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Displacement Solution for a Static Vertical Rigid Movement of an Interior Circular Disc in a Transversely Isotropic Tri-Material Full-Space

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Abstract**—**This article is concerned with the determination of the static interaction of a vertically loaded rigid circular disc embedded at the interface of a horizontal layer sandwiched in between two different transversely isotropic half-spaces called as tri-material full-space. The axes of symmetry of different regions are assumed to be normal to the horizontal interfaces and parallel to the movement direction. With the use of a potential function method, and by implementing Hankel integral transforms in the radial direction, the government partial differential equation for the solely scalar potential function is transformed to an ordinary 4th order differential equation, and the mixed boundary conditions are transformed into a pair of integral equations called dual integral equations, which can be reduced to a Fredholm integral equation of the second kind, which is solved analytically. Then, the displacements and stresses are given in the form of improper line integrals, which is due to inverse Hankel integral transforms. It is shown that the present solutions are in exact agreement with the existing solutions for a homogeneous full-space with transversely isotropic material. To confirm the accuracy of the numerical evaluation of the integrals involved, the numerical results are compared with the solutions exists for the homogeneous full-space. Then, some different cases with different degrees of material anisotropy are compared to portray the effect of degree of anisotropy.

Keywords**—**Transversely isotropic, rigid disc, elasticity, dual integral equations, tri-material full-space.

I. Introduction

BEACAUSE of mathematical difficulties and engineering application, the static interaction of an embedded rigid disc with an elastic medium has been a subject of active research for many years. The first research in the topic is probably related to the work done by Bousinesq in 1888 on indentation problem. The problem was reinvestigated by Harding and Sneddon [1], who used Love's stress function. Some other static or dynamic solutions were obtained on the response of a rigid disc resting on the surface of a half-space, as in Sneddon [2], Keer [3], Arnold et al. [4], Bycroft [5], Awajobi and Grootenhuis [6], Robertson [7], Gladwell [8], and Pak and Gobert [9]. In all cases, an integral transform has been used, which resulted in some dual integral equations, whose solution has been given in the numerical form in general case. The solution has been given analytically for some simpler cases.

This paper investigates the axisymmetric interaction of vertically loaded rigid disc embedded at the interface of a transversely isotropic tri-material full-space. The axes of symmetry of different regions are normal to the horizontal interfaces and parallel to the movement direction. Because of its completeness and simplicity, the potential function introduced in [10] is used to uncouple the equations of motions. Using the Hankel integral transforms [11], the relaxed mixed boundary-value problem considered here is transformed into a pair of integral equations named as dual integral equations in the literatures. With the aid of Nobel's multiplying factor procedure [12], the dual integral equations obtained in this paper are changed to the form of Fredholm integral equation of the second kind, which is numerically solved for the general layered full-space. It is shown that the numerical solution for homogeneous transversely isotropic full-space which is degenerated from the general solution presented here is identical to the numerical solution given in [13].

II. Boundary Value Problem and the Solution

A massless rigid disc of radius a, embedded at an interface of a layered, transversely isotropic, linearly elastic full-space is considered (see Fig. 1). The disc is assumed to be undergoing a forced rigid body translation Δ in the vertical direction. In view of the axial symmetry of the problem, a cylindrical coordinate system (r,θ,z) is installed at the center of the disc. Considering this coordinate system, the arrangement of different layers of full-space may be defined as an upper half-space with $z<-h$ called as Region I, an intermediate layer with $-h<z<0$ denoted as Region II and a lower half-space with $z>0$ showing by Region III. In the absence of body force, the axisymmetric equations of motion in terms of displacements in each region can be expressed as [10].

$$A_{q11}\left(\frac{\partial^2 u_q}{\partial r^2}+\frac{1}{r}\frac{\partial u_q}{\partial r}-\frac{u_q}{r^2}\right)+A_{q44}\frac{\partial^2 u_q}{\partial z^2}+(A_{q13}+A_{q44})\frac{\partial^2 w_q}{\partial r\,\partial z}=0 \quad (1a)$$

$$A_{q44}\left(\frac{\partial^2 w_q}{\partial r^2}+\frac{1}{r}\frac{\partial w_q}{\partial r}\right)+A_{q33}\frac{\partial^2 w_q}{\partial z^2}+(A_{q13}+A_{q44})\left(\frac{\partial^2 u_q}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial u_q}{\partial z}\right)=0 \quad (1b)$$

