

# Reliability Analysis of $k$ -out-of- $n : G$ System Using Triangular Intuitionistic Fuzzy Numbers

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**Abstract**—In the present paper, we analyze the vague reliability of  $k$ -out-of- $n : G$  system (particularly, series and parallel system) with independent and non-identically distributed components, where the reliability of the components are unknown. The reliability of each component has been estimated using statistical confidence interval approach. Then we converted these statistical confidence interval into triangular intuitionistic fuzzy numbers. Based on these triangular intuitionistic fuzzy numbers, the reliability of the  $k$ -out-of- $n : G$  system has been calculated. Further, in order to implement the proposed methodology and to analyze the results of  $k$ -out-of- $n : G$  system, a numerical example has been provided.

**Keywords**—Vague set, vague reliability, triangular intuitionistic fuzzy number,  $k$ -out-of- $n : G$  system, series and parallel system.

## I. INTRODUCTION

IN various disciplines of science and engineering, analyzing the reliability of a system which is assembled to perform a certain function play an important role. In general, reliability is defined as the probability that an element (that is, a component, subsystem or full system) will accomplish its assigned task within a specified time, which is designated as the interval  $t = [0, t_m]$ . There is a great interest in evaluating the reliability of  $k$ -out-of- $n : G$  (or  $k$ -out-of- $n : F$ ) systems, mainly because such systems are more general than series or parallel systems and some interconnection networks can be modeled using this technique. A system is said to be a  $k$ -out-of- $n : G$  system if it works if and only if at least  $k$  out of  $n$  components work. A dual concept called  $k$ -out-of- $n : F$  system defined as that it fails if and only if at least  $k$  out of  $n$  components fail. Based on these two definitions, a system is  $k$ -out-of- $n : G$  system if and only if it is  $(n - k + 1)$ -out-of- $n : F$  system. Likewise, a system is  $k$ -out-of- $n : F$  system if and only if it is  $(n - k + 1)$ -out-of- $n : G$  system.

It is well known that the conventional reliability analysis has been found to be inadequate to handle uncertainty of failure data and modeling. To overcome this problem, Onisawa and Kacprzyk [22] used fuzzy set theory in the evaluation of the reliability of a system. From a long period of time, efforts have been made in the design and development of reliable large-scale systems. In that period of time, considerable work has been done by researchers to build a systematic theory of reliability based on the probability theory. Cai, Wen, and Zhang [8] presented the following two fundamental assumptions in the conventional reliability theory i.e.,

- 1) Binary state assumption: the system is precisely defined as functioning or failing; and
- 2) Probability assumption: the system behavior is fully characterized in the context of probability measure.

However, because of the inaccuracy and uncertainties in data, the estimation of precise values of probability becomes very difficult in many systems. In this point of view, possibility measures have been proposed. For detailed discussions on possibility theory, we refer to Dubois and Prade [16] and Zadeh [29].

In order to understand the fuzzy states, consider a computer system that consists of three independent processing units. The system is fully functioning when all the three processing units are functioning simultaneously, and is fully failed when all three processing units are failed completely. However, when just one or two processing units are failed, the system will operate in a degraded situation. In this stage, the system is neither fully functioning nor fully failed, but is in some intermediate state. It may be noted that the assumption of the binary state for describing the system failure and success may be no longer appropriate. Consequently, we can fuzzify the definitions for system failure and success, and then characterize them in terms of the fuzzy sets. Now we are naturally in a position to consider the following two assumptions (Cai and Wen [7]; Cai et al. [8]; Cai, Wen, and Zhang [9], [10], [11]):

- 1) Fuzzy-state assumption: The meaning of system failure cannot be precisely defined in a reasonable way. At any time, system may be in one of the following two states: fuzzy success state or fuzzy failure state.
- 2) Possibility assumption: The system behavior can be fully characterized in the context of possibility measures.

## II. LITERATURE SURVEY

Profust reliability theory is based on the probability and fuzzy-state assumptions. In profust reliability theory, the system success and failure are characterized by fuzzy states, i.e., the meaning of system failure is not defined in a precise way, but in a fuzzy way. Cai and Wen [7] introduced the fuzzy success state and fuzzy failure state in which a transition between two fuzzy states was regarded as a fuzzy event. With the concept of fuzzy reliability, they made a comparison between two replacement policies, i.e., the block replacement policy under a non-fuzzy environment and the periodic replacement policy without repair at failures under a fuzzy environment. In the work of Cai et al. [8], the fuzzy system reliability was established based on the binary state

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and possibility assumptions. However, in the work of Cai et al. [9], the fuzzy system reliability was established based on the three-state and possibility assumptions. Further, Cai et al. [10] developed the fuzzy system reliability based on the basis of fuzzy state and probability assumptions. Next, Cai et al. [11] also discussed the system reliability for coherent system based on the fuzzy-state and probability assumptions. Cai et al. [10] presented a fuzzy set-based approach to failure rate and reliability analysis, where profust failure rate is defined in the context of statistics. Further, Singer [25] used a fuzzy set approach for fault tree and reliability analysis in which the relative frequencies of the basic events are considered as fuzzy numbers. Cheng and Mon [12] used interval of confidence in order to analyze fuzzy system reliability. Chen [13] presented a new method for fuzzy system reliability analysis using fuzzy number arithmetic operations in which the reliability of each component is considered as fuzzy number and used simplified fuzzy arithmetic operations rather than complicated interval fuzzy arithmetic operations of fuzzy numbers (Cheng and Mon [12]) or the complicated extended algebraic fuzzy numbers (Singer [25]).

