International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:2, 2014

Nullity of t-Tupple Graphs

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***Abstract*—**The nullity η(G) of a graph is the occurrence of zero as an eigenvalue in its spectra. A zero-sum weighting of a graph G is real valued function, say f from vertices of G to the set of real numbers, provided that for each vertex of G the summation of the weights f(w) over all neighborhood w of v is zero for each v in G.A high zero-sum weighting of G is one that uses maximum number of non-zero independent variables. If G is graph with an end vertex, and if H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then, η(G)= η(H). In this paper, a high zero-sum weighting technique and the endvertex procedure are applied to evaluate the nullity of t-tupple and generalized t-tupple graphs are derived and determined for some special types of graphs,

Also, we introduce and prove some important results about the t-tupple coalescence, Cartesian and Kronecker products of nut graphs.

***Keywords*—**Graph theory, Graph spectra, Nullity of graphs.

I. Introduction

THE eigenvalues of the adjacency matrix A(G) are said to be the eigenvalues of the graph G, the occurrence of zero as an eigenvalue in the spectrum of the graph G is called the "nullity" of G, it is denoted by η(G). Brown and others [2] proved that a graph G is singular if, and only if, G possesses a non-trivial zero-sum weighting, and asked, what causes a graph to be singular and what the effects of this on its properties are. Rashid [6] proved that the maximum number of non zero independent variables used in a high zero- sum weighting for a graph G, is equal to the nullity of G.

Definition 1:[2],[6, p.16] A *vertex weighting* of a graph G is a function f: V(G)→R where R is the set of real numbers, which assigns a real number (weight) to each vertex. The weighting of G is said to be *non-trivial* if there is at least one vertex v∈V(G) for which f(v) ≠ 0.

Definition 2:[2, p.16] A non-trivial vertex weighting of a graph G is called a *zero-sum weighting* provided that for each v∈V(G), ∑f(w) = 0, where the summation is taken over all w∈N_G(v).

Clearly, the following weighting for G is a non-trivial zero-sum weighting where x_1, x_2, x_3, x_4, and x_5 are weights and provided that (x_1, x_2, x_3, x_4, x_5) ≠ (0, 0, 0, 0, 0)as indicated in Fig.1.

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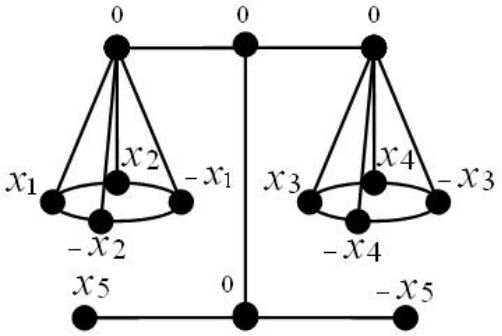

Fig. 1 A non-trivial zero-sum weighting for a graph G

Theorem 1: A graph G is Singular if, and only if there is a non-trivial zero sum weighting for G.■

Out of all zero-sum weightings of a graph G, a *high zero-sum weighting* of G is one that uses maximum number of non-zero independent variables.

Lemma 1: [6, p.35] In any graph G, the maximum number MV(G) of non zero independent variables in a high zero-sum weighting equals the number of zeros as an eigenvalues of the adjacency matrix of G, (i.e.MV(G)= η(G)).■

In Fig.1, the weighting for the graph G is a high zero-sum weighting that uses 5 independent variables, hence, η(G) = 5.

This is a very active method to characterize the degree of singularity (nullity) of a chemical compound Graph, the carbohydrate graph C_nH_{2n+1}, with n=5, has two bonding graphs, (a) where the 5 carbon atoms induces a path of order 5, η(G) = 7 this is a more stable case which is usually present in the nature, while in (b) where the 5 carbon atoms induces a star of order 5, with η(G) = 9 which has different physical properties as in case a,as well as more instability.

Lemma 2:[1, p.72],[3]

i. The eigenvalues of the cycle C_p are of the form $2\cos\frac{2\pi r}{p}$, r = 0, 1,…,p-1. According to this, η(C_P)= 2 if p=0(mod4) and 0 otherwise.
ii. The eigenvalues of the path P_p are of the form $2\cos\frac{\pi r}{p+1}$, r =1,2, … p. And thus, η(P_P)=1 if p is odd and 0 otherwise.
iii. The spectrum of the complete graph K_p, consists of p-1 and -1 with multiplicity p-1.
iv. The spectrum of the complete bipartite graph $K_{m,n}$, consists of $\sqrt{mn}$, $-\sqrt{mn}$ and zero m+n-2 times

Lemma 3: (Endvertex Lemma)[4, p.234] If G is graph with an end vertex, and if H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then η(G) = η(H). ■

Lemma 4: (Coneighbor Lemma) If two vertices u and v have the same neighbors in a graph G, then η(G)= η(G-u)+1= η(G-v)+1.■

Definition 3: [7] Let (G_1, u) and (G_2, v) be two graphs rooted at vertices u and v, respectively. We attach G_1 to G_2 (or G_2 to G_1) by identifying the vertex u of G_1 with the vertex v of G_2. Vertices u and v are called *vertices of attachment*. The vertex formed by their identification is called the *coalescence vertex*. The resulting graph $G_1 \circ G_2$ is called the *coalescence (vertex identification)* of G_1 and G_2.

Definition 4: [7] Let $\{(G_1, v_1), (G_2, v_2),\ldots,(G_t, v_t)\}$ be a family of not necessary distinct connected graphs with roots $v_1, v_2, \ldots, v_t$, respectively. A connected graph $G= G_1 \circ G_2 \circ \ldots \circ G_t$ is called the *multiple coalescence* of $G_1, G_2,\ldots,G_t$ provided that the vertices $v_1, v_2, \ldots, v_t$ are identified to reform the coalescence vertex v. The *t-tupple coalescence graph* is denoted by $\overset{|t|}{G}$ is the multiple coalescence of t isomorphic copies of a graph G. In the same ways $G_1 \circ \overset{|t|}{G}_2$ is the multiple coalescence of G_1 and t copies of G_2.

Remark 1: [7] All coalesced graphs have v as a common cut vertex. Some graphs and their operations will, herein, be illustrated in Fig.2.