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where $q = \mathrm{I}$, II and III, u and w are the displacement components in r and z directions, respectively and A_{ij} are the elasticity constants. For a transversely isotropic material, five independent elastic constants are needed to describe its behavior [10]. These elasticity constants which are different in different region are connected to the engineering constants E, E', υ, υ', G and G' through some relations (see [10] and [14] for details). Considering the vertical movement of the disc, a relaxed treatment of the mixed boundary-value problem can be stated in terms of the components of the Cauchy stress tensor σ_{ij} $(i, j = r, z)$ and the displacement components u and w as follows

$$w(r, z=0) = \Delta, \qquad r \le a \qquad (2)$$

$$w_{\mathrm{II}}(r, z=0^-) - w_{\mathrm{III}}(r, z=0^+) = 0, \qquad r \ge 0 \qquad (3)$$

$$u_{\mathrm{II}}(r, z=0^-) - u_{\mathrm{III}}(r, z=0^+) = 0, \qquad r \ge 0 \qquad (4)$$

$$\sigma_{zz\mathrm{II}}(r, z=0^-) - \sigma_{zz\mathrm{III}}(r, z=0^+) = R(r), \qquad r \le a \qquad (5)$$

$$\sigma_{zz\mathrm{II}}(r, z=0^-) - \sigma_{zz\mathrm{III}}(r, z=0^+) = 0, \qquad r > a \qquad (6)$$

$$\sigma_{rz\mathrm{II}}(r, z=0^-) - \sigma_{rz\mathrm{III}}(r, z=0^+) = 0, \qquad r \ge 0 \qquad (7)$$

$$w_{\mathrm{I}}(r, z=-h^-) - w_{\mathrm{II}}(r, z=-h^+) = 0, \qquad r \ge 0 \qquad (8)$$

$$u_{\mathrm{I}}(r, z=-h^-) - u_{\mathrm{II}}(r, z=-h^+) = 0, \qquad r \ge 0 \qquad (9)$$

$$\sigma_{zz\mathrm{I}}(r, z=-h^-) - \sigma_{zz\mathrm{II}}(r, z=-h^+) = 0, \qquad r \ge 0 \qquad (10)$$

$$\sigma_{rz\mathrm{I}}(r, z=-h^-) - \sigma_{rz\mathrm{II}}(r, z=-h^+) = 0. \qquad r \ge 0 \qquad (11)$$

Here, the function $R(r)$ is the resultant vertical stress applied on the rigid disc.

Utilizing the scalar potential function, F, introduced by Lekhnitskii [10] as the static case of the generalized potential function given by Eskandari-Ghadi [15], the difficulty in dealing with the coupled partial differential equations in (1) is avoided. The displacement- and the stress-potential function relationships are given in Hankel integral transformed space shortly. Thus, the solutions for the potential functions in different regions in Hankel space are [16]:

$$F_{\mathrm{I}}^{(0)}(\xi, z) = A_{\mathrm{I}}(\xi) e^{s_{\mathrm{I}1}\xi z} + C_{\mathrm{I}}(\xi) e^{s_{\mathrm{I}2}\xi z}, \qquad z < -h \qquad (12)$$

$$F_{\mathrm{II}}^{(0)}(\xi, z) = A_{\mathrm{II}}(\xi) e^{s_{\mathrm{II}1}\xi z} + B_{\mathrm{II}}(\xi) e^{-s_{\mathrm{II}1}\xi z} + C_{\mathrm{II}}(\xi) e^{s_{\mathrm{II}2}\xi z} + D_{\mathrm{II}}(\xi) e^{-s_{\mathrm{II}2}\xi z}, \qquad -h < z < 0 \qquad (13)$$

$$F_{\mathrm{III}}^{(0)}(\xi, z) = B_{\mathrm{III}}(\xi) e^{-s_{\mathrm{III}1}\xi z} + D_{\mathrm{III}}(\xi) e^{-s_{\mathrm{III}2}\xi z}. \qquad z > 0 \qquad (14)$$

where the regularity condition has been applied in both vertical and horizontal directions. It is emphasized that in the above solutions, ξ is the Hankel's parameter, $A_{\mathrm{I}}(\xi)$ to $D_{\mathrm{III}}(\xi)$ are unknown functions to be determined using both the continuity and boundary conditions, and s_{q1} and s_{q2} are the roots of (15).