A lot of generalization of the fuzzy set theory has been proposed, among which there are Intuitionistic Fuzzy Sets (IFSs) [2], Interval-Valued Intuitionistic Fuzzy Sets (IVIFSs) [3], Vague Sets [18], R-Fuzzy Sets [27] and Interval-Valued Fuzzy Sets (IVFSs) [28]. From the references ([6], [14], [15]), it may be noted that IVFS theory is equivalent to IFS theory, which in its turn is equivalent to Vague Set theory, and IVIFS theory extends IFS theory. The implementation of intuitionistic (vague) fuzzy set theory instead of fuzzy set theory means the introduction of another degree of freedom into a set description. Such a generalization of fuzzy set theory gives us an additional possibility to represent imperfect knowledge that leads to describing many real problems in a more adequate way. Presently, intuitionistic fuzzy sets are being studied and used in different fields of science and engineering. Burillo [5] studied perturbations of intuitionistic fuzzy number and their properties of the correlation between these numbers. Mahapatra and Roy [20] presented a method to analyze the fuzzy reliability of the series and parallel system using triangular intuitionistic fuzzy numbers (TIFNs) arithmetic operations. Shing Yao et al., [26] applied a statistical methodology in fuzzy system reliability analysis.

We studied some basics of  $k$ -out-of- $n$  system with identical or non-identical components and intuitionistic (vague) set theory in Section II. In Section III, we use the concept of the statistical confidence interval to estimate the reliability of each component of the system. In literature, the domain of the confidence level is taken to be one which is of less practical significance because highest level of confidence of domain experts lies in between  $[0, 1]$  according to the experts knowledge. Therefore, in order to handle the problem in a broader sense, the statistical confidence intervals is being converted to a triangular intuitionistic fuzzy numbers. Then we analyze and discussed the reliability of the  $k$ -out-of- $n : G$  in the intuitionistic fuzzy sense. In Section IV, we compare the obtained results using proposed methodology and existing methodology with the help of a numerical example.

### III. PRELIMINARIES

There are several efficient algorithms available for computing the reliability of a non-repairable  $k$ -out-of- $n$  system with identical or non-identical components. For details, we refer to Misra [21], Rushdi ([23], [24]), Dutuit and Rauzy [17], and Kuo and Zuo [19]. These algorithms are independent of the failure distribution of the components (that is, in these algorithms the reliability of each component is considered to be known), but they use the independent assumption among the components's failure behavior and in order to evaluate the reliability of  $k$ -out-of- $n$  the following assumptions were made:

- 1) System consists of  $n$  mutually statistically independent components.
- 2) Initially (at time  $t = 0$ ), all components are working and all are new.
- 3) The system function if and only if there are at least  $k$  working components.
- 4) There is no repair policy.
- 5) Reliability of each component is known and the components of the system are numbered from 1 to  $n$ .
- 6) Failure time of each component can follow any arbitrary distribution and considered the following two cases:
  - (i) Identical components: all components are identical and follow the same failure distribution.
  - (ii) Non-identical components: Non-identical components: all or some of the components are non-identical and may follow different failure distributions

#### A. Independent Identically Distributed $k$ -out-of- $n$ System

Consider a system with  $n$  independent and identically distributed (i.i.d.) components, and the system reliability  $R(t)$  can be determined by component reliability  $p_i(t)$ ,  $i = 1, 2, \dots, n$ . We write  $R(t)$  a function of  $p_1(t), p_2(t), \dots, p_n(t)$  as

$$R(t) = \phi(p_1(t), p_2(t), \dots, p_n(t)), \quad (1)$$

where the structure function  $\phi$  is decided by the structure of the system.

In a  $k$ -out-of- $n : G$  system with i.i.d. components, the number of working components follows the binomial distribution with parameter  $(n, p)$ . Then we have

$$\text{Prob(exactly } i \text{ components work)} = \binom{n}{i} [p(t)]^i [q(t)]^{n-i}. \quad (2)$$

The reliability of the system is equal to the probability that the number of working components is greater than or equal to  $k$ :

$$R_G(n, p; t) = \sum_{i=k}^n \binom{n}{i} [p(t)]^i [q(t)]^{n-i}. \quad (3)$$

The reliability of a  $k$ -out-of- $n : F$  system with independently and identically distributed (i.i.d.) components is equal to the probability that the number of failing components is less than or equal to  $k - 1$ .

$$R_F(n, p; t) = \sum_{i=0}^{k-1} \binom{n}{i} [p(t)]^{n-i} [q(t)]^i, \quad (4)$$

As a  $k$ -out-of- $n$  :  $F$  system is equivalent to a  $n - k + 1$ -out-of- $n$  :  $G$  system, equation 4 is equivalent to

$$\sum_{j=n-k+1}^n \binom{n}{j} [p(t)]^j [q(t)]^{n-j}. \quad (5)$$

If we denote  $R_G(n, k; t)$  the reliability of a  $k$ -out-of- $n$  :  $G$  system and  $R_F(n, n - k + 1; t)$  the reliability of a  $k$ -out-of- $n$  :  $F$  system, then we have

$$R_G(n, k; t) = R_F(n, n - k + 1; t).$$

Both series and parallel systems are special cases of the  $k$ -out-of- $n$  :  $F$  (or  $k$ -out-of- $n$  :  $G$ ) system. A series system is equivalent to 1-out-of- $n$  :  $F$  (or  $n$ -out-of- $n$  :  $G$ ) system, while a parallel system is equivalent to  $n$ -out-of- $n$  :  $F$  (or 1-out-of- $n$  :  $G$ ) system.