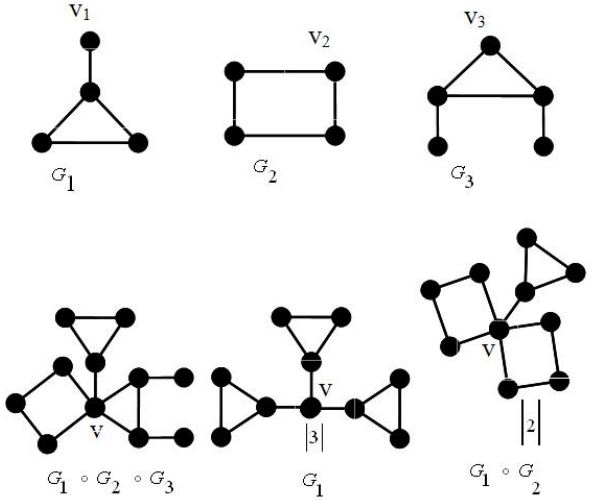

Fig. 2 Multiple coalescence $G_1 \circ G_2 \circ G_3$, t-tupple coalescence $\overset{|3|}{G}_1$ and coalescence of both $G_1 \circ \overset{|2|}{G}_2$

Definition 5: [7] Let G be a graph consisting of n vertices and L = $\{H_1, H_2, \ldots,H_n\}$ be a family of rooted graphs. Then the graph formed by attaching H_k to the k-th ($1 \le k \le n$) vertex of G is called the *generalized rooted product* and is denoted by G(L); G itself is called the core of G(L). If each member of L is isomorphic to the rooted graph H, then the graph G(L) is denoted by G(H). Recall G_1, G_2 and G_3 from Fig. 3. Then, we have

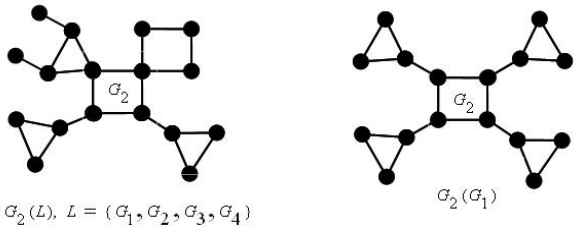

Fig. 3 Generalized rooted product graphs

II. Nullity of t-Tupple Graphs

In this section, we determine the nullity of *t-Tupple* graphs of some known graphs such as C_p, P_p, K_p and $K_{m,n}$.

A. t-Tupple Coalescence for Cycles

t-tupple coalescence graph $\overset{|t|}{C}_p$ has order $t(p-1)+1$ and size tp, and the diameter of $\overset{|t|}{C}_p$ is equal to $2diam(C_p)$, for $t \ge 2$. The nullity of $\overset{|t|}{C}_p$ is determined in the next proposition.

*Proposition 1:*For a t-tupple coalescence graph $\overset{|t|}{C}_p$, $t \ge 1$, we have:

i. If $p = 4n$, $n = 1, 2, \ldots$, then $\eta(\overset{|t|}{C}_{4n}) = t+1$.

ii. If $p = 4n+2$, $n = 1, 2, \ldots$, then $\eta(\overset{|t|}{C}_{4n+2}) = t-1$.

iii. If p is odd, then $\eta(\overset{|t|}{C}_p) = 0$.

Proof:

i. Let $x_{i,j}$, $i = 1, 2, ..., t$ and $j = 1, 2, ..., 4n$, where $x_{1,1} = x_{2,1} = ... = x_{t,1}$, be a weighting of a t-tupple coalescence graph $\overset{|t|}{C}_{4n}$, $n = 1, 2, ...$, as indicated in Fig. 4.

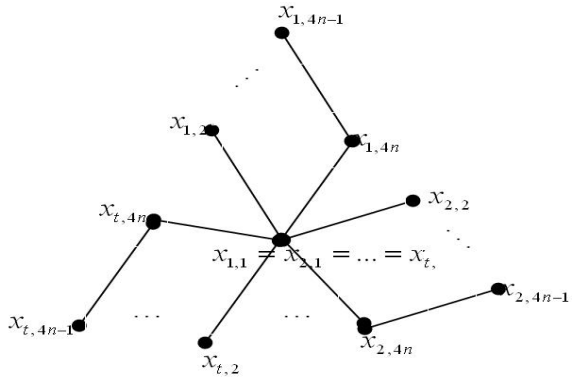

Fig. 4 A weighting of $\overset{|t|}{C}_{4n}$, , $n = 1, 2, \ldots$

Then, from the condition that $\sum_{w\in N_G(v)} f(w)=0$, for all v in $\overset{|t|}{C}_{4n}$, $n=1,2,\ldots$, we have:

for $i=1,2,\ldots,t$ and $j=1,2,\ldots,4n-3$

$$x_{i,\,j}+x_{i,\,j+2}=0\Rightarrow x_{i,\,j}=-x_{i,\,j+2} \quad (1)$$

since,

$$x_{1,1}=x_{2,1}=\ldots=x_{t,1} \quad (2)$$

then, from (1) and (2), we get:

for $i=1,2,\ldots,t$ and $j=1,5,\ldots,4n-3$

$$x_{i,\,j}=x_{1,1} \quad (3)$$

and, for $i=1,2,\ldots,t$ and $j=3,7,\ldots,4n-1$,

$$x_{i,\,j}=-x_{1,1} \quad (4)$$

Also, from the condition that $\sum_{w\in N_G(v)} f(w)=0$, for all v in $\overset{|t|}{C}_{4n}$, $n=1,2,\ldots$, we have:

for $i=1,2,\ldots,t$ and $j=2,4,\ldots,4n-2$

$$x_{i,\,j}+x_{i,\,j+2}=0\Rightarrow x_{i,\,j}=-x_{i,\,j+2} \quad (5)$$

thus, from (5), we get:

for $j=2,6,\ldots,4n-2$

$$\begin{aligned} x_{1,\,j}&=x_{1,2}\\ x_{2,\,j}&=x_{2,2}\\ &\vdots\\ x_{t,\,j}&=x_{t,\,2}\end{aligned} \quad (6)$$

and, for $j=4,8,\ldots,4n$

$$\begin{aligned} x_{1,\,j}&=-x_{1,2}\\ x_{2,\,j}&=-x_{2,2}\\ &\vdots\\ x_{t,\,j}&=-x_{t,\,2}\end{aligned} \quad (7)$$

Then, from (3) and (4) we use only one independent variable, and from (6) and (7), we use t independent variables, for a zero-sum weighting of $\overset{|t|}{C}_{4n}$, $n=1,2,\ldots$.

Thus, the maximum number of non zero independent variables used in a high zero-sum weighting of $\overset{|t|}{C}_{4n}$, $n=1,2,\ldots$, is equal to $t+1$. Hence, by Proposition 1.4, $\eta(\overset{|t|}{C}_{4n})=t+1$.

On the other hand, $\overset{|t|}{C}_{4n}$, $n=1,2,\ldots$, is multiple coalescence of t isomorphic copies of a graph C_{4n}, $n=1,2,\ldots$, since $\eta(C_{4n})=2$, by Lemma 2 (i). Then, we need $2t$ variables for a weighting of C_{4n}. But by (3) and (4), one variable is occurred t times. In such a case, we must use this variable exactly once. Therefore, $\eta(\overset{|t|}{C}_{4n})=2t-(t-1)=t+1$.

ii. There exists a high zero-sum weighting for $\overset{|t|}{C}_{4n+2}$, $n=1,2,\ldots$, which uses $t-1$ independent variables. Hence, by Lemma 1, $\eta(\overset{|t|}{C}_{4n+2})=t-1$.

iii. If p is odd, there exists no non-trivial zero-sum weighting for $\overset{|t|}{C}_{p}$. Thus, by Lemma 1, $\overset{|t|}{C}_{p}$ is non singular. ■

B. t-Tupple Coalescence for Paths

t-Tupple coalescence graphs $\overset{|t|}{P}_p$ have order $t(p-1)+1$ and size $t(p-1)$, and the $diam(\overset{|t|}{P}_p)\le 2diam(P_p)$. Besides, equality holds where the rooted vertex of P_p is an end vertex.