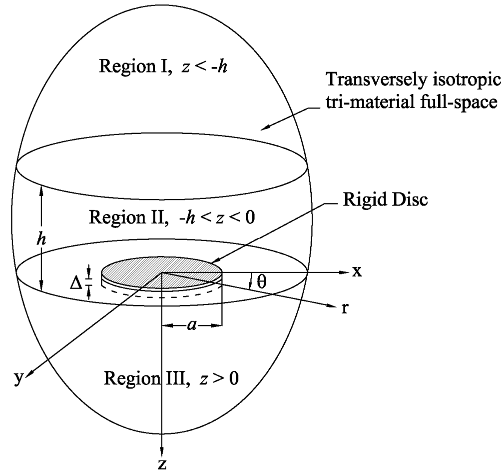

Fig. 1 Axisymmetric transversely isotropic tri-material full-space containing a rigid disc

$$A_{q33}A_{q44}s_q^4 + (A_{q13}^2 + 2A_{q13}A_{q44} - A_{q11}A_{q33})s_q^2 + A_{q11}A_{q44} = 0 \qquad (15)$$

The displacement components and stresses in each region can be expressed in terms of the potential function F, as follows [16].

$$u_q^{(1)} = \alpha_{q3}\xi \frac{dF_q^{(0)}}{dz}, \quad w_q^{(0)} = [\alpha_{q2}\frac{\partial^2}{\partial z^2} - \xi^2(1+\alpha_{q1})]F_q^{(0)}, \qquad (16)$$

$$\sigma_{qzr}^{(1)} = A_{q44}\xi\left[\left(\alpha_{q3} - \alpha_{q2}\right)\frac{d^2}{dz^2} + \xi^2(1+\alpha_{q1})\right]F_q^{(0)},$$
$$\sigma_{qzz}^{(0)} = \frac{d}{dz}\left[\alpha_{q3}A_{q13}\xi^2 - A_{q33}\xi^2(1+\alpha_{q1}) + A_{q33}\alpha_{q2}\frac{d^2}{dz^2}\right]F_q^{(0)}, \qquad (17)$$

where

$$\alpha_{q1} = \frac{A_{q12} + A_{q66}}{A_{q66}}, \quad \alpha_{q2} = \frac{A_{q44}}{A_{q66}}, \quad \alpha_{q3} = \frac{A_{q13} + A_{q66}}{A_{q66}}, \qquad (18)$$

and $q = \mathrm{I}$, II and III.

Using (12)-(14) in (16) and (17), it is possible to write the continuity and boundary conditions in the matrix form as [17].

$$I(\xi)\,[\;A_{\mathrm{I}}(\xi) \quad C_{\mathrm{I}}(\xi) \quad A_{\mathrm{II}}(\xi) \quad B_{\mathrm{II}}(\xi) \quad C_{\mathrm{II}}(\xi) \quad D_{\mathrm{II}}(\xi) \quad B_{\mathrm{III}}(\xi) \quad D_{\mathrm{III}}(\xi)]^T = s_l^c(\xi) \tag{19}$$

$$s_l^c(\xi) = [\quad 0 \quad R^{(0)}(\xi) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad]^T \tag{20}$$