The reliability of the series system is given by

$$R(t) = \prod_{i=1}^n p_i(t). \quad (6)$$

The reliability of the parallel system is given by

$$R(t) = 1 - \prod_{i=1}^n (1 - p_i(t)). \quad (7)$$

#### B. A Non-i.i.d. $k$ -out-of- $n$ System

For the general case with non-identical components, computing the system reliability is somewhat more difficult. There are several algorithms to compute the reliability of a  $k$ -out-of- $n$  system with non-identical components [19]. We consider a well known algorithm that was originally proposed by Barlow and Heidtmann [4] and Rushdi ([23], [24]). We also utilize the iterative implementation provided in [17].

This algorithm has  $O(n(nk+1))$  computational complexity and requires less memory than other algorithms [19]. The algorithm is based on the following recursive relationship. Let  $H(r, m)$  be the probability of at least  $r$  components out of the first  $m$  components are good. Then, we have

$$H(r, m) = \begin{cases} R(k, n) = H(k, n) & \\ \begin{cases} p_m \cdot H(r-1, m-1) + q_m \cdot H(r, m-1) & \text{if } 1 \leq r \leq m, \\ 1 & \text{if } r = 0, m \geq 0, \\ 0 & \text{if } r = m+1, m \geq 0. \end{cases} \end{cases} \quad (8)$$

Although  $H(r, m)$  is a two-dimensional array, at any given time, we need to store only a few of these values. In the following iterative algorithm, only  $k+1$  values of  $H$  are stored in the one-dimensional array  $K$ .

#### Algorithm:

```

K[0] = 1;
for j = 1 to k do K[j] = 0;
done
for i = 1 to n
  for j = k down to 1 do
    K[j] = p_i · K[j - 1] + q_i · K[j]
  done
done

```

At the end of the algorithm, for  $1 \leq j \leq k$ , the reliability results for a  $j$ -out-of- $n$  system will be accumulated in  $K[j]$ . Hence, the reliability of a  $k$ -out-of- $n$  system is equivalent to  $K[k]$ .

#### C. Basic Concepts of VSs and IFSs

**Definition 1: (Fuzzy Set)** A fuzzy set  $A = \{\langle x, \mu(x) \rangle | x \in X\}$  in a universe of discourse  $X$  is characterized by a membership function  $\mu_A$  as follows:

$$\mu_A : X \rightarrow [0, 1]. \quad (9)$$

#### Definition 2: (Vague Set [18])

A vague  $V = \{\langle x, [\mu_{\tilde{V}}(x), 1 - \nu_{\tilde{V}}(x)] \rangle : x \in X\}$ , on the universal set  $X$  is characterized by a true membership function  $\mu_{\tilde{V}} : X \rightarrow [0, 1]$  and a false membership (non-membership) function  $\nu_{\tilde{V}} : X \rightarrow [0, 1]$ . The values  $\mu_{\tilde{V}}(x)$  and  $\nu_{\tilde{V}}(x)$  represents the degree of truth membership and degree of false membership of  $x$  and always satisfies the condition  $0 \leq \mu_{\tilde{V}}(x) + \nu_{\tilde{V}}(x) \leq 1$ , for all  $x \in X$ . The value  $1 - \mu_{\tilde{V}}(x) - \nu_{\tilde{V}}(x)$  represents the degree of hesitation of  $x \in X$ .

The value  $\mu_{\tilde{V}}(x)$  is considered as the lower bound of the grade of membership of  $x$  derived from the evidence for  $x$  and  $\nu_{\tilde{V}}(x)$  is the lower bound of the grade of membership of the negation of  $x$  derived from the evidence against  $x$ . Thus, the grade of membership of  $x$  in the vague set  $\tilde{A}$  is bounded by a sub-interval  $[\mu(x), 1 - \nu(x)]$  of  $[0, 1]$ . For example, if the membership value of  $x$  in vague set  $\tilde{V}$  on the universal set  $X$  is  $[0.5, 0.7]$ , then  $\mu_{\tilde{V}}(x) = 0.5$  and  $1 - \nu_{\tilde{V}}(x) = 0.7$  or  $\nu_{\tilde{V}}(x) = 0.3$ . This means that  $x$  belong to vague set  $\tilde{V}$  with accept evidence is 0.5, decline evidence is 0.3.

**Definition 3: (Intuitionistic Fuzzy Set ([1], [2]))** Let  $X$  be the universe of discourse. Then an IFS  $\tilde{A}$  in  $X$  is given by

$$\tilde{A} = \{\langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle : x \in X\}, \quad (10)$$

where  $\mu_{\tilde{A}} : X \rightarrow [0, 1]$  and  $\nu_{\tilde{A}} : X \rightarrow [0, 1]$  with the condition  $0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1, \forall x \in X$ . The numbers  $\mu_{\tilde{A}}(x)$  and  $\nu_{\tilde{A}}(x)$  denotes the degree of membership and non-membership of an element  $x$  to a set  $\tilde{A}$  respectively. For each element  $x \in X$ , the amount  $\pi_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$  is called the degree of indeterminacy (hesitation part). It is the degree of uncertainty whether  $x$  belongs to  $\tilde{A}$  or not.

