 2013.

[3] S. H. Hosseini-Hashemi, H. Salehipuor, S. R. Atashipour, and R. Sburlati, "On the Exact In-Plane and Out-Plane Free Vibration Analysis of Thick Functionally Graded Rectangular Plates: Explicit 3-D Elastic Solutions," *Composites Part B: Engineering*, vol. 46 (1), pp. 108-115, 2013.

[4] R. C. Batra, "Material Tailoring and Universal Relations for Axis-Symmetric Deformations of Functionally Graded Rubberlike Cylinders and Spheres," *Mathematics and Mechanics of Solids,* vol. 16, 729-738, 2011.
[5] S. R. Buskirk, S. Venkataraman, P. G. Ifju, and A. J. Rapoff, "Functionally Graded Biomimetic Plate with Hole," *SEM Annual Conference & Exposition on Experimental and Applied Mechanics*, 2002.
[6] S. Nagpal, S. Jain, and S. Sayal, "Stress Concentration and Its Mitigation Techniques in a Flat Plate with Singularities: A Critical Review," *Engineering Journal*, 16 (1), 2012.
[7] J. H. Kim, and A. Kc, "A Generalized Interaction Integral Method for the Evaluation of T-Stress in Orthotropic Functionally Graded Materials under Thermal Loading," *Journal of Applied Mechanics Transactions ASME*, 75, pp. 1-11, 2008.
[8] X. Zhao, Y. Y. Lee, and K. M. Liew, "Free Vibration Analysis of Functionally Graded Plates Using Element-Free kp-Ritz Method," *Journal of Sound Vibration*, vol. 319, pp. 918-939, 2009.
[9] J. Ying, C. F. Lu, and W. Q. Chen, "Two-Dimensional Elastic Solutions for Functionally Graded Beams Resting on Elastic Foundation," *Composite Structures*, vol. 84, pp. 209-219, 2008.
[10] M. N. M. Allam, A. M. Zenkour, and H. F. El-Mekawy, "Stress Concentrations in a Viscoelastic Composite Plate Weakened by a Triangular Hole," *Composite Structures*, vol. 79, pp. 1-11, 2007.
[11] L. Dai, W. Guo, and X. Wang, "Stress Concentration at an Elliptical Hole in Transversely Isotropic Piezoelectric Solids," *International Journal of Solids and Structures,* vol. 43, pp. 1818-1831, 2006.
[12] A. S. Kosmodamianskii, N. M. Neskorodev, and L. N. Profatilo, "Bending of an Anisotropic Plate with Holes of Arbitrary Shape," *International Applied Mechanics*, vol. 33, pp. 491-496, 1997.
[13] V. V. Meglinskii, "Bending of an Elliptical Anisotropic Plate with Two Elliptical Holes," *Prikladnaya Mekhanika,* vol. 2, pp. 19-27, 1966.
[14] S. Santhanam, J. N. Majerus, K.P. Jen, and D. S. Karanth, "Analytical/Numerical Solution for a Single Hole-Edge Crack in a Narrow Plate under Pure Bending," *Engineering Fracture Mechanics,* vol. 46, pp. 751-761, 1993.
[15] Q. Yangn, C. F. Gao, and W. Chen, "Stress Analysis of a Functionally Graded Plate with Circular Hole," *Archive Applied* Mechanics, vol. 80, pp. 895-907, 2009.
[16] D. V. Kubair and B. Bhanu-Chandar, "Stress Concentration Factor Due to a Circular Hole in a Functionally Graded Panels under Uniaxial Tension," *International Journal of Mechanical Sciences*, vol. 50, pp. 732-742, 2008.
[17] M. Mohammadi, J. R. Dryden, and L. Jiang, "Stress Concentration around a Hole in a Radially Inhomogeneous Plate," *International Journal of Solids and Structures*, vol. 48, pp. 483-491, 2011.
[18] R. Sburlati, "Stress Concentration Factor Due to a Functionally Graded Ring around a Hole in an Isotropic Plate," *International Journal of Solids and Structures*, 2013.
[19] N. I. Muskhelishvili, "Some Basic Problem of Mathematical Theory of Elasticity," *Noordhoof, Leyden*, 1975.
[20] G. N. Savin, "Stress Concentration around Holes," Pergamon Press, New York, 1961.