Improved Exponential Stability Analysis for Delayed Recurrent Neural Networks

Miaomiao Yang, Shouming Zhong

Abstract—This paper studies the problem of exponential stability analysis for recurrent neural networks with time-varying delay.By establishing a suitable augmented LyapunovCKrasovskii function and a novel sufficient condition is obtained to guarantee the exponential stability of the considered system.In order to get a less conservative results of the condition,zero equalities and reciprocally convex approach are employed. The several exponential stability criterion proposed in this paper is simpler and effective. A numerical example is provided to demonstrate the feasibility and effectiveness of our results.

Keywords—Exponential stability , Neural networks, Linear matrix inequality, Lyapunov-Krasovskii, Time-varying.

I. INTRODUCTION

RECURRENT neural networks including Hopfield neural networks (HNNs) and cellular neural networks (CNNs) have been studied extensively over the recent decades [1-14] and have been widely applied within various engineering fields such as neuro- biology, population dynamics,and computing technology.Although neural networks can be implemented by very large scale integrated circuits, there inevitably exist some delays in neural networks due to the limitation of the speed of transmission and switching of signals. It is well known that time-delay is usually a cause of instability and oscillations of recurrent neural networks. Therefore, the problem of stability of recurrent neural networks with time delay is of importance in both theory and practice.

The problem of global exponential stability analysis for delay neural network has been studied by many investigators in the past years. In [6],some sufficient conditions are obtained for existence and global exponential stability of a unique equilibrium point of competitive neural networks.In [11],the authors discussed the exponential stabilization of recurrent neural networks with time-varying and distributed delays. A control law was obtained by means of linear matrix inequality.By constructing a new Lyapunov functional and using a S-procedure, both delay-dependent and delay-independent stability conditions were developed for static recurrent neural networks with interval time-varying delays in [15].

Motivated by these observations, it is of great importance to further investigate the stabilization problem of delayed neural

Miaomiao Yang and Shouming Zhong are with the School of Mathematics Science, University Electronic Science and Technology of China, Chengdu 611731, PR China. (e-mail address:miaomiaoyang1989@163.com).

Shouming Zhong is with Key Laboratory for NeuroInformation of Ministry of Education, University of Electronic Science and Technology of China, Chengdu 611731, PR China.

networks by making use of the delay interval of neurons. In this paper,our attention focuses on the exponential stabilization problem of a class of recurrent neural networks with time delay.By choosing a new Lyapunov functional which fractions delay interval and employing different free-weighting matrices in the upper bounds of integral terms to guarantee the stability of the delayed neural networks. It is shown that this obtained conditions have less conservatism. Finally, two numerical examples are given to show the usefulness of the proposed criteria.

notation:Throughout this paper, the superscripts $'-1'$ and $'T'$ stand for the inverse and transpose of a matrix, respectively;$\Re^n$ denotes an n-dimensional Euclidean space;$\Re^{m \times n}$ is the set of all $m \times n$ real matrices; $P > 0$ means that the matrix P is symmetric positive definite,$diag(\cdot, \cdot, \cdot)$ denotes a block diagonal matrix.In block symmetric matrix or long matrix expression,we use $(*)$ to represent a term that is induced by symmetry,I is an appropriately dimensional identity matrix.

II. PROBLEM STATEMENT

Consider the following neural networks with time-varying delays:

$$\dot{z}(t) = -Cz(t) + Ag(z(t)) + Bg(z(t - \tau(t))) + \mu \qquad (1)$$

$$z(t) = \phi(t), t \in [-\tau, 0] \qquad (2)$$

where $z(t) = [z_1(t), z_2(t), \cdots, z_n(t)]^T \in \Re^n$ is neuron vector $g(z(t))) = [g_1(z_1(t)), g_2(z_2(t)), \cdots, g_n(z_n(t))]^T \in \Re^n$ denotes the neuron activation function,$C = diag\{c_1, c_2, \cdots, c_n\} > 0$ $A \in \Re^{n \times n}, B \in \Re^{n \times n}$ are the connection weight matrices and the delayed connection weight matrices,respectively, $\mu = [\mu_1, \mu_2, \cdots, \mu_n]^T$ is constant input vector,and $\tau(t)$ is a continuous time-varying function which satisfies.

$$0 \leq \tau(t) \leq \tau, \dot{\tau}(t) \leq u \qquad (3)$$

where τ and u are constants.

The following assumption is made in this paper.
Assumption 1. The neuron activation functions $g_i(t)$ in (1) are bounded and satisfy

$$\gamma_i^- \leq \frac{g_i(x) - g_i(y)}{x - y} \leq \gamma_i^+, x, y \in \Re, x \neq y, i = 1, 2, \cdots, n \qquad (4)$$

Where $\gamma_i^-, \gamma_i^+ (i = 1, 2, \cdots, n)$ are positive constants.