The nullity of $\overset{|t|}{P}_p$ is determined in the next proposition.

Proposition 2: For a t-tupple coalescence graph $\overset{|t|}{P}_p$, $t\ge 1$, we have:

i. If p is even, $p=2n$, $n=1,2,\ldots$, and the attachment is at any vertex, then $\eta(\overset{|t|}{P}_{2n})=t-1$.

ii. If p is odd, $p=2n+1$, $n=1,2,\ldots$, and the attachment is at a vertex with zero weight ,then $\eta(\overset{|t|}{P}_{2n+1})=2t-1$.

iii. If p is odd, $p=2n+1$, $n=1,2,\ldots$, and the attachment is at a vertex with non-zero weight ,then $\eta(\overset{|t|}{P}_{2n+1})=1$.

Proof:

i. Let the attachment is at end vertex, and let $x_{i,\,j}$, $i=1,2,\ldots,t$ and $j=1,2,\ldots,2n+1$, where

$x_{1,1} = x_{2,1} = ... = x_{t,1}$, be a weighting of $P_{2n}^{|t|}$, $n = 1, 2, ...,$ as indicated in Fig. 5.

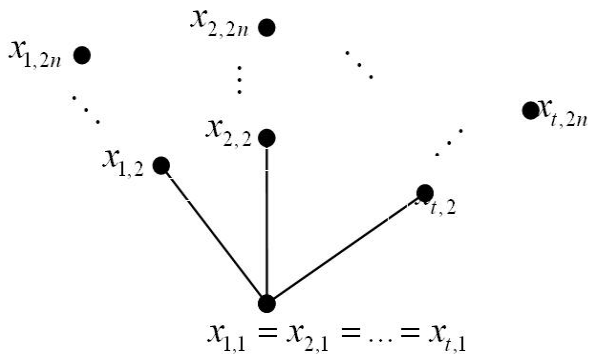

Fig. 5 A weighting of $P_{2n}^{|t|}$, $n = 1, 2, ...$, where the attachment is an end vertex

Then, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in $P_{2n}^{|t|}$, $n = 1, 2, ...,$ we have:

$$x_{1,2n-1} = x_{2,2n-1} = ... = x_{t,2n-1} = 0 \quad (8)$$

and, for $i = 1, 2, ..., t$ and $j = 1, 3, ..., 2n-3$

$$x_{i,j} + x_{i,j+2} = 0 \Rightarrow x_{i,j} = -x_{i,j+2} \quad (9)$$

Hence, from (8) and (9), we get:
for $i = 1, 2, ..., t$ and $j = 1, 3, ..., 2n-1$

$$x_{i,j} = 0 \quad (10)$$

Also, for $i = 1, 2, ..., t$ and $j = 2, 4, ..., 2n-2$

$$x_{i,j} + x_{i,j+2} = 0 \Rightarrow x_{i,j} = -x_{i,j+2} \quad (11)$$

and

$$x_{1,2} + x_{2,2} + ... + x_{t,2} = 0. \quad (12)$$

Then,

$$x_{t,2} = -x_{1,2} - x_{2,2} + ... - x_{t-1,2} = -\sum_{i=1}^{t-1} x_{i,2}. \quad (13)$$

Thus, from (11) and (13), we use only $t-1$ independent variables for a high zero-sum weighting of $P_{2n}^{|t|}$. Hence, by Lemma 1, $P_{2n}^{|t|} = t-1$.

ii. Let the attachment vertex be a neighbor of an end vertex, and let $x_{i,j}$, $i = 1, 2, ..., t$ and $j = 1, 2, ..., 2n+1$, where $x_{1,2} = x_{2,2} = ... = x_{t,2}$, be a weighting of a t-tupple coalescence graph $P_{2n+1}^{|t|}$, $n = 1, 2, ...,$ we use $(t-1)+t$ independent variable for a high zero-sum weighting of $P_{2n+1}^{|t|}$. Hence $P_{2n+1}^{|t|} = (t-1)+t = 2t-1$, where the attachment is a vertex with zero weight.

iii. Let the attachment be at an end vertex, and let $x_{i,j}$, $i = 1, 2, ..., t$ and $j = 1, 2, ..., 2n+1$, where $x_{1,1} = x_{2,1} = ... = x_{t,1}$, be a weighting of the graph $P_{2n+1}^{|t|}$, $n = 1, 2, ...,$

Then, a high zero-sum weighting of $P_{2n+1}^{|t|}$ is obtained which uses exactly one independent variable.

Therefore, by Lemma 1, $\eta(P_{2n+1}^{|t|}) = 1$. ■

C. t-Tupple Coalescence for Complete Graphs

The t-tupple coalescence graph $K_p^{|t|}$ is not a complete graph, and it contains one cut vertex of degree t(p-1), with order $t(p-1)+1$ and size tq, $tq = \frac{tp(p-1)}{2}$ and $diam(K_p^{|t|}) = 2$, for $t \geq 2$.

The nullity of $K_p^{|t|}$ is determined in the next proposition.

Proposition 3: For a t-tupple coalescence graph $K_p^{|t|}$, $t \geq 1$, we have:

i. If $p = 2$, then $\eta(K_2^{|t|}) = t-1$. .

ii. If $p \geq 3$, then $\eta(K_p^{|t|}) = 0$.

Proof:

i. For $p = 2$, then, $K_2 \equiv P_2$, therefore, the proof is a special case of Prop.2 (i).

ii. For $p \geq 3$, there exists no non-trivial zero-sum weighting of the graph $K_p^{|t|}$, thus by Th. 1, $K_p^{|t|}$ is a non singular. Hence, $\eta(K_p^{|t|}) = 0$. ■

D.t-Tupple Coalescence for Complete Bipartite Graphs

The t-tupple coalescence graph $\overset{|t|}{K}_{m,n}$ has order $t(m+n-1)+1$ and size $t(mn)$ and $diam(\overset{|t|}{K}_{m,n}) \le 2diam(K_{m,n})$, in which strictly holds where $K_{m,n}$ is a star graph and the coalescence vertex is the central. And equality holds otherwise. The nullity of $\overset{|t|}{K}_{m,n}$ is determined in the next proposition.