$$I(\xi) = \begin{bmatrix} 0 & 0 & \eta_{\mathrm{II}1} & \eta_{\mathrm{II}1} & \eta_{\mathrm{II}2} & \eta_{\mathrm{II}2} & -\eta_{\mathrm{III}1} & -\eta_{\mathrm{III}2} \\ 0 & 0 & -\nu_{\mathrm{II}1} & \nu_{\mathrm{II}1} & -\nu_{\mathrm{II}2} & \nu_{\mathrm{II}2} & -\nu_{\mathrm{III}1} & -\nu_{\mathrm{III}2} \\ \alpha_{\mathrm{I}3}s_{\mathrm{I}1}\xi e^{-s_{\mathrm{I}1}\xi h} & \alpha_{\mathrm{I}3}s_{\mathrm{I}2}e^{-s_{\mathrm{I}2}\xi h} & -\alpha_{\mathrm{II}3}s_{\mathrm{II}1}e^{-s_{\mathrm{II}1}\xi h} & \alpha_{\mathrm{II}3}s_{\mathrm{II}1}e^{s_{\mathrm{II}1}\xi h} & -\alpha_{\mathrm{II}3}\lambda_{\mathrm{II}2}e^{-s_{\mathrm{II}2}\xi h} & \alpha_{\mathrm{II}3}s_{\mathrm{II}2}e^{s_{\mathrm{II}2}\xi h} & 0 & 0 \\ \varphi_{\mathrm{I}1}e^{-s_{\mathrm{I}1}\xi h} & \varphi_{\mathrm{I}2}e^{-s_{\mathrm{I}2}\xi h} & -\varphi_{\mathrm{II}1}e^{-s_{\mathrm{II}1}\xi h} & -\varphi_{\mathrm{II}1}e^{s_{\mathrm{II}1}\xi h} & -\varphi_{\mathrm{II}2}e^{-s_{\mathrm{II}2}\xi h} & -\varphi_{\mathrm{II}2}e^{s_{\mathrm{II}2}\xi h} & 0 & 0 \\ \eta_{\mathrm{I}1}e^{-s_{\mathrm{I}1}\xi h} & \eta_{\mathrm{I}2}e^{-s_{\mathrm{I}2}\xi h} & -\eta_{\mathrm{II}1}e^{-s_{\mathrm{II}1}\xi h} & -\eta_{\mathrm{II}1}e^{s_{\mathrm{II}1}\xi h} & -\eta_{\mathrm{II}2}e^{-s_{\mathrm{II}2}\xi h} & -\eta_{\mathrm{II}2}e^{s_{\mathrm{II}2}\xi h} & 0 & 0 \\ -\nu_{\mathrm{I}1}e^{-s_{\mathrm{I}1}\xi h} & -\nu_{\mathrm{I}2}e^{-s_{\mathrm{I}2}\xi h} & \nu_{\mathrm{II}1}e^{-s_{\mathrm{II}1}\xi h} & -\nu_{\mathrm{II}1}e^{s_{\mathrm{II}1}\xi h} & \nu_{\mathrm{II}2}e^{-s_{\mathrm{II}2}\xi h} & -\nu_{\mathrm{II}2}e^{s_{\mathrm{II}2}\xi h} & 0 & 0 \\ 0 & 0 & \alpha_{\mathrm{II}3}s_{\mathrm{II}1}\xi & -\alpha_{\mathrm{II}3}s_{\mathrm{II}1}\xi & \alpha_{\mathrm{II}3}s_{\mathrm{II}2}\xi & -\alpha_{\mathrm{II}3}s_{\mathrm{II}2}\xi & \alpha_{\mathrm{III}3}s_{\mathrm{III}1}\xi & \alpha_{\mathrm{III}3}s_{\mathrm{III}2}\xi \\ 0 & 0 & \varphi_{\mathrm{II}1} & \varphi_{\mathrm{II}1} & \varphi_{\mathrm{II}2} & \varphi_{\mathrm{II}2} & -\varphi_{\mathrm{III}1} & -\varphi_{\mathrm{III}2} \end{bmatrix} \tag{21}$$

where

$$\begin{aligned} \eta_{qi} &= A_{q44}\left[\left(\alpha_{q3}-\alpha_{q2}\right)s_{qi}^2+\alpha_{q1}+1\right] \\ \varphi_{qi} &= \alpha_{q2}s_{qi}^2-\alpha_{q1}-1 \\ \nu_{qi} &= \left[\left(1+\alpha_{q1}-\alpha_{q2}s_{qi}^2\right)A_{q33}-\alpha_{q3}A_{q13}\right]s_{qi} \end{aligned} \tag{22}$$

Solving (19) it is possible to find

$$\begin{aligned} &\frac{A_{\mathrm{I}}(\xi)}{R^{(0)}(\xi)} = I_{12}^{-1}(\xi),\ \frac{C_{\mathrm{I}}(\xi)}{R^{(0)}(\xi)} = I_{22}^{-1}(\xi),\ \frac{A_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)} = I_{32}^{-1}(\xi), \\ &\frac{B_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)} = I_{42}^{-1}(\xi),\ \frac{C_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)} = I_{52}^{-1}(\xi),\ \frac{D_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)} = I_{62}^{-1}(\xi), \\ &\frac{B_{\mathrm{III}}(\xi)}{R^{(0)}(\xi)} = I_{72}^{-1}(\xi),\ \frac{D_{\mathrm{III}}(\xi)}{R^{(0)}(\xi)} = I_{82}^{-1}(\xi), \end{aligned} \tag{23}$$

where $I_{i2}^{-1}(\boldsymbol{\xi})=$ the element at the i[th] row and the second column of the inverse matrix of $I(\boldsymbol{\xi})$ and i=1,2,…,8.