Assumption 1 guarantees the existence of an equilibrium point of system(1) [13].Denote that$z^* = [z_1^*, z_2^*, \cdots, z_n^*]^T$ is the

equilibrium point. Using the transformation $x(\cdot) = z(\cdot) - z^*$ system (1) can be converted to the following error system:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau(t))) \quad (5)$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \Re^n$ is the neuron vector, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t))]^T \in \Re^n$ denotes the neuron activation function.
let $f_i(x(t)) = g_i(z_i(\cdot)) - g_i(z_i^*), i = 1, 2, \cdots, n$.we can get

$$\gamma_i^- \leq \frac{f_i(x_i(t))}{x_i(t)} \leq \gamma_i^+, f_i(0) = 0, i = 1, 2, \cdots, n \quad (6)$$

Definition 1.The equilibrium point of system (5) is said to be globally exponentially stable,if there exist scalars $k \geq 0$ and $\beta > 0$ such that

$$\|x(t)\| \leq \beta e^{-kt} \sup_{-\tau \leq s \leq 0} \|\phi(s) - z^*\|, \forall t > 0. \quad (7)$$

Lemma 1 [14].For any constant matrix $Z \in \Re^{n \times n}, Z = Z^T > 0$, scalars $h_2 > h_1 > 0$ such that the following integrations are well defined, then

$$(h_2 - h_1) \int_{h_2}^{h_1} x^T(s)Zx(s)ds \geq \int_{h_2}^{h_1} x^T(s)ds Z \int_{h_2}^{h_1} x(s)ds \quad (8)$$

Lemma 2 [17]. For all real vectors a, b and all matrix $Q > 0$ with appropriate dimensions,if follows that:

$$2a^T b \leq a^T Qa + b^T Q^{-1}b \quad (9)$$

Lemma 3.By (6) the following inequalities hold

$$0 \leq \int_0^{x_i(t)} [f_i(s) - \gamma_i^- s]ds \leq [f_i(x_i(t)) - \gamma_i^- x_i(t)]x_i(t) \quad (10)$$

$$0 \leq \int_0^{x_i(t)} [\gamma_i^+ s - f_i(s)]ds \leq [\gamma_i^- x_i(t) + f_i(x_i(t))]x_i(t) \quad (11)$$

Proof: Let $F_i(s) = f_i(s) - \gamma_i^- s$,we have
$$\frac{F_i(s)}{s} = \frac{f_i(s)}{s} - \gamma_i^- \geq 0$$
Therefore,
$$\int_0^{x_i(t)} [f_i(s) - \gamma_i^- s]ds \geq 0$$
By Assumption 1
$$D^+ F_i(s) = \varlimsup_{\psi \to 0^+} \frac{F_i(s + \psi) - F_i(s)}{\psi}$$
$$= \varlimsup_{\psi \to 0^+} \frac{f_i(s + \psi) - \gamma_i^-(s + \psi) - f_i(s) + \gamma_i^- s}{\psi}$$
$$= \varlimsup_{\psi \to 0^+} (\frac{f_i(s + \psi) - f_i(s)}{\psi} - \gamma_i^-) \geq 0$$

then function $F_i(s)$ is a monotone nondecreasing.we have the following inequality

$$0 \leq \int_0^{x_i(t)} [f_i(s) - \gamma_i^- s]ds \leq [f_i(x_i(t)) - \gamma_i^- x_i(t)]x_i(t)$$

(11) is similar to proof (10) and is omitted here. ∎

III. MAIN RESULTS

In this section, we propose a new exponential criterion for the neural networks with time-varying delays system. Now, we have the following main results.

Theorem 1.For given scalars $\Gamma_1 = diag(\gamma_1^-, \gamma_2^-, \cdots, \gamma_n^-)$,$\Gamma_2 = diag(\gamma_1^+, \gamma_2^+, \cdots, \gamma_n^+), u \leq 1$,the system(5) is globally exponentially stable with the exponential convergence rate index k if there exist symmetric positive definite matrices $P, Q_i(i = 1, 2, \cdots, 4), R_i(i = 1, 2, 3), S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ * & S_{22} & S_{23} \\ * & * & S_{33} \end{bmatrix}$ positive diagonal matrices $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n), \Delta = diag(\delta_1, \delta_2, \cdots, \delta_n)$, M_1, M_2,such that the following LMIs hold:

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & S_{13} & 0 \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 \\ * & * & * & e_{44} & e_{45} & 0 & 0 \\ * & * & * & * & e_{55} & e_{56} & e_{57} \\ * & * & * & * & * & e_{66} & e_{67} \\ * & * & * & * & * & * & e_{77} \end{bmatrix} \leq 0 \quad (12)$$

$$F = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & f_{15} & S_{13} & 0 \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 \\ * & * & * & f_{44} & f_{45} & f_{46} & 0 \\ * & * & * & * & f_{55} & f_{56} & e_{57} \\ * & * & * & * & * & f_{66} & e_{67} \\ * & * & * & * & * & * & e_{77} \end{bmatrix} \leq 0 \quad (13)$$

$$G = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & f_{15} & S_{13} & 0 \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 \\ * & * & * & g_{44} & 0 & g_{46} & g_{47} \\ * & * & * & * & e_{55} & e_{56} & e_{57} \\ * & * & * & * & * & e_{66} & g_{67} \\ * & * & * & * & * & * & e_{77} \end{bmatrix} \leq 0 \quad (14)$$

where

$$e_{11} = 2kP - PC - CP - 4k\Gamma_1\Lambda - C(\Gamma_2\Delta - \Gamma_1\Lambda)$$
$$+ (\frac{\tau}{3})^2 C(R_1 + R_2 + R_3)C - 2\Gamma_1 M_1 \Gamma_2 - (\Gamma_2\Delta - \Gamma_1\Lambda)C$$
$$+ \sum_{i=1}^{4} Q_i + S_{11} - e^{-\frac{2}{3}k\tau}R_3 + 4k\Gamma_2\Delta,$$

$$e_{12} = PA + 2k\Lambda - 2k\Delta - (\Lambda - \Delta)C + (\Gamma_2\Delta - \Gamma_1\Lambda)A$$
$$- (\frac{\tau}{3})^2 C(R_1 + R_2 + R_3)A + M_1(\Gamma_1 + \Gamma_2),$$