Proposition 4: For a t-tupple coalescence graph $\overset{|t|}{K}_{m,n}$, $t \ge 1$, $m,n \ge 1$,we have: the following cases:

i. If $(m,n)=(1,1)$, then $\eta(\overset{|t|}{K}_{1,1}) = \eta(\overset{|t|}{P}_2) = t-1$.

ii. If $(m,n)=(1,2)$, we have two cases:

a) If the attachment is at a vertex in a set that has one vertex, then $\eta(\overset{|t|}{K}_{1,2}) = \eta(\overset{|t|}{P}_3) = 2t-1$.

b) If the attachment is at a vertex in a set that has two vertices, then $\eta(\overset{|t|}{K}_{1,2}) = \eta(\overset{|t|}{P}_3) = 1$.

iii. for, $m,n \ge 2$, if $(m,n)=(2,2)$, then $\eta(\overset{|t|}{K}_{2,2}) = \eta(\overset{|t|}{C}_4) = t+1$, and $\eta(\overset{|t|}{K}_{m,n}) = t(m+n-3)+1$, otherwise.

Proof: Parts (i) and (ii) are special cases of Prop. 2.

iii. For $(m,n)=(2,2)$, then $K_{2,2} \equiv C_4$, which is a special case of Prop.3 (i). For $m \ge 2$ and $n \ge 3$, let $x_{i,j}$ and $y_{r,s}$, $i,r = 1,2,...,t$, $j = 1,2,...,m$, and $s = 1,2,...,n$. Where, $x_{1,1} = x_{2,1} = ... = x_{t,1}$ be a weighting of a t-tupple coalescence graph $\overset{|t|}{K}_{m,n}$, as indicated in Fig.6.

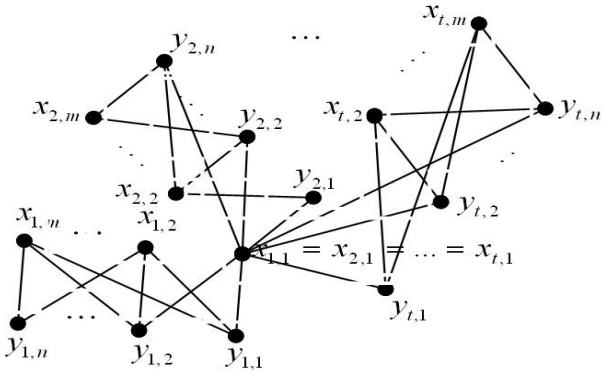

Fig. 6 A weighting of $\overset{|t|}{K}_{m,n}$, for $m \ge 2$ and $n \ge 3$

Then, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in $\overset{|t|}{K}_{m,n}$, we have:

$$\begin{aligned} y_{1,1} + y_{1,2} + ... + y_{1,n} &= 0 \\ y_{2,1} + y_{2,2} + ... + y_{2,n} &= 0 \\ &\vdots \\ y_{t,1} + y_{t,2} + ... + y_{t,n} &= 0 \end{aligned}$$

then,

$$\begin{aligned} y_{1,n} &= -y_{1,1} - y_{1,2} - ... - y_{1,n-1} = -\sum_{s=1}^{n-1} y_{1,s} \\ y_{2,n} &= -y_{2,1} - y_{2,2} - ... - y_{2,n-1} = -\sum_{s=1}^{n-1} y_{2,s} \\ &\vdots \\ y_{t,n} &= -y_{t,1} - y_{t,2} - ... - y_{t,n-1} = \sum_{s=1}^{n-1} y_{t,s} \end{aligned} \quad (14)$$

and, we have:

$$\begin{aligned} x_{1,1} + x_{1,2} + ... + x_{1,m} &= 0 \\ x_{2,1} + x_{2,2} + ... + x_{2,m} &= 0 \\ &\vdots \\ x_{t,1} + x_{t,2} + ... + x_{t,m} &= 0 \end{aligned}$$

then,

$$\begin{aligned} x_{1,m} &= -x_{1,1} - x_{1,2} - ... - x_{1,m-1} = -\sum_{j=1}^{m-1} x_{1,j} \\ x_{2,m} &= -x_{2,1} - x_{2,2} - ... - x_{2,m-1} = -\sum_{j=1}^{m-1} x_{2,j} \\ &\vdots \\ x_{t,m} &= -x_{t,1} - x_{t,2} - ... - x_{t,m-1} = \sum_{j=1}^{m-1} x_{t,j} \end{aligned} \quad (15)$$

with, $x_{1,1} = x_{2,1} = ... = x_{t,1}$.

Then, from (14) we use $(n-1)$,t times, and from (15), we use $(m-1)$, one times and $(m-2)$, $(t-1)$ times. Thus, the maximum number of non-zero independent variables used in a high zero-sum weighting of $\overset{|t|}{K}_{m,n}$,$(m,n) \ne (1,1)$, is equal to $t(n-1)+t(m-2)+1$.

Therefore, by Lemma 4 (iv),

$$\eta(\overset{|t|}{K}_{m,n}) = t(n-1)+t(m-2)+1 = t(n+m-3)+1,$$

where, $(m,n) \ne (1,1)$ .On the other hand, $\overset{|t|}{K}_{m,n}$, $(m,n) \ne (1,1)$ is a multiple coalescence of t isomorphic copies of a graph $K_{m,n}$, then, we need $t(m+n-2)$

variables for a weighting of $K_{m,n}^{|t|}$. But by (15), one variable is occurred t times. In such a case, we must use this variable exactly once. Therefore,

$$\eta(K_{m,n}^{|t|}) = t(m+n-2)-(t-1) = t(m+n-3)+1.\blacksquare$$

E. t-Tupple Coalescence for Star Graphs

The t-tupple coalescence graph $S_{1,n-1}^{|t|}$ has order $t(n-1)+1$ and size $t(n-1)$ and the diameter of $S_{1,n-1}^{|t|}$ is equal to 2 if the rooted vertex of $S_{1,n-1}$ is a central vertex and equals 4 if the rooted vertex of $S_{1,n-1}$ is a non central vertex.

The nullity of $S_{1,n-1}^{|t|}$ is determined in the next proposition.

Proposition 5: For a t-tupple coalescence graph $S_{1,n-1}^{|t|}$, $t \geq 1$, $n \geq 4$, we have:

i. If the attachment is at central vertex, then: $\eta(S_{1,n-1}^{|t|}) = t(n-1)-1$.
ii. If the attachment is at non-central vertex, then: $\eta(S_{1,n-1}^{|t|}) = t(n-3)+1$.

Proof:

i. If the attachment vertex is the central vertex, then the t-tupple $S_{1,n-1}^{|t|}$ is a star graph with order t(n-1)+1 and nullity t(n-1)-1.
ii. The proof is similar to that of Prop. 4.■

III. Nullity of Generalized Rooted Graphs

In this section, we study the nullity of generalized rooted product graphs, for some known graphs such as C_p, P_p, K_p and $K_{m,n}$.