Equations (2) and (6) result in a pair of integral equations called dual integral equations. For the treatment of these equations, the procedure introduced by Noble [12] is used. To do so, it is convenient to write the equations in the following form

$$\int_0^\infty \xi^{-1}[1+H(\xi)]B(\xi)J_0(r\xi)d\xi = \frac{\Delta}{l}, \quad r \le a \tag{24a}$$

$$\int_0^\infty B(\xi)J_0(r\xi)d\xi = 0, \quad r > a \tag{24b}$$

where

$$H(\xi) = -1+\frac{1}{l}\left[\varphi_{\mathrm{III}1}\frac{B_{\mathrm{III}}(\xi)}{R^{(0)}(\xi)}+\varphi_{\mathrm{III}2}\frac{D_{\mathrm{III}}(\xi)}{R^{(0)}(\xi)}\right]\frac{\xi}{\vartheta(\xi)}, \tag{25}$$

$$B(\xi) = \xi\vartheta(\xi)R^{(0)}(\xi), \tag{26}$$

$$\begin{aligned} \vartheta(\xi) = \nu_{\mathrm{II}1}&\left[-\frac{A_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)}+\frac{B_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)}\right]+\nu_{\mathrm{II}2}\left[-\frac{C_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)}\right. \\ &\left.+\frac{D_{\mathrm{II}}(\xi)}{R^{(0)}(\xi)}\right]-\nu_{\mathrm{III}1}\frac{B_{\mathrm{III}}(\xi)}{R^{(0)}(\xi)}-\nu_{\mathrm{III}2}\frac{D_{\mathrm{III}}(\xi)}{R^{(0)}(\xi)}, \end{aligned} \tag{27}$$

and l is a modifier needed for the function $H(\xi)$ to make the condition at infinity to be satisfied [18]. This modifier is defined as $l = 1+H(\infty)$. By solving the dual integral equations (24) for $R^{(0)}(\xi)$, this function is known, and the remaining functions $A_{\mathrm{I}}(\xi)$ to $D_{\mathrm{III}}(\xi)$ are then known from (23).

III. Fredholm Integral Equation

As seen in [12], with the aid of Hankel inversion theorem, the governing dual integral equations (24) can be reduced to a Fredholm integral equation as

$$\theta(r)+\frac{1}{\pi}\int_0^a M(r,\rho)\theta(\rho)d\rho = \frac{\Delta}{l}, \quad 0 \le r < a \tag{28}$$

where

$$M\left(r,\rho\right) = 2\int_0^\infty H(\xi)\cos(r\xi)\cos(\rho\xi)d\xi. \quad 0 \le r,\ \rho < a \tag{29}$$

and the function $R^{(0)}(\boldsymbol{\xi})$ is expressed as [19]

$$R^{(0)}(\xi) = \frac{2}{\pi\vartheta(\xi)}\int_0^a \theta(\rho)Cos(\rho\xi)d\rho. \tag{30}$$

Substituting (30) in (23), the functions $A_{\mathrm{I}}(\xi)$ to $D_{\mathrm{III}}(\xi)$ can be found. Furthermore, the displacements and stresses can be obtained in terms of $\theta(r)$ by substituting these eight functions into the expression of the potential function F, and then using the displacement- and stress- potential relations. Equations (28) with (29) can be numerically solved for $\theta(r)$.

IV. Numerical Evaluation of the Fredholm Integral Equation

In the previous section, the Fredholm integral equation was expressed in terms of $\theta(r)$. Generally, the Fredholm integral (29) cannot be analytically solved. As a result, the equation may be converted to a set of linear algebraic equations of the form

$$K_{ij}\theta_j = \frac{\Delta}{l}, \quad i, j = 1, 2, ..., n \tag{31}$$

where

$$K_{ij} = \delta_{ij} + \frac{1}{\pi} W_j M(r_i, \rho_j), \quad i, j = 1, 2, ..., n \tag{32}$$

moreover, W_j is the weight function for transforming an integral to a summation, and n is the number of points selected on the disc for numerical evaluation [20]. To evaluate $M_{ij} = M(r_i, \rho_j)$ from (29), an adaptive numerical quadrature method is adopted and coded in MATHEMATICA software. After determining M_{ij}, the function θ is found from (31) at the selected points. Then, replacing $\theta_j = \theta(\rho_j)$ in (29) and integrating, the function θ is found as a function of horizontal distance, r, after which $R^{(0)}(\xi)$ is obtained from (30).