$$e_{13} = PB + (\Gamma_2\Delta - \Gamma_1\Lambda)B - (\frac{\tau}{3})^2 C(R_1 + R_2 + R_3)B,$$

$$e_{14} = 3/2e^{-\frac{2}{3}k\tau}R_3, e_{15} = S_{12},$$

$$e_{22} = (\Lambda - \Delta)A + A^T(\Lambda - \Delta) + (\frac{\tau}{3})^2 A^T(R_1 + R_2 + R_3)A$$
$$- 2M_1,$$

$$e_{23} = (\Lambda - \Delta)B + (\frac{\tau}{3})^2 A^T(R_1 + R_2 + R_3)B,$$

$$e_{33} = (\frac{\tau}{3})^2 B^T(R_1 + R_2 + R_3)B - 2M_2,$$

$$e_{34} = M_2(\Gamma_1 + \Gamma_2), e_{45} = 2e^{-\frac{2}{3}k\tau}R_3,$$

$e_{44} = -e^{-2k\tau}(1-u)Q_1 - 3e^{-\frac{2}{3}k\tau}R_3 - 2\Gamma_1 M_2 \Gamma_2,$

$e_{55} = -e^{-\frac{2}{3}k\tau}Q_2 + S_{22} - e^{-\frac{2}{3}k\tau}S_{11} - e^{-\frac{2}{3}k\tau}R_3,$

$\quad - e^{-\frac{4}{3}k\tau}R_2, e_{57} = -e^{-\frac{2}{3}k\tau}S_{13},$

$e_{56} = S_{23} - e^{-\frac{2}{3}k\tau}S_{12} + e^{-\frac{4}{3}k\tau}R_2,$

$e_{66} = -e^{-\frac{4}{3}k\tau}Q_3 + S_{33} - e^{-\frac{4}{3}k\tau}R_2,$

$\quad - e^{-2k\tau}R_1 - e^{-\frac{2}{3}k\tau}S_{22},$

$e_{67} = -e^{\frac{2}{3}k\tau}S_{23} + e^{-2k\tau}R_1,$

$e_{77} = -e^{-2k\tau}Q_4 - e^{-\frac{2}{3}k\tau}S_{33} - e^{-2k\tau}R_1,$

$f_{15} = S_{12} + e^{-\frac{2}{3}k\tau}R_3, f_{45} = e^{-\frac{4}{3}k\tau} \times \frac{3}{2}R_2,$

$f_{44} = -e^{-2k\tau}(1-u)Q_1 - 3e^{-\frac{4}{3}k\tau}2R_2$

$\quad - 2\Gamma_1 M_2 \Gamma_2, f_{46} = e^{-\frac{4}{3}k\tau} \times 2R_2,$

$f_{55} = -e^{-\frac{2}{3}k\tau}Q_2 + S_{22} - e^{-\frac{2}{3}k\tau}S_{11} - e^{-\frac{2}{3}k\tau}R_3,$

$\quad - e^{-\frac{4}{3}k\tau}R_2, f_{56} = S_{23} - e^{-\frac{2}{3}k\tau}S_{12},$

$f_{66} = -e^{-\frac{4}{3}k\tau}Q_3 + S_{33} - e^{-\frac{4}{3}k\tau}R_2,$

$\quad - e^{-2k\tau}R_1 - e^{-\frac{2}{3}k\tau}S_{22},$

$g_{46} = e^{-2k\tau} \times \frac{3}{2}R_1, g_{47} = 2e^{-2k\tau}R_1,$

$g_{44} = -e^{-2k\tau}(1-u)Q_1 - 3e^{-2k\tau}R_1$

$\quad - 2\Gamma_1 M_2 \Gamma_2$

$g_{67} = -e^{\frac{2}{3}k\tau}S_{23}$

Proof: Construct a Lyapunov-Krasovskii function as follows:

$$V(x_t) = \sum_{i=1}^{5} V_i(x_t)$$

where

$$V_1(x_t) = e^{2kt}x^T(t)Px(t)$$

$$V_2(x_t) = 2e^{2kt}\sum_{i=1}^{n}[\int_0^{x_i(t)} \lambda_i(f_i(s) - \gamma_i^- s)ds$$

$$+ \int_0^{x_i(t)} \delta_i(\gamma_i^+ s - f_i(s))ds]$$

$$V_3(x_t) = \int_{t-\tau(t)}^{t} e^{2ks}x^T(s)Q_1 x(s)ds$$

$$+ \int_{t-\frac{\tau}{3}}^{t} e^{2ks}x^T(s)Q_2 x(s)ds$$

$$+ \int_{t-\frac{2}{3}\tau}^{t} e^{2ks}x^T(s)Q_3 x(s)ds$$

$$+ \int_{t-\tau}^{t} e^{2ks}x^T(s)Q_4 x(s)ds$$

$$V_4(x_t) = \int_{t-\frac{\tau}{3}}^{t} e^{2ks} \begin{bmatrix} x(s) \\ x(s-\frac{\tau}{3}) \\ x(s-\frac{2}{3}\tau) \end{bmatrix}^T S \begin{bmatrix} x(s) \\ x(s-\frac{\tau}{3}) \\ x(s-\frac{2}{3}\tau) \end{bmatrix} ds$$

$$V_5(x_t) = \frac{\tau}{3}\int_{-\tau}^{-\frac{2}{3}\tau} \int_{t+\theta}^{t} e^{2ks}\dot{x}(s)^T R_1 \dot{x}(s)dsd\theta$$

$$+ \frac{\tau}{3}\int_{-\frac{2}{3}\tau}^{-\frac{\tau}{3}} \int_{t+\theta}^{t} e^{2ks}\dot{x}(s)^T R_2 \dot{x}(s)dsd\theta$$

$$+ \frac{\tau}{3}\int_{-\frac{\tau}{3}}^{0} \int_{t+\theta}^{t} e^{2ks}\dot{x}(s)^T R_3 \dot{x}(s)dsd\theta$$