A. Generalized Rooted Product Graphs for Cycles

A generalized rooted product graph G(H) where $G = C_p$ has order $p.p(H)$ and size $p + pq(H)$ with $diam(C_p(H)) \leq \frac{p}{2} + 2diam(H)$ if p is even, and $diam(C_p(H)) \leq \frac{p-1}{2} + 2diam(H)$ if p is odd. Equality holds in both cases if coalescence vertices of H are diametrical vertices.

Proposition 6: Let $k, t \in Z^+$, then:

i. If the rooted vertex of P_{2t+1} has zero weight, then, $\eta(C_p(P_{2t+1})) = \eta(C_p(P_{2t+1}) - E(C_p)) = p$
ii. $\eta(C_p(P_{2t})) = 0$.
iii. If the rooted vertex of P_{2t+1} has non-zero weight, then, $\eta(C_{4k}(P_{2t+1})) = 2$.

Proof:

i. Let $e_1, e_2, \ldots, e_p$ be the edges of the graph C_p, then we remove all these edges to obtain the graph $(C_p(P_{2t+1}) - E(C_p))$. Clearly, $(C_p(P_{2t+1}) - E(C_p)) = P_{2t+1} \cup P_{2t+1} \cup \ldots \cup P_{2t+1}$ is the union of p paths each of order $2t+1$. See Fig. 7.

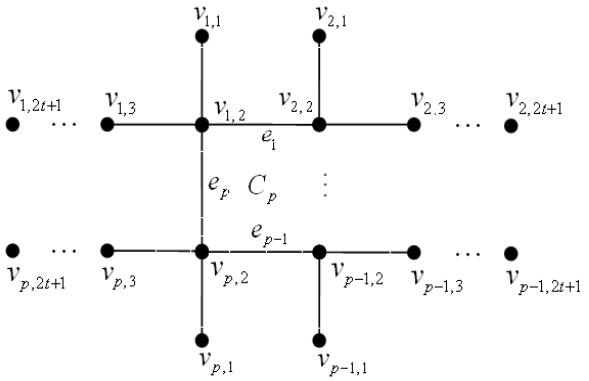

Fig. 7 The graph $C_p(P_{2t+1})$

In a high zero-sum weighting of $C_p(P_{2t+1})$, we can use exactly p non-zero independent variables; hence, $\eta(C_p(P_{2t+1})) = p$. On the right side, it is clear that $C_p(P_{2t+1}) - E(C_p) = P_{2t+1} \cup P_{2t+1} \cup \ldots \cup P_{2t+1}$.

Hence,

$$\eta(C_p(P_{2t+1}) - E(C_p)) = \eta(P_{2t+1} \cup P_{2t+1} \cup \ldots \cup P_{2t+1})$$

Since, $\eta(P_{2t+1}) = 1$ by Lemma 2 (ii) and $\eta(P_{2t+1} \cup P_{2t+1} \cup \ldots \cup P_{2t+1}) = \eta(P_{2t+1}) + \eta(P_{2t+1}) + \ldots + \eta(P_{2t+1})$. Therefore, $\eta(C_p(P_{2t+1}) - E(C_p)) = 1+1+\ldots+1 = p$.

ii. The proof is similar to that of Prop.6 (iii).
iii. Let $x_{i,j}$, $i = 1, 2, \ldots, 4k$ and $j = 1, 2, \ldots, 2t+1$, be a weighting of a generalized rooted product graph $C_{4k}(P_{2t+1})$, $k, t \in Z^+$, Thus, we use 2 independent variables, for a high zero-sum weighting of $C_{4k}(P_{2t+1})$, $k, t \in Z^+$. Hence, $\eta(C_{4k}(P_{2t+1})) = 2$.■

Proposition 7: Let $G = C_p$ and $H = K_n$, $n > 1$, we have:

i. $\eta(C_p(K_3)) = 1$. .
ii. If $n \neq 3$, then $\eta(C_p(K_n)) = 0$.

Proof: The proof of (i) and (ii) are determined by weights technique.■

Proposition 8: Let $G = C_p$ and $H = K_{m,n}, m,n \geq 2$, we consider the following cases:

i. If $p = 4k$, $k \in Z^+$, then $\eta(C_{4k}(K_{m,n})) = 4k(m+n-3)+2$.

ii. If $p \neq 4k$, $k \in Z^+$, then

$$\eta(C_p(K_{m,n})) = \eta(C_p(K_{m,n}) - V(C_p)) = p(m+n-3).$$

Proof:

i. Let $x_{i,j}$, $y_{r,s}$, where, $i,r = 1,2,...,4k$, $j = 1,2,...,m$, and $s = 1,2,...,n$ be a weighting for $C_{4k}(K_{m,n})$, $k \in Z^+$ as indicated in Fig. 8.

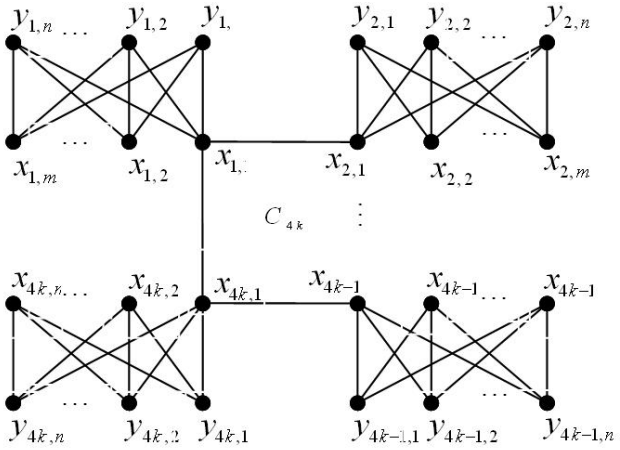

Fig. 8 A weighting of the generalized rooted product graph $C_{4k}(K_{m,n})$

Then, from the condition that $\sum_{w \in N_G(v)} f(w) = 0$, for all v in $C_{4k}(K_{m,n})$, $k \in Z^+$, we have:

$$\begin{aligned} x_{1,1} + x_{1,2} + ... + x_{1,m} &= 0 \\ x_{2,1} + x_{2,2} + ... + x_{2,m} &= 0 \\ &\vdots \\ x_{4k,1} + x_{4k,2} + ... + x_{4k,m} &= 0 \end{aligned}$$

then,

$$\begin{aligned} x_{1,m} &= -x_{1,1} - x_{1,2} - ... - x_{1,m-1} \\ x_{2,m} &= -x_{2,1} - x_{2,2} - ... - x_{2,m-1} \\ &\vdots \\ x_{4k,m} &= -x_{4k,1} - x_{4k,2} - ... - x_{4k,m-1} \end{aligned} \quad (16)$$

and,

$$\begin{aligned} y_{1,1} + y_{1,2} + ... + y_{1,n} &= 0 \\ y_{2,1} + y_{2,2} + ... + y_{2,n} &= 0 \\ &\vdots \\ y_{4k,1} + y_{4k,2} + ... + y_{4k,n} &= 0 \end{aligned}$$

then,

$$\begin{aligned} y_{1,n} &= -y_{1,1} - y_{1,2} - ... - y_{1,n-1} \\ y_{2,n} &= -y_{2,1} - y_{2,2} - ... - y_{2,n-1} \\ &\vdots \\ y_{4k,n} &= -y_{4k,1} - y_{4k,2} - ... - y_{4k,n-1} \end{aligned} \quad (17)$$