We can see that the difference between vague set and intuitionistic fuzzy set is due to the definition of membership intervals. We have  $[\mu_{\tilde{V}}(x), 1 - \nu_{\tilde{V}}(x)]$  for  $x$  in  $\tilde{V}$  but  $\langle \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle$  for  $x$  in  $\tilde{A}$ . Here the semantics of  $\mu_{\tilde{A}}$  is same as with  $\mu_{\tilde{V}}$  and  $\nu_{\tilde{A}}$  is the same as with  $\nu_{\tilde{V}}$ . However, the boundary  $(1 - \nu_{\tilde{V}}(x))$  is able to indicate the possible existence of a data value, as already mentioned by Bustince and Burillo in [6]. This subtle difference gives rise to a simpler but meaningful graphical view of data sets. We now depict a VS in Fig. 1 and an IFS in Fig. 2, respectively. It can be seen that, the shaded part formed by the boundary in a given VS in Fig. 1 represents the possible existence of data. Thus, this hesitation region corresponds to the intuition of representing vague data.

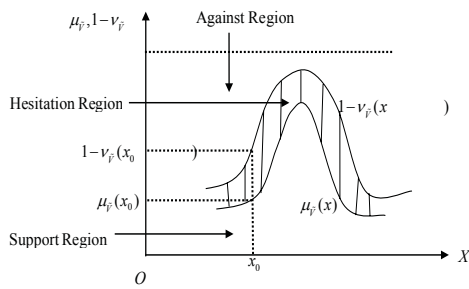


Fig. 1. Vague set

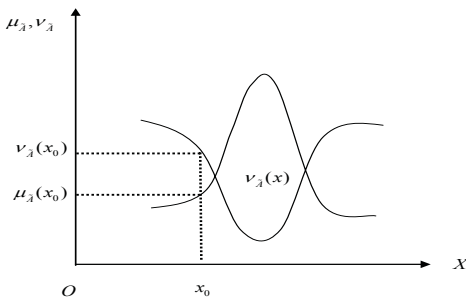


Fig. 2. Intuitionistic fuzzy set

**Definition 4: ( $\alpha$ -Cut of the Vague Set or IFS)** The  $\alpha$ -cut of a membership function, is a crisp set which consists of elements of  $\tilde{A}$  having at least degree  $\alpha$ . It is denoted by  $\tilde{A}^\mu(\alpha)$  and is defined mathematically as

$$\tilde{A}^\mu(\alpha) = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X\}, \alpha \in [0, 1], \quad (11)$$

while for the non-membership function, it is defined as

$$\tilde{A}^\nu(\alpha) = \{x : 1 - \nu_{\tilde{A}}(x) \geq \alpha, x \in X\}, \alpha \in [0, 1]. \quad (12)$$

**Definition 5: (Intuitionistic Fuzzy Number)** An intuitionistic fuzzy subset

$\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle : x \in X \}$  of the real line  $\mathcal{R}$  is called an intuitionistic fuzzy number if the following axioms hold:

(i)  $\tilde{A}$  is normal, i.e., there at least two points  $x_1, x_2 \in \mathcal{R}$  such that  $\mu_{\tilde{A}}(x_1) = 1$  and  $\nu_{\tilde{A}}(x_2) = 1$ ;

(ii) The membership function  $\mu_{\tilde{A}}$  is fuzzy-convex i.e.,

$$\mu_{\tilde{A}}(\lambda \cdot x_1 + (1 - \lambda) \cdot x_2) \geq \min \{ \mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2) \},$$

for all  $x_1, x_2 \in X, \lambda \in [0, 1]$ ;

(iii) The non-membership function  $\nu_{\tilde{A}}$  is fuzzy-concave i.e.,

$$\nu_{\tilde{A}}(\lambda \cdot x_1 + (1 - \lambda) \cdot x_2) \leq \max \{ \nu_{\tilde{A}}(x_1), \nu_{\tilde{A}}(x_2) \},$$

for all  $x_1, x_2 \in X, \lambda \in [0, 1]$ ;

(iv)  $\mu_{\tilde{A}}$  is upper semi-continuous and  $\nu_{\tilde{A}}$  is lower semi-continuous.

**Definition 6: (Triangular Intuitionistic Number)** Let  $\tilde{A}$  be an IFS denoted by  $\tilde{A} = \langle [a, b, c]; \mu, \nu \rangle$ , where  $a, b, c \in \mathcal{R}$ , then the set  $\tilde{A}$  is said to be triangular intuitionistic fuzzy

number if its membership function and the non-membership function are given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu \left( \frac{x-a}{b-a} \right) & \text{for } a \leq x \leq b, \\ \mu & \text{for } x = b, \\ \mu \left( \frac{c-x}{c-b} \right) & \text{for } b \leq x \leq c, \\ 0 & \text{otherwise,} \end{cases} \quad (13)$$

and

$$1 - \nu_{\tilde{A}}(x) = \begin{cases} (1 - \nu) \left( \frac{x-a}{b-a} \right) & \text{for } a \leq x \leq b, \\ (1 - \nu) & \text{for } x = b, \\ (1 - \nu) \left( \frac{c-x}{c-b} \right) & \text{for } b \leq x \leq c, \\ 0 & \text{otherwise,} \end{cases} \quad (14)$$

where the parameter  $b$  gives the modal value of  $\tilde{A}$  such that  $\mu_{\tilde{A}}(b) = \mu, 1 - \nu_{\tilde{A}}(b) = 1 - \nu$ , and  $a, c$  are the lower and upper bounds of available area for the evaluation data.