The time derivative of $V(x_t)$ along the trajectory of system (5) is given by

$$\dot{V}(x_t) = \sum_{i=1}^{5} \dot{V}_i(x_t)$$

where

$$\dot{V}_1(x_t) = 2ke^{2kt}x^T(t)Px(t) + 2e^{2kt}x^T(t)P\dot{x}(t) \tag{15}$$

$$\dot{V}_2(x_t) = 4ke^{2kt}\sum_{i=1}^{n}[\int_0^{x_i(t)} \lambda_i(f_i(s) - \gamma_i^- s)ds$$

$$+ \int_0^{x_i(t)} \delta_i(\gamma_i^+ s - f_i(s))ds] + 2e^{2kt}[(f^T(x(t))$$

$$- x^T(t)\Gamma_1)\Lambda\dot{x}(t) + (x^T(t)\Gamma_2 - f^T(x(t)))\Delta\dot{x}(t)]$$

$$\leq 4ke^{2k\tau}[(f(x(t)) - x^T(t)\Gamma_1)\Lambda x(t) + (x^T(t)\Gamma_2$$

$$- f^T(x(t)))\Delta x(t)] + 2e^{2k\tau}[f^T(x(t))(\Lambda - \Delta)$$

$$+ x^T(t)(\Gamma_2\Delta - \Gamma_1\Lambda)]\dot{x}(t) \tag{16}$$

$$\dot{V}_3(x_t) \leq e^{2kt}x^T(t)(\sum_{i=1}^{4} Q_i)x(t)$$

$$- e^{2k(t-\tau)}(1-u)x^T(t-\tau(t))Q_1 x(t-\tau(t))$$

$$- e^{2k(t-\frac{\tau}{3})}x^T(t-\frac{\tau}{3})Q_2 x(t-\frac{\tau}{3})$$

$$- e^{2k(t-\frac{2\tau}{3})}x^T(t-\frac{2\tau}{3})Q_3 x(t-\frac{2\tau}{3})$$

$$- e^{2k(t-\tau)}x^T(t-\tau)Q_4 x(t-\tau) \tag{17}$$

$$\dot{V}_4(x_t) = e^{2kt} \begin{bmatrix} x(t) \\ x(t-\frac{\tau}{3}) \\ x(t-\frac{2}{3}\tau) \end{bmatrix}^T S \begin{bmatrix} x(t) \\ x(t-\frac{\tau}{3}) \\ x(t-\frac{2}{3}\tau) \end{bmatrix}$$

$$- e^{2k(t-\frac{\tau}{3})} \begin{bmatrix} x(t-\frac{\tau}{3}) \\ x(t-\frac{2}{3}\tau) \\ x(t-\tau) \end{bmatrix}^T S \begin{bmatrix} x(t-\frac{\tau}{3}) \\ x(t-\frac{2}{3}\tau) \\ x(t-\tau) \end{bmatrix} \tag{18}$$

$$\dot{V}_5(x_t) = (\frac{\tau}{3})^2 e^{2kt}\dot{x}^T(t)(R_1 + R_2 + R_3)\dot{x}(t)$$

$$- \frac{\tau}{3}\int_{t-\frac{\tau}{3}}^{t} e^{2ks}\dot{x}^T(s)R_3\dot{x}(s)ds$$

$$- \frac{\tau}{3}\int_{t-\frac{2}{3}\tau}^{t-\frac{\tau}{3}} e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds$$

$$- \frac{\tau}{3}\int_{t-\tau}^{t-\frac{2}{3}\tau} e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \tag{19}$$

(1) When $0 \leq \tau(t) \leq \frac{\tau}{3}$.Based on the bounds lemma of[16],we have

$$-\frac{\tau}{3}\int_{t-\frac{\tau}{3}}^{t} e^{2ks}\dot{x}^T(s)R_3\dot{x}(s)ds \leq e^{2k(t-\frac{\tau}{3})}\begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ x(t-\frac{\tau}{3}) \end{bmatrix}^T$$
$$\times \begin{bmatrix} -R_3 & 2R_3 & 0 \\ R_3 & -3R_3 & 2R_3 \\ 0 & 2R_3 & -R_3 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ x(t-\frac{\tau}{3}) \end{bmatrix} \tag{20}$$

$$-\frac{\tau}{3}\int_{t-\tau}^{t-\frac{2}{3}\tau} e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \leq -e^{2k(t-\tau)}$$
$$\times [x(t-\frac{2}{3}\tau)-x(t-\tau)]^T R_1[x(t-\frac{2}{3}\tau)-x(t-\tau)] \tag{21}$$

$$-\frac{\tau}{3}\int_{t-\frac{2}{3}\tau}^{t-\frac{\tau}{3}} e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds \leq -e^{2k(t-\frac{2}{3})\tau}$$
$$\times [x(t-\frac{\tau}{3})-x(t-\frac{2}{3}\tau)]^T R_2[x(t-\frac{\tau}{3})-x(t-\frac{2}{3}\tau)] \tag{22}$$