Also, we have: For $i = 2,4,...,4k-2$

$$x_{i,1} + x_{i+2,1} = 0 \Rightarrow x_{i,1} = -x_{i+2,1} \quad (18)$$

and, for $i = 1,3,...,4k-3$,

$$x_{i,1} + x_{i+2,1} = 0 \Rightarrow x_{i,1} = -x_{i+2,1} \quad (19)$$

Thus, from (16) and (17), we use $4k(m-1) + 4k(n-1)$ variables. But from (18) and (19), $4t-2$ variables are iterated. Therefore, the maximum number of non-zero independent variables used in a high zero-sum weighting for $C_{4k}(K_{m,n})$, $k \in Z^+$, is equal to:

$$\begin{aligned} &4k(m-1) + 4k(n-1) - (4k-2) \\ &= 4k(m-1) + 4k(n-1) - 4k + 2 \\ &= 4k(m+n-3) + 2. \end{aligned}$$

Hence, by Lemma 1,

$$\eta(C_{4k}(K_{m,n})) = 4k(m+n-3) + 2.$$

ii. The proof is similar to that of Proposition 6 (i).■

Corollary 1: [5, p.44] If the rooted vertex of a graph $S_{1,n-1}$ is a central vertex, then:

$$\eta(C_p(S_{1,n-1})) = \eta(C_p(S_{1,n-1}) - E(C_p)) = p(n-2).$$

Proof: Follows by applying End vertex Corollary.■

B. Generalized Rooted Product Graphs for Paths

A generalized rooted product graph G(H), where $G = P_p$ has order p|H| and size $p + pq(H) - 1$ and $diam(P_p(H)) \leq p + 2diam(H) - 1$. Equality holds if coalescence vertices of H are diametrical vertices.

Proposition 9: Let $k, t \in Z^+$, then:

i. $\eta(P_{2t}(C_{4k})) = \eta(P_{2t}(C_{4k}) - V(P_{2t})) = 2t$.

ii. $\eta(P_{2t+1}(C_{4k})) = \eta(P_{2t+1}(C_{4k}) - V(P_{2t+1})) + 1$
$= 2t + 2.$

iii. For each and $n \neq 4k$, $\eta(P_p(C_n)) = 0.$

Proof:

i. We can use exactly $2t$ distinct independent variables for a high zero-sum weighting of $P_{2t}(C_{4k})$., we have $\eta(P_{2t}(C_{4k})) = 2t$.

On the other hand, $(P_{2t}(C_{4k}) - V(P_{2t})) = (P_{4k-1} \cup P_{4k-1} \cup ... \cup P_{4k-1})$..

Hence, $\eta(P_{2t}(C_{4k}) - V(P_{2t})) = \eta(P_{4k-1} \cup P_{4k-1} \cup ... \cup P_{4k-1})$.

Since, $\eta(P_{4k-1}) = 1$ by Lemma 2 (ii), and

$$\eta(P_{4k-1} \cup P_{4k-1} \cup ... \cup P_{4k-1})$$
$$= \eta(P_{4k-1}) + \eta(P_{4k-1}) + ... + \eta(P_{4k-1})$$
$$= 1 + 1 + ... + 1 + 1 = 2t$$

Therefore, $\eta(P_{2t}(C_{4k})) = 2t.$

ii. The proof is similar to part (i).

iii. The proof is similar to that of Prop.8 (ii).■

*Proposition 10:*Let $G = P_p$ and $H = K_n$, $n > 1$ then $\eta(P_p(K_n)) = 0$.

Proof: The proof follows from the fact that in any high zero-sum weighting, each vertex of $P_p(K_n)$ must be zero weighted, and then $P_p(K_n)$ has no non-trivial zero-sum weighting.■

*Proposition 11:*Let $m, n \geq 2$, $t \in Z^+$, then:

i. $\eta(P_{2t}(K_{m,n})) = \eta(P_{2t}(K_{m,n}) - V(P_{2t}))$
$= 2t(m + n - 3).$

ii. $\eta(P_{2t+1}(K_{m,n})) = \eta(P_{2t+1}(K_{m,n}) - V(P_{2t+1})) + 1$
$= 2t(m + n - 3) + 1.$

Proof:

i. If the rooted vertex of $K_{m,n}$ belongs to the set of vertices with cardinality m, then from the removal of all vertices of P_{2t} we obtain a disconnected graph with $2t$ components and each component is $K_{m-1,n}$, and by Lemma 4 (iv), $\eta(K_{m-1,n}) = (m + n - 3)$. Therefore, we can use exactly $(m+n-3)+(m+n-3)+...+(m+n-3) = 2t(m+n-3)$. independent variables for a high zero-sum weighting of $P_{2t}(K_{m,n})$. Therefore, by Lemma 1, we have: $\eta(P_{2t}(K_{m,n})) = 2t(m + n - 3)$.■

Note: It is clear that if $G = P_p$, $H = S_{1,n-1}, n \geq 3$, and the rooted vertex of a graph $S_{1,n-1}$ is a central vertex , then $\eta(P_p(S_{1,n-1})) = \eta(P_p(S_{1,n-1}) - E(P_p)) = p(n-2).$

C. Generalized Rooted Product Graphs for Complete Graphs

A generalized rooted product graph G(H) where $G = K_p$ has order $p(p(H))$ and size $\frac{p(p-1)}{2} + pq(H)$ and $diam(K_p(H)) \leq 1 + 2diam(H)$. Equality holds if coalescence vertices are diametrical vertices.

*Proposition 12:*Let $t \in Z^+$, then:

i. $\eta(K_p(C_{4t})) = \eta(K_p(C_{4t}) - V(K_p)) = p.$

ii. $\eta(K_3(C_{4t-1})) = 1.$.

iii. $\eta(K_p(C_n)) = 0,$, for $p = 0$ and $n = 4t+1, 4t+2$.

Proof:
The proof of (i), (ii), and (iii) are similar to that of Proposition 5 (i), 7 (iii) and 3 (ii), respectively.■

*Proposition 13:*Let $t \in Z^+$, then:

i. If the rooted vertex of P_{2t+1} has zero weight, then, $\eta(K_p(P_{2t+1})) = \eta(K_p(P_{2t+1}) - E(K_p)) = p.$

ii. $\eta(K_p(P_n)) = 0$, otherwise.