The  $\alpha$ -cut of a triangular intuitionistic fuzzy number  $\tilde{A}$  is defined as follows:

**Definition 7: ( $\alpha$ -Cut of Intuitionistic Fuzzy Number)** Let us consider an IFN  $\tilde{A} = \langle (a, b, c); \mu, \nu \rangle$  defined on the real line  $\mathcal{R}$ . The  $\alpha$ -cut representation of an IFN  $\tilde{A}$  generates the following pair of intervals and is denoted by

$$(\tilde{A}(\alpha_\mu) = [A^l(\alpha_\mu), A^r(\alpha_\mu)]; \tilde{A}(\alpha_\nu) = [A^l(\alpha_\nu), A^r(\alpha_\nu)]), \quad (15)$$

where  $A^l(\alpha_\mu), A^l(\alpha_\nu)$  are the increasing functions and  $A^r(\alpha_\mu), A^r(\alpha_\nu)$  are decreasing functions of  $\alpha_\mu$  and  $\alpha_\nu$ , respectively. The interval of confidence defined by the  $\alpha$ -cut of TIFN  $\tilde{A}$  are defined as

$$\tilde{A}(\alpha_\mu) = \left[ a + \frac{\alpha_\mu}{\mu}(b-a), c - \frac{\alpha_\mu}{\mu}(c-b) \right], \forall \alpha_\mu \in [0, \mu], \quad (16)$$

and

$$\tilde{A}(\alpha_\nu) = \left[ a + \frac{\alpha_\nu}{(1-\nu)}(b-a), c - \frac{\alpha_\nu}{(1-\nu)}(c-b) \right], \forall \alpha_\nu \in [0, 1-\nu]. \quad (17)$$

#### IV. RELIABILITY ANALYSIS USING TIFNS

In this section, we presented an intuitionistic fuzzy statistical approach for evaluating the reliability of a  $k$ -out-of- $n$  :  $G$  system with independent and non-i.i.d. components, where the reliability of the components are unknown.

Consider a system of  $n$  independent and non-identical components with unknown reliability  $R_i(t)$ ,  $i = 1, 2, \dots, n$ . To analyze the reliability of the system the most important consideration is that the values of  $R_i(t)$  are not fixed. Since they are extracted from various sources such as historical records, reliability databases, and system reliability experts opinion, uncertainty in the values is an undeniable fact. For example, based on an independent sample, the intervals between consequent failures are measured of the  $i^{th}$  component and the result is  $\{45, 230, 105, 150, 115\}$ . Then  $\lambda_i = 5/45+230+105+150+115 = 0.0077519$  and reliability of the component associated with the exponential distribution at time  $t = 30$  is 0.79250. But, if we have new observation of failure like 30 hour, then  $\lambda_i = 6/45 + 230 + 105 + 150 + 115 + 30 = 0.0088889$  and the reliability of the component at  $t = 30$  is 0.76593 which is very different. If we use the point estimate  $\bar{R}_i$  to estimate  $R_i$  from the statistical data in the past, then we don't know the probability of the error  $\bar{R}_i - R_i$ . Moreover, the reliability of the system may fluctuate around the estimated value  $\bar{R}_i$  during a time interval.



It follows that to use the point estimation to estimate the reliability of the components is not suitable for the real cases. Therefore, it is more desirable to use interval estimation to obtain (statistical confidence interval) the probability distribution of the error between the estimated value  $\bar{R}_i$  and the actual value  $R_i$ .

The  $(1 - \gamma)\%$  confidence interval of  $R_i$  is

$$[\bar{R}_i - t_{n_i-1}(\gamma_1) \frac{s_i}{\sqrt{n_i}}, \bar{R}_i + t_{n_i-1}(\gamma_2) \frac{s_i}{\sqrt{n_i}}], i = 1, 2, \dots, n, \quad (18)$$

where  $\gamma_1 + \gamma_2 = \gamma$ ,  $0 < \gamma_1, \gamma_2, \gamma < 1$  and

$$s_i^2 = \frac{1}{(n_i-1)} \sum_{j=1}^{n_i} (R_{ij} - \bar{R}_i)^2.$$

Let  $T$  be a  $t$ -distributed random variable with  $n_i - 1$  degree of freedom. Then  $t_{n_i-1}(\gamma_k)$  satisfies the condition  $p(T \geq t_{n_i-1}(\gamma_k)) = \gamma_k$ ,  $k = 1, 2$ .