(2) When $\frac{\tau}{3} \leq \tau(t) \leq \frac{2}{3}\tau$.we have

$$-\frac{\tau}{3}\int_{t-\frac{2}{3}\tau}^{t-\frac{\tau}{3}} e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds \leq e^{2k(t-\frac{2}{3}\tau)}\begin{bmatrix} x(t-\frac{\tau}{3}) \\ x(t-\tau(t)) \\ x(t-\frac{2}{3}\tau) \end{bmatrix}^T$$
$$\times \begin{bmatrix} -R_2 & 2R_2 & 0 \\ R_2 & -3R_2 & 2R_2 \\ 0 & 2R_2 & -R_2 \end{bmatrix} \begin{bmatrix} x(t-\frac{\tau}{3}) \\ x(t-\tau(t)) \\ x(t-\frac{2}{3}\tau) \end{bmatrix} \tag{23}$$

$$-\frac{\tau}{3}\int_{t-\tau}^{t-\frac{2}{3}\tau} e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \leq -e^{2k(t-\tau)}$$
$$\times [x(t-\frac{2}{3}\tau)-x(t-\tau)]^T R_1[x(t-\frac{2}{3}\tau)-x(t-\tau)] \tag{24}$$

$$-\frac{\tau}{3}\int_{t-\frac{\tau}{3}}^{t} e^{2ks}\dot{x}^T(s)R_3\dot{x}(s)ds \leq -e^{2k(t-\frac{\tau}{3})}$$
$$\times [x(t)-x(t-\frac{\tau}{3})]^T R_3[x(t)-x(t-\frac{\tau}{3})] \tag{25}$$

(3) When $\frac{2}{3}\tau \leq \tau(t) \leq \tau$.Based on the bounds lemma of[16],we have

$$-\frac{\tau}{3}\int_{t-\tau}^{t-\frac{2}{3}\tau} e^{2ks}\dot{x}^T(s)R_1\dot{x}(s)ds \leq e^{2k(t-\tau)}\begin{bmatrix} x(t-\frac{2}{3}\tau) \\ x(t-\tau(t)) \\ x(t-\tau) \end{bmatrix}^T$$
$$\times \begin{bmatrix} -R_1 & 2R_1 & 0 \\ R_1 & -3R_1 & 2R_1 \\ 0 & 2R_1 & -R_1 \end{bmatrix} \begin{bmatrix} x(t-\frac{2}{3}\tau) \\ x(t-\tau(t)) \\ x(t-\tau) \end{bmatrix} \tag{26}$$

$$-\frac{\tau}{3}\int_{t-\frac{\tau}{3}}^{t} e^{2ks}\dot{x}^T(s)R_3\dot{x}(s)ds \leq -e^{2k(t-\frac{\tau}{3})}$$
$$\times [x(t)-x(t-\frac{\tau}{3}]^T R_3[x(t)-x(t-\frac{\tau}{3})] \tag{27}$$

$$-\frac{\tau}{3}\int_{t-\frac{2}{3}\tau}^{t-\frac{\tau}{3}} e^{2ks}\dot{x}^T(s)R_2\dot{x}(s)ds \leq -e^{2k(t-\frac{2}{3})\tau}$$
$$\times [x(t-\frac{\tau}{3})-x(t-\frac{2}{3}\tau)]^T R_2[x(t-\frac{\tau}{3})-x(t-\frac{2}{3}\tau)] \tag{28}$$

In order to derive less conservative results,By (6) we add the following inequalities with positive diagonal matrices M_1, M_2

$$e^{2kt}[-2f^T(x(t))M_1 f(x(t)) + 2x^T(t)M_1(\Gamma_1+\Gamma_2)f(x(t))$$
$$- 2x^T(t)\Gamma_1 M_1\Gamma_2 x(t)] \geq 0 \tag{29}$$

$$e^{2kt}[-2f^T(x(t-\tau(t)))M_2 f(x(t-\tau(t))) + 2x^T(t-\tau(t))M_2$$
$$\times (\Gamma_1+\Gamma_2)f(x(t-\tau(t)) - 2x^T(t-\tau(t))\Gamma_1 M_2\Gamma_2$$
$$\times x(t-\tau(t))] \geq 0 \tag{30}$$

(1) When $0 \leq \tau(t) \leq \frac{\tau}{3}$. According to (12),from $(15) - (22), (29) - (30)$.then one can obtain

$$\dot{V}(x_t) \leq e^{2kt}[\xi^T(t)E\xi(t)] \leq 0 \tag{31}$$

where

$$\xi^T(t) = [x^T(t), f^T(x(t)), f^T(x(t-\tau(t))), x^T(t-\tau(t)),$$
$$x^T(t-\frac{\tau}{3}), x^T(t-\frac{2}{3}\tau), x^T(t-\tau)]$$

(2) When $\frac{\tau}{3} \leq \tau(t) \leq \frac{2}{3}\tau$. According to (13),from $(15) - (19), (23) - (25), (29) - (30)$.then one can obtain

$$\dot{V}(x_t) \leq e^{2kt}[\xi^T(t)F\xi(t)] \leq 0 \tag{32}$$

(3) When $\frac{2}{3}\tau \leq \tau(t) \leq \tau$, According to (14),from $(15) - (19), (26) - (30)$.then one can obtain

$$\dot{V}(x_t) \leq e^{2kt}[\xi^T(t)G\xi(t)] \leq 0 \tag{33}$$

Therefore

$$V(x_t) \leq V(x_0)$$

On the other hand,

$$V_1(x_0) \leq \lambda_{\max}(P)\|x(0)\|^2 \leq \lambda_{\max}(P) \sup_{-\tau \leq s \leq 0} \|x(s)\|^2 \tag{34}$$