In order to validate the present study, numerical solutions presented in [13] for a homogeneous full-space are evaluated here and used for comparison. To understand the importance of relative material moduli in the problem, 5 sets of material constants as listed in Table I are considered.

TABLE I
Synthetic Material Engineering Constants

Material	E (N/mm^2)	E' (N/mm^2)	G (N/mm^2)	G' (N/mm^2)	v	v'
1 (Transversely Isotropic)	45,000	20,000	20,000	20,000	0.25	0.25
2 (Transversely Isotropic)	50,000	25,000	20,000	20,000	0.25	0.25
3 (Transversely Isotropic)	50,000	55,000	20,000	20,000	0.25	0.25
4 (Transversely Isotropic)	50,000	100,000	20,000	20,000	0.25	0.25
5 (Transversely Isotropic)	50,000	150,000	20,000	20,000	0.25	0.25

With these materials, three different cases are defined as below to demonstrate the solution:

Case 1: $A_{\mathrm{I}ij} = Mat5 \quad A_{\mathrm{II}ij} = Mat5 \quad A_{\mathrm{III}ij} = Mat5$

Case 2: $A_{\mathrm{I}ij} = Mat1 \quad A_{\mathrm{II}ij} = Mat2 \quad A_{\mathrm{III}ij} = Mat5$

Case 3: $A_{\mathrm{I}ij} = Mat1 \quad A_{\mathrm{II}ij} = Mat3 \quad A_{\mathrm{III}ij} = Mat5$

Case 4: $A_{\mathrm{I}ij} = Mat1 \quad A_{\mathrm{II}ij} = Mat4 \quad A_{\mathrm{III}ij} = Mat5$

The last three cases are defined with different ratio of $E'_{\mathrm{II}} / E_{\mathrm{II}}$, to portrait the effect of degree of anisotropy. As it is seen $E'_{\mathrm{II}} / E_{\mathrm{II}}$ for Mat2, Mat3 and Mat4 is 0.5, ~1 and 2, respectively. On the other hand, the first case is a homogenous full-space, which is used for verification.

For generality, all numerical results presented here are dimensionless. Therefore, the stress σ_{zz} and the vertical displacement are presented in the form of dimensionless parameters as $\sigma_{zz} / (F / \pi a^2)$ and w / Δ, respectively. Here F is the vertical resultant force, applied on the disc, which itself is evaluated by the integral $F = 2\pi \int_0^a R(r) r dr$. F is also evaluated directly in terms of the Fredholm equation as $F = 4a \int_0^a \theta(\rho) d\rho$ [21].

Fig. 2 illustrates the stress difference $R(r)$ for the different cases listed above. As indicated in the figure, there exists a singular behavior at the vicinity of the edge of the disc. A very precise attention has been paid in the numerical evaluation, however, there exists a small error in the evaluation of the stress difference outside the disc, where the stress difference should be zero. This error is due to the number of points selected for evaluations of the stress in (31), and the truncation point selected in (29). Fig. 3 indicates a depth wise variation of the stress σ_{zz} for different cases. As seen in each curve, there exists a jump at $z = 0$, where the disc is located, which is equal to $R(r = 0, z = 0)$. Since E'_{II} in Case 4 is the largest among the tri-material cases, the stress σ_{zz} in this case for constant Δ is the largest one. The solution presented by Ardeshir-Behrestaghi and Eskandari-Ghadi [13] is used as a benchmark, to provide a comparison with the results in this paper and proof validity of the numerical evaluations. To this end, Case 1 with the same material properties for all the layers is defined. Fig. 3 also shows the results for a homogenous full-space for both studies, where an excellent agreement is discovered between the two solutions.