The decision maker not only chooses  $\gamma_1$  and  $\gamma_2$  to satisfy the condition

$\gamma_1 + \gamma_2 = \gamma$ ,  $0 < \gamma_1, \gamma_2, \gamma < 1$ , but also satisfies the following conditions:

$$0 < \bar{R}_i - t_{n_i-1}(\gamma_1) \frac{s_i}{\sqrt{n_i}} < 1 \quad (19)$$

and

$$0 < \bar{R}_i + t_{n_i-1}(\gamma_2) \frac{s_i}{\sqrt{n_i}} < 1, i = 1, 2, \dots, n. \quad (20)$$

Shing Yao et al., [26] transferred the statistical confidence intervals into the triangular fuzzy numbers. Through these triangular fuzzy numbers, fuzzy reliability of the system is computed at zero degree of hesitation between the membership functions. Moreover, the domain of the confidence level is taken to be one, that is,  $\alpha = 1$ . Therefore, the results computed by fuzzy numbers have not practically significance, because highest level of confidence of domain experts lies in between  $[0, 1]$  according to the experts knowledge. Therefore, we could not consider this problem using fuzzy point of view only. In our approach, we transferred the statistical confidence interval into triangular intuitionistic (vague) fuzzy number to overcome the above-mention shortcoming by considering some degree of hesitation between the degree of membership and non-membership functions.

Therefore, we transferred the confidence interval in equation 18 to the intuitionistic fuzzy numbers as follows:

$$\tilde{R}_i = \left\langle \left( \bar{R}_i - t_{n_i-1}(\gamma_1) \frac{s_i}{\sqrt{n_i}}, \bar{R}_i, \bar{R}_i + t_{n_i-1}(\gamma_2) \frac{s_i}{\sqrt{n_i}} \right); \mu_i, \nu_i \right\rangle. \quad (21)$$

The  $\alpha$ -level sets of  $\tilde{R}_i$ ,  $i = 1, 2, \dots, n$  generates the following pair of intervals:

$$(\tilde{R}_i(\alpha_\mu) = [R_i^l(\alpha_\mu); R_i^u(\alpha_\mu)], \tilde{R}_i(\alpha_\nu) = [R_i^l(\alpha_\nu), R_i^u(\alpha_\nu)]),$$

where

$$R_i^l(\alpha_\mu) = \bar{R}_i - \left(1 - \frac{\alpha_\mu}{\mu_i}\right) t_{n_i-1}(\gamma_1) \frac{s_i}{\sqrt{n_i}},$$

$$R_i^u(\alpha_\mu) = \bar{R}_i + \left(1 - \frac{\alpha_\mu}{\mu_i}\right) t_{n_i-1}(\gamma_2) \frac{s_i}{\sqrt{n_i}},$$

$$R_i^l(\alpha_\nu) = \bar{R}_i - \left(1 - \frac{\alpha_\nu}{(1 - \nu_i)}\right) t_{n_i-1}(\gamma_1) \frac{s_i}{\sqrt{n_i}},$$

$$R_i^u(\alpha_\nu) = \bar{R}_i + \left(1 - \frac{\alpha_\nu}{(1 - \nu_i)}\right) t_{n_i-1}(\gamma_2) \frac{s_i}{\sqrt{n_i}},$$

for all  $\alpha_\mu \in [0, \mu_i]$ ,  $\alpha_\nu \in [0, 1 - \nu_i]$ .

Finally, the intuitionistic fuzzy reliability of the  $k$ -out-of- $n$  :  $G$  system is calculated to invoke the algorithm given in section 2, for both left and right end points of the  $\alpha$ -level sets for different values of  $\alpha$ . By the decomposition theorem, we constructed intuitionistic fuzzy reliability of the  $k$ -out-of- $n$  :  $G$  system as

$$\tilde{R}_s = \left[ \bigcup_{0 \leq \alpha_\mu \leq \mu_s} [R_s^l(\alpha_\mu), R_s^u(\alpha_\mu)]; \bigcup_{0 \leq \alpha_\nu \leq 1 - \nu_s} [R_s^l(\alpha_\nu), R_s^u(\alpha_\nu)] \right]. \quad (22)$$

## V. NUMERICAL EXAMPLE

*Example 1:* (Shing Yao et al., [26]) Consider the following statistical data for each component in Table I of the  $k$ -out-of- $n$  :  $G$  system consisting three non-i.i.d. components.

TABLE I  
STATISTICAL DATA

Components	Sample size	Sample mean	Sample standard deviation
$C_1$	$n_1 = 10$	$R_1 = 0.80$	$s_1 = 0.02$
$C_2$	$n_2 = 20$	$R_2 = 0.75$	$s_2 = 0.03$
$C_3$	$n_3 = 15$	$R_3 = 0.90$	$s_3 = 0.01$

Let  $\gamma = 0.02$ ,  $\gamma_1 = 0.011$  and  $\gamma_2 = 0.009$ . Then from the table of the  $t$ -distribution with  $n_i - 1$  degrees of freedom,  $i = 1, 2, 3$ , we get the following data:  $t_9(\gamma_1) = 2.7017$ ,  $t_{19}(\gamma_1) = 2.5212$ ,  $t_{14}(\gamma_1) = 2.5921$ ,  $t_9(\gamma_2) = 2.9068$ ,  $t_{19}(\gamma_2) = 2.6034$ ,  $t_{14}(\gamma_2) = 2.6946$ .

Using the above statistical information, we found end points of the statistical confidence interval for each component which is given in Table II.