$$V_2(x_0) \leq 2[f(x(0))-\Gamma_1 x(0)]^T \Lambda x(0) + 2[\Gamma_2 x(0)$$
$$- f(x(0))]^T \Delta x(0)$$
$$\leq 2\lambda_{\max}(\Gamma_2-\Gamma_1)(\lambda_{\max}(\Lambda)+\lambda_{\max}(\Delta))$$
$$\times \sup_{-\tau \leq s \leq 0} \|x(s)\|^2 \tag{35}$$

$$V_3(x_0) \leq (\frac{\tau}{3}\lambda_{\max}(Q_1) + \frac{2\tau}{3}\lambda_{\max}(Q_2) + \tau\lambda_{\max}(Q_3)$$
$$+ \frac{\tau}{3}\lambda_{\max}(Q_4) + \frac{2\tau}{3}\lambda_{\max}(Q_5) + \tau\lambda_{\max}(Q_6))$$
$$\times \sup_{-\tau \leq s \leq 0} \|x(s)\|^2 \tag{36}$$

$$V_4(x_0) \leq \int_{-\frac{\tau}{3}}^{0} e^{2ks}\begin{bmatrix} x(s) \\ x(s-\frac{\tau}{3}) \\ x(s-\frac{2\tau}{3}) \end{bmatrix}^T S \begin{bmatrix} x(s) \\ x(s-\frac{\tau}{3}) \\ x(s-\frac{2\tau}{3}) \end{bmatrix} ds \tag{37}$$

$$\leq \int_{-\frac{\tau}{3}}^{0} [x^T(s)S_{11}x(s) + 2x^T(s)S_{12}x(s - \frac{\tau}{3})$$

$$+ 2x^T(s)S_{13}x(s - \frac{2}{3}\tau) + x^T(s - \frac{\tau}{3})S_{22}x(s - \frac{\tau}{3})$$

$$+ 2x^T(s - \frac{\tau}{3})S_{23}x(s - \frac{2}{3}\tau) + x^T(s - \frac{2}{3}\tau)S_{33}$$

$$\times x(s - \frac{2}{3}\tau)]ds \qquad (38)$$

According to Lemma3

$$V_4(x_0) \leq \frac{\tau}{3}[\lambda_{\max}(S_{11}) + \lambda_{\max}(S_{12}) + \lambda_{\max}(S_{13})]\|x(s)\|^2$$

$$+ \frac{\tau}{3}[\lambda_{\max}(S_{12}) + \lambda_{\max}(S_{13}) + \lambda_{\max}(S_{22})]$$

$$\times \|x(s - \frac{\tau}{3})\|^2 + \frac{\tau}{3}[\lambda_{\max}(S_{22}) + \lambda_{\max}(S_{23})$$

$$+ \lambda_{\max}(S_{33})]\|x(s)\|^2$$

$$\leq [\lambda_{\max}(S_{11}) + 2\lambda_{\max}(S_{12}) + 2\lambda_{\max}(S_{13})$$

$$+ \lambda_{\max}(S_{22}) + 2\lambda_{\max}(S_{23}) + \lambda_{\max}(S_{33})]$$

$$\times \sup_{-\tau \leq s \leq 0} \|x(s)\|^2 \qquad (39)$$

$$V_5(x_0) \leq \frac{\tau}{3}\lambda_{\max}(R_1) \int_{-\tau}^{-\frac{2\tau}{3}} \int_{\theta}^{0} \dot{x}^T(s)\dot{x}(s)dsd\theta$$

$$+ \frac{\tau}{3}\lambda_{\max}(R_2) \int_{-\frac{2\tau}{3}}^{-\frac{\tau}{3}} \int_{\theta}^{0} \dot{x}^T(s)\dot{x}(s)dsd\theta$$

$$+ \frac{\tau}{3}\lambda_{\max}(R_3) \int_{-\frac{\tau}{3}}^{0} \int_{\theta}^{0} \dot{x}^T(s)\dot{x}(s)dsd\theta$$

$$\leq \tau^3[\lambda_{\max}(R_1) \times \frac{5}{18} + \lambda_{\max}(R_2) \times \frac{1}{6}$$

$$+ \lambda_{\max}(R_3) \times \frac{1}{18}][\lambda_{\max}(C^TC) + \gamma^2\lambda_{\max}(A^TA)$$

$$+ \gamma^2\lambda_{\max}(B^TB)] \sup_{-\tau \leq s \leq 0} \|x(s)\|^2 \qquad (40)$$

According to $(34) - (39)$,we can get the following inequalities:

$$V(x(0)) \leq \omega \sup_{-\tau \leq s \leq 0} \|x(s)\|^2 \qquad (41)$$

where

$$\omega = \lambda_{\max}(P) + 2\lambda_{\max}(\Gamma_2 - \Gamma_1)(\lambda_{\max}(\Lambda) + \lambda_{\max}(\Delta))$$

$$+ \frac{\tau}{3}\lambda_{\max}Q_1 + \frac{2\tau}{3}\lambda_{\max}Q_2 + \tau\lambda_{\max}Q_3 + \frac{\tau}{3}\lambda_{\max}Q_4$$

$$+ \frac{2\tau}{3}\lambda_{\max}Q_5 + \tau\lambda_{\max}Q_6 + \lambda_{\max}(S_{11}) + 2\lambda_{\max}(S_{12})$$

$$+ 2\lambda_{\max}(S_{13}) + \lambda_{\max}(S_{22}) + 2\lambda_{\max}(S_{23}) + \lambda_{\max}(S_{33})$$

$$+ \tau^3[\lambda_{\max}(R_1) \times \frac{5}{18} + \lambda_{\max}(R_2) \times \frac{1}{6} + \lambda_{\max}(R_3) \times \frac{1}{18}]$$

$$\times [\lambda_{\max}(C^TC) + \gamma^2\lambda_{\max}(A^TA) + \gamma^2\lambda_{\max}(B^TB)] \qquad (42)$$