Proof: The proof of (i) and (ii) are determined by applying End vertex Lemma.■

*Proposition 14:*Let $p>1$, $m,n \geq 2$, then $\eta(K_p(K_{m,n})) = \eta(K_p(K_{m,n}) - V(K_p)) = p(m + n - 3).$

Proof: The proof follows by applying Coneighbour Lemma.■

Note: If $n \geq 3$ and the rooted vertex of $S_{1,n-1}$ is the central vertex, then $\eta(K_p(S_{1,n-1})) = \eta(K_p(S_{1,n-1}) - E(K_p)) = p(n-2).$

D. Generalized Rooted Product Graphs for Complete Bipartite Graphs

The necessary and sufficient condition for a connected bipartite graph G to be a complete bipartite graph is that G does not contain a path of order four as an induced subgraph. Moreover, the generalized rooted product graph G(H) where $G = K_{m,n}$ has order $(m + n)p(H)$ and size $mn+(m+n)q(H)$ and $diam(K_{m,n}(H)) \leq diam(K_{m,n}) + 2diam(H).$ Equality holds if coalescence vertex of H is diametrical vertex.

*Proposition 15:*Let $t \in Z^+$, and $m, n \geq 2$, then:

i. If $(m, n) = (2, 2)$, then: $\eta(K_{2,2}(C_{4t-1})) = 1.$

ii. for, $m,n \geq 2$, if $(m,n)=(2,2)$, then $\eta(\overset{|t|}{K}_{2,2}(C_{4t}))=6$, and $\eta(\overset{|t|}{K}_{m,n}(C_{4t}))=\eta(\overset{|t|}{K}_{m,n}(C_{4t})-E)=(m+n-2)+(m+n)$, otherwise.

where E is the set of edges of C_{4t} which are adjacent with $K_{m,n}$ after the coalescence.

i. $\eta(K_{m,n}(C_p))=0$, otherwise.

Proof:

i. The proof is similar to that of Proposition 7 (i).

ii. Since $K_{2,2} \approx C_4$, then the proof is similar to that of Proposition 7(i), we can use exactly $2(m+n)-2$ +(m+n) independent variables for a high zero-sum weighting of $K_{m,n}(C_{4t})$.

Therefore, by Lemma1, we have: $\eta(K_{m,n}(C_{4t}))=(m+n-2)+(m+n)$.

On the right side, it is obvious that: $(K_{m,n}(C_{4t})-E)$ $=(K_{m,n}\cup P_{4t-1}\cup P_{4t-1}\cup\ldots\cup P_{4t-1})$.

Hence, $\eta(K_{m,n}(C_{4t})-E)=\eta(K_{m,n}\cup P_{4t-1}\cup P_{4t-1}\cup\ldots\cup P_{4t-1})$.

Since, $\eta(K_{m,n})=m+n-2$ by Lemma 2 (iv), and $\eta(P_{4t-1})=1$ by Lemma 2 (ii). Then:

$\eta(K_{m,n}\cup P_{4t-1}\cup P_{4t-1}\cup\ldots\cup P_{4t-1})$

$=\eta(K_{m,n})+\eta(P_{4t-1})+\eta(P_{4t-1})+\ldots+\eta(P_{4t-1})$

$=(m+n-2)+1+1+\ldots+1=2(m+n)-2$.

Therefore,

$\eta(K_{m,n}(C_{4t}))=(m+n-2)+(m+n)=\eta(K_{m,n}(C_{4t})-E)$.

iii. The proof is similar to that of Proposition 8 (ii).■

Proposition 16:Let $t \in Z^+$, and $m,n \geq 2$, then:

i. If the rooted vertex of P_{2t+1} has zero-weight, then: $\eta(K_{m,n}(P_{2t+1}))=\eta(K_{m,n}(P_{2t+1})-E(K_{m,n}))=m+n$.

ii. If the rooted vertex of P_{2t+1} has non-zero weight, then: $\eta(K_{m,n}(P_{2t+1}))=\eta(K_{m,n}(P_{2t+1})-E(K_{m,n}))-2$ $=m+n-2$.

iii. $\eta(K_{m,n}(P_{2t}))=0$.

Proof: The proof of (i) and (ii) follows by applying Endvertex Lemma m+n times which leaves such number of odd paths. The prove of (iii) is similar to that of Proposition 8.■

A. More Generalized Rooted Products

We are going to study more generalized rooted product graphs

Proposition 17:Let $t \in Z^+$, and $u_1,u_2,\ldots,u_{p(G)}$ be the vertices of the non-trivial graph G, then:

i. If the rooted vertex of P_{2t+1} has zero weight, then: $\eta(G(P_{2t+1}))=\eta(G(P_{2t+1})-E(G))=p(G)$.

ii. If the rooted vertex of P_{2t+1} has non-zero weight, then: $\eta(G(P_{2t+1}))=\eta(G)$.

iii. $\eta(G(P_{2t}))=0$.

Proof: The proof of (i), (ii) and (iii) follows by applying Endvertex Lemma. ■

We are going to prove more generalized rooted products to Prop, 8, 11 and 14.

Proposition 18: Let $u_1,u_2,\ldots,u_{p(G)}$ be the vertices of a graph G and $H=K_{m,n}$, then: $\eta(G(K_{m,n}))=p(G)(m+n-3)+\eta(G)$.

Proof: If G is non-singular, then each vertex u_i, $i=1,2,\ldots p(G)$, has a zero weight in a high zero-sum weighting for $G(K_{m,n})$. Therefore, removal of all vertices of G will not change the number of independent variables in any high zero-sum weighting of the graph $G(K_{m,n})$, which is $(m+n-3)+(m+n-3)+\ldots+(m+n-3)$ $p(G)$ times. Hence,

$\eta(G(K_{m,n}))=\eta(G(K_{m,n})-V(G))=p(G)(m+n-3)$

But, if $\eta(G)>0$, then $\exists$ more η independent variables in a high zero-sum weighting for $G(K_{m,n})$.■

IV. Nut Graphs and Their Coalescence

In this section, we introduce and prove some results on the nullity of coalescence of nut graphs such as coalescence at vertex, generalization rooted, edge introduced, Cartesian product and tensor product. It is clear that it is clear that in [6, Theorem 4] that a nut graph is connected with no end vertex and it is not bipartite. We can assume that K_1 is also a nut graph. The smallest nut graph has order 7.

Some basic nut graphs with order 7 is C_5oC_3, with order 3m, $m \geq 3$ is $C_m(C_3)$ and whose orders 8, 10 and are illustrated in the next figure.