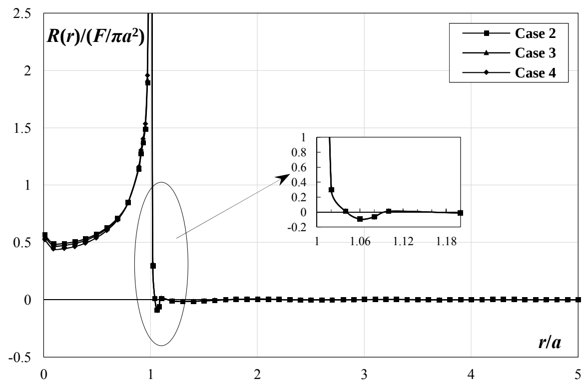

Fig. 2 Normalized resultant vertical stress applied on the rigid disc in terms of horizontal distance

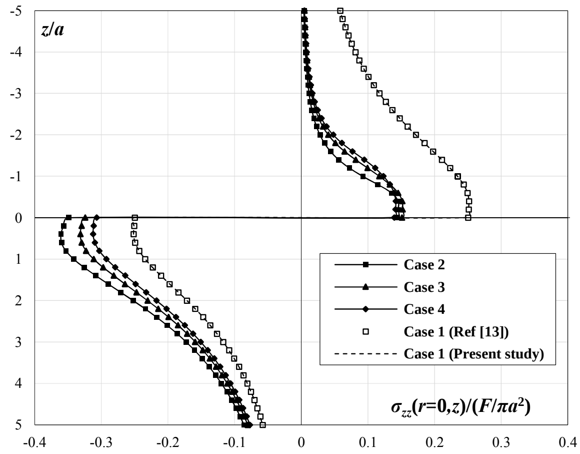

Fig. 3 The stress σ_{zz} at $r = 0$ in terms of depth z

Fig. 4 illustrates the normalized vertical displacement at $z = 0$ in terms of radial distance. The vertical displacement from zero to $r = a$ should be equal to Δ as inferred from (2). As seen in Fig. 5, although the displacement is continuous at $z = 0$, its derivative with respect to depth is not continuous as indicated in (5) and the strain-displacement and stress-strain relationships. Since E'_{II} in Case 4 is the largest, the vertical displacement in this case for constant Δ is the largest one.

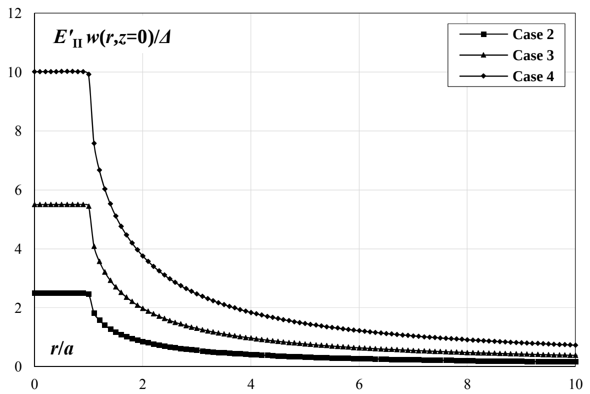

Fig. 4 Variation of vertical displacement at z=0 in terms of horizontal distance

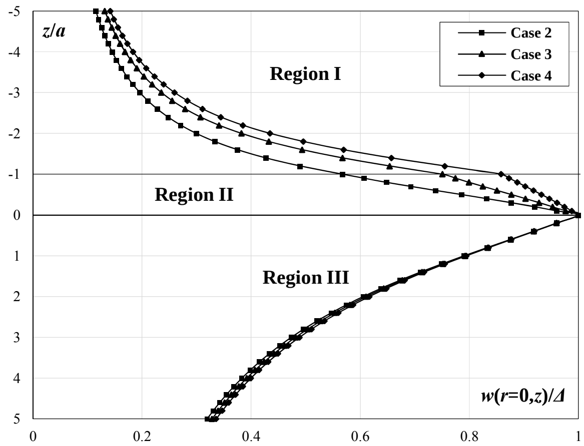

Fig. 5 Depth-wise variation of vertical displacement at r=0

V. Conclusion

With the use of a scalar potential function and Hankel integral transforms, the relaxed mixed boundary value problem for a vertical movement of a rigid disc embedded in a tri-material full-space considered in this paper have been transformed to a Fredholm integral equation of the second kind, which have been solved numerically. The solution is collapsed on the homogenous full-space reported in the literature, which proves the validity of the solution presented in this paper. Some different arrangements have been compared to see the effect of different degree of anisotropy.

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