On the other hand,we have

$$V(x(t)) \geq e^{2kt}\lambda_{\min}(P)\|x(t)\|^2 \qquad (43)$$

Therefore

$$\|x(t)\| \leq \sqrt{\frac{\omega}{\lambda_{\min}(P)}} e^{-kt} \sup_{-\tau \leq s \leq 0} \|x(s)\| \qquad (44)$$

Thus,according to definition1 the system (5) is exponentially stable ,the proof is completed. ∎

Corollary 1. For given scalars $\Gamma_1 = diag(\gamma_1^-, \gamma_2^-, \cdots, \gamma_n^-)$,$\Gamma_2 = diag(\gamma_1^+, \gamma_2^+, \cdots, \gamma_n^+)$,$\tau \leq 1$,the system(5) is globally exponentially stable with the exponential convergence rate index k if there exist symmetric positive definite matrices P,$Q_i(i = 2, 3, 4)$,$R_i(i = 1, 2, 3)$,and$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ * & S_{22} & S_{23} \\ * & * & S_{33} \end{bmatrix}$ positive diagonal matrices $M_1, M_2, \Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$, $\Delta = diag(\delta_1, \delta_2, \cdots, \delta_n)$ such that the following LMIs hold:

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & S_{13} & 0 \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 \\ * & * & * & e_{44} & e_{45} & 0 & 0 \\ * & * & * & * & e_{55} & e_{56} & e_{57} \\ * & * & * & * & * & e_{66} & e_{67} \\ * & * & * & * & * & * & e_{77} \end{bmatrix} \leq 0 \qquad (45)$$

$$F = \begin{bmatrix} f_{11} & e_{12} & e_{13} & 0 & f_{15} & S_{13} & 0 \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 \\ * & * & * & f_{44} & f_{45} & f_{46} & 0 \\ * & * & * & * & f_{55} & f_{56} & f_{57} \\ * & * & * & * & * & f_{66} & e_{67} \\ * & * & * & * & * & * & e_{77} \end{bmatrix} \leq 0 \qquad (46)$$

$$G = \begin{bmatrix} f_{11} & e_{12} & e_{13} & 0 & f_{15} & S_{13} & 0 \\ * & e_{22} & e_{23} & 0 & 0 & 0 & 0 \\ * & * & e_{33} & e_{34} & 0 & 0 & 0 \\ * & * & * & g_{44} & 0 & g_{46} & g_{47} \\ * & * & * & * & e_{55} & e_{56} & e_{57} \\ * & * & * & * & * & e_{66} & g_{67} \\ * & * & * & * & * & * & e_{77} \end{bmatrix} \leq 0 \qquad (47)$$

where

$$e_{11} = 2kP - PC - CP - 4k\Gamma_1\Lambda - C(\Gamma_2\Delta - \Gamma_1\Lambda)$$

$$+ (\frac{\tau}{3})^2 C(R_1 + R_2 + R_3)C - (\Gamma_2\Delta - \Gamma_1\Lambda)C$$

$$+ \sum_{i=2}^{4} Q_i + S_{11} - e^{-\frac{2}{3}k\tau}R_3 + 4k\Gamma_2\Delta - 2\Gamma_1 M_1 \Gamma_2$$

$$e_{12} = PA + 2k\Lambda - 2k\Delta - (\Lambda - \Delta)C + (\Gamma_2\Delta - \Gamma_1\Lambda)A$$

$$- (\frac{\tau}{3})^2 C(R_1 + R_2 + R_3)A + M_1(\Gamma_1 + \Gamma_2)$$

$$e_{13} = PB + (\Gamma_2\Delta - \Gamma_1\Lambda)B - (\frac{\tau}{3})^2 C(R_1 + R_2 + R_3)B$$

$$e_{14} = 3/2e^{-\frac{2}{3}k\tau}R_3, e_{15} = S_{12}$$

$$e_{22} = (\Lambda - \Delta)A - 2M_1 + (\frac{\tau}{3})^2 A^T(R_1 + R_2 + R_3)A$$

$$+ A^T(\Lambda - \Delta), e_{44} = -3e^{-\frac{2}{3}k\tau}R_3 - 2\Gamma_1 M_2 \Gamma_2$$

$$e_{23} = (\Lambda - \Delta)B + (\frac{\tau}{3})^2 A^T(R_1 + R_2 + R_3)B$$

$$e_{33} = (\frac{\tau}{3})^2 B^T(R_1 + R_2 + R_3)B - 2M_2$$

$$e_{34} = M_2(\Gamma_1 + \Gamma_2), e_{45} = 2e^{-\frac{2}{3}k\tau}R_3$$

$$e_{55} = -e^{-\frac{2}{3}k\tau}Q_2 + S_{22} - e^{-\frac{2}{3}k\tau}S_{11} - e^{-\frac{2}{3}k\tau}R_3$$
$$\quad - e^{-\frac{4}{3}k\tau}R_2, e_{57} = -e^{-\frac{2}{3}k\tau}S_{13}$$

$$e_{56} = S_{23} - e^{-\frac{2}{3}k\tau}S_{12} + e^{-\frac{4}{3}k\tau}R_2,$$

$$e_{66} = -e^{-\frac{4}{3}k\tau}Q_3 + S_{33} - e^{-\frac{4}{3}k\tau}R_2$$
$$\quad - e^{-2k\tau}R_1 - e^{-\frac{2}{3}k\tau}S_{22}$$