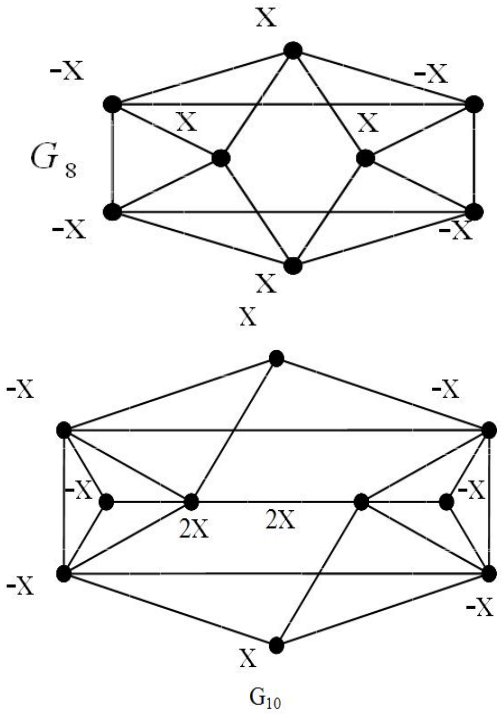

Fig. 9 Some basic nut graphs

Lemma 5: In the above nut graphs, if T or T_i, i=1, 2, 3 is replaced by C_{4k-1}, $k \geq 1$, then the resulting graph is again a nut graph.

Proof: The weighting of the triangle T {or any T_i} is {-x, x, x} can be replaced by {-x, x, x, -x, -x, x, x,…,-x, -x, x, x} in the weighting of the cycle C_{4k-1}.■

Theorem 2: For any two rooted nut graphs (G_1, u) and (G_2, v), the coalescence graph $G = G_1 \circ G_2$ is a nut graph.

Proof: Let $f(u_i) = x_i$, $i = 1, 2, ..., p_1$ and $f(v_j) = y_j$, $j = 1, 2, ..., p_2$ be a non-zero weighting for G_1 and G_2. Without loss of generality, let $f(u) = x$ and $f(v) = y$. In G, $u \equiv v$ and $\sum_{w \in N_G(u \equiv v)} f(w) = \sum_{w \in N_{G_1}(u)} f(w) + \sum_{w \in N_{G_2}(v)} f(w) = 0$. Now, for each $u_i \in N_{G_1}(u)$ and each $v_j \in N_{G_2}(v)$, we have:

$$-x = \sum_{w \neq u \in N_{G1}(u_i)} f(w) \text{ and } -y = \sum_{w \neq u \in N_{G_2}(v_j)} f(w).$$

Hence, $x = y$ and a non-trivial zero-sum weighting for G exists which uses exactly one variable say x. See Fig. 10. Therefore, G is a nut graph.■

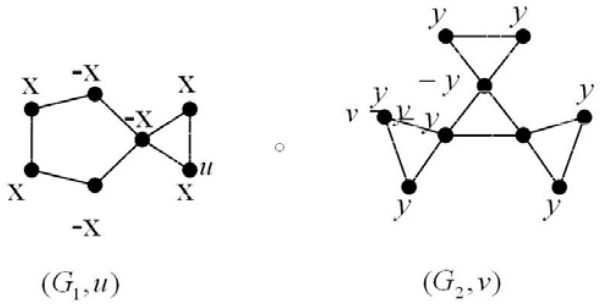

Fig. 10 The coalescence of two nut graphs

Thus, due to Lemma 5, and above basic nut graphs, given any number $n \geq 7$, there exists a nut graph of order n. Now we assert and answer the following question.

Question: Given a number n can one construct a nut graph of order n?

The answer: Due to Lemma 5, Th.2 and basic nut graphs given in Fig. 9, there exist a nut graph of order n, for each positive integer n, $n \geq 7$. Because the nut graphs of order 7, 8, 9 and 10 are given in the Fig. 9 where m=3. Nut graphs of order 11 are obtained from G_7, 12 are obtained from G_{3m}, while of order 13 and 14 are obtained from the coalescence of G_7 and G_7, G_7 and G_8, and all other orders are obtained by replacing a cycle C_{4n-1} by a cycle C_{4n+3} in the last above 4 cases.

Theorem 3: For any two rooted nut graphs (G_1, u) and (G_2, v), both generalized rooted graphs $G_1(G_2)$ and $G_2(G_1)$ are nut graphs.

Proof: The prove is an extension of that of Th.2, and hence is omitted.■

Theorem 4: For any two rooted nut graphs (G_1, u) and (G_2, v), the edge introduced graph $G = G_1 : G_2$ (obtained by introducing the edge uv) is a non-singular graph.

Proof: Weights G_1 and G_2 as in Th.2, then:

$$\sum_{w \in N_G(u)} f(w) = \sum_{w \in N_{G_1}(u)} f(w) + y = 0 \Rightarrow y = 0$$

and

$$\sum_{w \in N_G(v)} f(w) = \sum_{w \in N_{G_2}(v)} f(w) + x = 0 \Rightarrow x = 0.$$

Hence, $x = y = 0$ and there exist no non-trivial zero-sum weighting for G. Thus, G is non-singular by Th.1 ■

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint non-trivial graphs. The *Cartesian product* $G_1 \times G_2$ of the two graphs G_1 and G_2 is the graph with a vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if, and only if, $[u_1 = u_2$ and $v_1 v_2 \in E(G_2)]$ or $[u_1 u_2 \in E(G_1)$ and $v_1 = v_2]$. See [8].

In [5, Th. 3.4.4], it is proved that the Cartesian product $G = G_1 \times G_2$ of two singular graphs is a singular graph.

Hence, it follows that the Cartesian product of any two nut graphs is a singular graph. A necessary condition for the Cartesian product to be a nut graph is given by the following Theorem:

Theorem 5:If for all eigenvalues $\lambda_i, i = 1, 2, ..., p_1$ of the nut graph (G_1, u) and all eigenvalues $\mu_j, j = 1, 2, ..., p_2$ of the nut graph (G_2, v), $\lambda_i \neq -\mu_j$ except the zero eigenvalue for both, then the Cartesian product $G = G_1 \times G_2$ of the nut graphs G_1 and G_2 is a nut graph.

Proof: The proof follows from Th. 1, and [5, Lemma 3.4.3].■

This means that the Cartesian product of bipartite graph by itself is never a nut graph.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint non-trivial graphs. The *tensor product* $G_1 \otimes G_2$ of two graphs G_1 and G_2 is the graph with vertex set $V(G_1 \otimes G_2) = V_1 \times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \otimes G_2$ if, and only if, $[u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)]$. It is also called the *direct product* or *the Kronecker product*. See [8].

The next theorem is a result that follows from the definition of tensor product $G_1 \otimes G_2$ and nut graph.

Theorem 6: The tensor product $G = G_1 \otimes G_2$ of two nut graphs cannot be a nut graph, except where $G_1 = G_2 = K_1$.

Proof: Assume that G_1 and G_2 are nut graphs and $G = G_1 \otimes G_2$.

Then by [5, Th. 3.2.9], $\eta(G) = p_1\eta_2 + p_2\eta_1 - \eta_1\eta_2 = p_1 + p_2 - 1$.

If G is a nut graph, then $\eta(G) = 1$.Since, $p_1 + p_2 - 1 = 1$

$\Rightarrow \quad p_1 + p_2 = 2$, hence $p_1 = p_2 = 1$.■

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