TABLE II  
TWO END POINTS

i	Degree of freedom	$R_i - t_{n_i-1}(\gamma_1) \frac{s_i}{\sqrt{n_i}}$	$R_i + t_{n_i-1}(\gamma_2) \frac{s_i}{\sqrt{n_i}}$
1	9	0.7829	0.8184
2	19	0.7331	0.7675
3	14	0.8933	0.9070

Using the Table II, we construct triangular intuitionistic fuzzy numbers by considering 0.2 degree of hesitation as follows:

$$\tilde{R}_1 = \langle (0.7829, 0.80, 0.8184); 0.6, 0.2 \rangle,$$

$$\tilde{R}_2 = \langle (0.7331, 0.75, 0.7675); 0.4, 0.4 \rangle,$$

$$\tilde{R}_3 = \langle (0.8933, 0.90, 0.9070); 0.7, 0.1 \rangle,$$

The  $\alpha$  level sets of  $\tilde{R}_i$ ,  $i = 1, 2, 3$  are given by

$$\tilde{R}_1(\alpha_\mu) = [0.7829 + 0.0285\alpha_\mu, 0.8184 - 0.0307\alpha_\mu], \forall \alpha_\mu \in [0, 0.6],$$

$$\tilde{R}_2(\alpha_\mu) = [0.7331 + 0.0423\alpha_\mu, 0.7675 - 0.0438\alpha_\mu], \forall \alpha_\mu \in [0, 0.4],$$

$$\tilde{R}_3(\alpha_\mu) = [0.8933 + 0.0096\alpha_\mu, 0.9070 - 0.0100\alpha_\mu], \forall \alpha_\mu \in [0, 0.7],$$

and

$$\begin{aligned}\tilde{R}_1(\alpha_\nu) &= [0.7829 + 0.0214\alpha_\nu, 0.8184 - 0.0230\alpha_\nu], \forall \alpha_\nu \in [0, 0.8], \\ \tilde{R}_2(\alpha_\nu) &= [0.7331 + 0.0282\alpha_\nu, 0.7675 - 0.0292\alpha_\nu], \forall \alpha_\nu \in [0, 0.6], \\ \tilde{R}_3(\alpha_\nu) &= [0.8933 + 0.0074\alpha_\nu, 0.9070 - 0.0078\alpha_\nu], \forall \alpha_\nu \in [0, 0.9].\end{aligned}$$

Using the algorithm given in section 2, we obtained intuitionistic (vague) fuzzy reliability of the  $k$ -out-of-3 :  $G$ ,  $k = 1, 2, 3$  in both existing methods and proposed method for different values of  $\alpha$  and results are shown in Tables III, IV and V, respectively.

TABLE III  
1-OUT-OF-3 SYSTEM RESULTS

$\alpha$	Crisp	Shing Yao et.al.(2008)			Proposed approach				
		$a$	$b$	$c$	$a'$	$a$	$b$	$c$	$c'$
0.0	0.9950	0.9938	0.9950	0.9961	0.9938	0.9938	0.9950	0.9961	0.9961
0.1	0.9950	0.9939	0.9950	0.9960	0.9940	0.9940	0.9950	0.9959	0.9959
0.2	0.9950	0.9941	0.9950	0.9959	0.9943	0.9941	0.9950	0.9958	0.9957
0.3	0.9950	0.9942	0.9950	0.9958	0.9945	0.9943	0.9950	0.9957	0.9955
0.4	0.9950	0.9943	0.9950	0.9957	0.9947	0.9945	0.9950	0.9955	0.9953
0.5	0.9950	0.9944	0.9950	0.9956	0.9949	0.9946	0.9950	0.9954	0.9951
0.6	0.9950	0.9945	0.9950	0.9954	—	0.9948	0.9950	0.9952	—
0.7	0.9950	0.9947	0.9950	0.9953	—	0.9949	0.9950	0.9951	—
0.8	0.9950	0.9948	0.9950	0.9952	—	—	0.9950	—	—
0.9	0.9950	0.9949	0.9950	0.9951	—	—	0.9950	—	—
1.0	0.9950	0.9950	0.9950	0.9950	—	—	0.9950	—	—

TABLE IV  
2-OUT-OF-3 SYSTEM RESULTS

$\alpha$	Crisp	Shing Yao et.al.(2008)			Proposed approach				
		$a$	$b$	$c$	$a'$	$a$	$b$	$c$	$c'$
0.0	0.9150	0.9028	0.9150	0.9271	0.9028	0.9028	0.9150	0.9271	0.9271
0.1	0.9150	0.9040	0.9150	0.9259	0.9052	0.9045	0.9150	0.9255	0.9249
0.2	0.9150	0.9053	0.9150	0.9248	0.9076	0.9062	0.9150	0.9239	0.9226
0.3	0.9150	0.9065	0.9150	0.9236	0.9099	0.9079	0.9150	0.9222	0.9202
0.4	0.9150	0.9078	0.9150	0.9224	0.9123	0.9096	0.9150	0.9206	0.9179
0.5	0.9150	0.9090	0.9150	0.9212	0.9146	0.9113	0.9150	0.9189	0.9155
0.6	0.9150	0.9102	0.9150	0.9199	—	0.9129	0.9150	0.9172	—
0.7	0.9150	0.9114	0.9150	0.9187	—	0.9146	0.9150	0.9155	—
0.8	0.9150	0.9126	0.9150	0.9175	—	—	0.9150	—	—
0.9	0.9150	0.9138	0.9150	0.9162	—	—	0.9150	—	—
1.0	0.9150	0.9150	0.9150	0.9150	—	—	0.9150	—	—

TABLE V  
3-OUT-OF-3 SYSTEM RESULTS

$\alpha$	Crisp	Shing Yao et.al.(2008)			Proposed approach				
		$a$	$b$	$c$	$a'$	$a$	$b$	$c$	$c'$
0.0	0.5400	0.5127	0.5400	0.5697	0.5127	0.5127	0.5400	0.5697	0.5697
0.1	0.5400	0.5154	0.5400	0.5667	0.5181	0.5165	0.5400	0.5655	0.5637
0.2	0.5400	0.5181	0.5400	0.5637	0.5235	0.5203	0.5400	0.5612	0.5577
0.3	0.5400	0.5208	0.5400	0.5607	0.5290	0.5242	0.5400	0.5570	0.5518
0.4	0.5400	0.5235	0.5400	0.5577	0.5345	0.5280	0.5400	0.5528	0.5459
0.5	0.5400	0.5262	0.5400	0.5547	0.5400	0.5319	0.5400	0.5486	0.5401
0.6	0.5400	0.5290	0.5400	0.5518	—	0.5358	0.5400	0.5445	—
0.7	0.5400	0.5317	0.5400	0.5488	—	0.5397	0.5400	0.5404	—
0.8	0.5400	0.5345	0.5400	0.5459	—	—	0.5400	—	—
0.9	0.5400	0.5372	0.5400	0.5429	—	—	0.5400	—	—
1.0	0.5400	0.5400	0.5400	0.5400	—	—	0.5400	—	—

The true membership and false membership functions corresponding to the obtained results ( $k$ -out-of-3 :  $G$ ,  $k = 1, 2, 3$  system) are shown in Fig. 3.

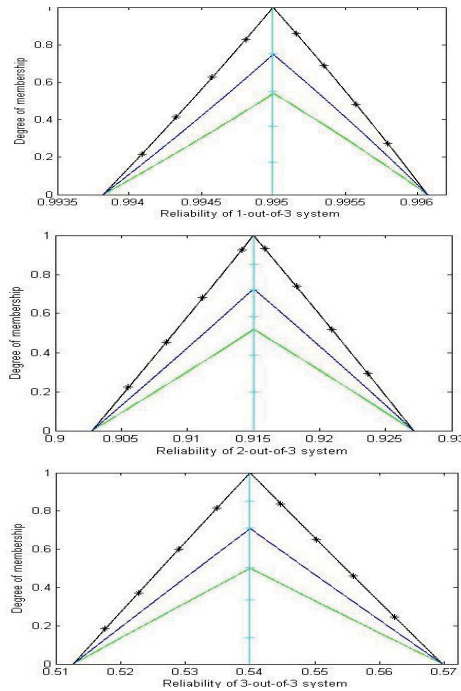


Fig. 3. Reliability of  $k$ -out-of-3 System:  $k = 1, 2, 3$

#### A. Comparison and Discussion

The intuitionistic (vague) fuzzy reliability results using the existing method and proposed method compared as follows:

- 1) Using the point estimate method, the reliability of the series (3-out-of-3 :  $G$ ) system is equal to 0.54 for all values of  $\alpha$ , its means that in this method any vagueness does not consider in the data. Moreover, point estimate method can be suitable where the data are precise and certain, also it does not consider the confidence level of the domain experts.
- 2) Using the Table III in the Shing Yao et al., [26] method, it can be easily seen that the degree of truth membership and false membership correspond to the reliability value 0.9943 are 0.4 and 0.6 respectively. It may be noted that the degree of hesitation has not been considered in the computation. Moreover, Shing Yao et al., [26] do not consider the confidence level of domain experts that lies in the interval  $[0, 1]$ .
- 3) Using the Table III in the proposed method, it can be seen that the degree of truth membership and false membership values corresponding to the crisp reliability 0.9943 are 0.2 and 0.3 respectively. There is 0.10 degree of hesitation that the value of reliability is 0.9943 which was not considered in Shing Yao et al. [26] method. Moreover, the reliability of the system in view of the Shing Yao et al., [26] is being represented just by one number (which represents the evidences both in favor/against for reliability of the system). On the other hand, the computed reliability of the system by proposed method is being represented by two numbers (which represent

the evidence in favor/against and an indeterminacy part for reliability of the system). As the proposed method also considers the confidence level of domain experts ( $\alpha \leq 0.8$ ), therefore, the proposed method is more flexible and realistic.

## VI. CONCLUSION AND FUTURE WORK

The intuitionistic fuzzy reliability of  $k$ -out-of- $n$  :  $G$  system with independent and non-identically distributed components, where the reliability of the components are unknown, has been analyzed. The reliability of each component has been estimated using statistical confidence interval approach. Considering the highest level of confidence of domain experts that belongs to the interval  $[0, 1]$ , we converted these statistical confidence interval into triangular intuitionistic fuzzy numbers. The reliability of the  $k$ -out-of- $n$  :  $G$  system has been calculated and discussed on the basis of these triangular intuitionistic fuzzy numbers with the help of a numerical example. On similar pattern, the intuitionistic fuzzy reliability of the real-time repairable  $k$ -out-of- $n$  system may be computed with the help of Markov Chain.

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