$$e_{67} = -e^{\frac{2}{3}k\tau}S_{23} + e^{-2k\tau}R_1, e_{57} = -e^{-\frac{2}{3}k\tau}S_{13},$$

$$e_{77} = -e^{-2k\tau}Q_4 - e^{-\frac{2}{3}k\tau}S_{33} - e^{-2k\tau}R_1$$

$$f_{15} = S_{12} + e^{-\frac{2}{3}k\tau}R_3, f_{45} = e^{-\frac{4}{3}k\tau} \times \frac{3}{2}R_2,$$

$$f_{44} = -3e^{-\frac{4}{3}k\tau}2R_2 - 2\Gamma_1 M_2 \Gamma_2$$

$$f_{55} = -e^{-\frac{2}{3}k\tau}Q_2 + S_{22} - e^{-\frac{2}{3}k\tau}S_{11} - e^{-\frac{2}{3}k\tau}R_3$$
$$\quad - e^{-\frac{4}{3}k\tau}R_2, f_{46} = e^{-\frac{4}{3}k\tau} \times 2R_2$$

$$f_{56} = S_{23} - e^{-\frac{2}{3}k\tau}S_{12}, g_{15} = S_{12} + e^{-\frac{2}{3}k\tau}R_3$$

$$f_{66} = -e^{-\frac{4}{3}k\tau}Q_3 + S_{33} - e^{-\frac{4}{3}k\tau}R_2 - e^{-2k\tau}R_1 - e^{-\frac{2}{3}k\tau}S_{22}$$

$$g_{47} = 2e^{-2k\tau}R_1, g_{44} = -3e^{-2k\tau}R_1 - 2\Gamma_1 M_2 \Gamma_2,$$

$$g_{67} = -e^{\frac{2}{3}k\tau}S_{23}$$

Proof: Choosing $Q_1 = 0$ in Theorem 1,one can easily obtains this result. ∎

Remark 1. This paper not only divides the delay interval$[0, d]$ into $[0, d/2]$ and $[d/2, d]$,but also divides $[0, d]$ into $[0, d/3], [d/3, 2d/3]$ and $[2d/3, d]$.Each segments has a different Lyapunov matrix,which have potential to yield less conservative results.

IV. EXAMPLES

In this section,we provide the simulation of examples to illustrate the effectiveness of our method.
Example 1.Consider the system(5) with the following parameters:

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 3.5 \end{bmatrix}, A = \begin{bmatrix} -1 & 0.5 \\ 0.5 & -1 \end{bmatrix}, B = \begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

$$\Gamma_1 = diag(0, 0), \Gamma_2 = diag(1, 1)$$

First,the maximum delay bounds τ are shown under different k are list in Table I.

Second, let $u = 0$,the maximum exponential convergence rate k are shown under different τ are list in Table II

TABLE I
ALLOWABLE UPPER BOUND OF τ WITH VARIOUS k

k	0.5	1
[18]	2.59	0.97
[19]	2.82	1.18
[20]	2.90	1.32
[22]	2.94	1.35
this works	3.58	1.58

TABLE II
ALLOWABLE UPPER BOUND OF k WITH VARIOUS τ

τ	0.5	1	1.5
[18]	1.35	0.98	0.75
this works	1.45	1.17	1.01

Then,when $k = 0.25, u = 0.8$ or unknown,the maximum delay bounds τ and various u in Table III.

TABLE III
ALLOWABLE UPPER BOUND OF τ WITH VARIOUS u

u	0.8	Unknown(corollary)
[19]	2.8	1.04
[21]	2.9	1.40
[22]	3.5	2.53
this works	4.76	4.69

Example 2.Consider the system(5) with the following parameters:

$$C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}, A = \begin{bmatrix} 1.2 & -0.8 & 0.6 \\ 0.5 & -1.5 & 0.7 \\ -0.8 & -1.2 & -1.4 \end{bmatrix}$$

$$B = \begin{bmatrix} -1.4 & 0.9 & 0.5 \\ -0.6 & 1.2 & 0.8 \\ 0.5 & -0.7 & 1.1 \end{bmatrix}$$

$$\Gamma_1 = diag(-1.2, 0, -2.4), \Gamma_2 = diag(0, 1.4, 0)$$

For various τ, u,the maximum exponential convergence rate k are shown in Table IV.

TABLE IV
ALLOWABLE UPPER BOUND OF k FOR EXAMPLE 2

(τ, u)	(0.5,0)	(0.5,0.5)	(0.6,0.5)	(0.8,0.5)
Theorem 1	0.57	0.11	0.09	0.07

V. CONCLUSION

In this paper,a improved global exponential stability criterion for recurrent neural networks with time-varying delay is proposed. A suitable Lyapunov functional has been proposed to derive some less conservative delay-dependent stability criteria by using the free-weighting matrices method and the convex combination theorem. Finally, numerical examples have been given to illustrate the effectiveness of the proposed method.

REFERENCES

[1] S. Arik Daly,An analysis of exponential stability of delayed neural networks with time-varying delays. Neural Networks.17(2004)
[2] J.D.Cao,Periodic-solutions and exponential stability in delayed cellular neural networks, Phy. Rev. E60(3)(1999) 3244-3248.
[3] R.Cau,C.Lien,J.W.Hsieh,Global exponential stability for uncertain cellular neural networks with multiple time-varying delays via LMI approach,Chaos Solitons Fractals 31(4)(2007) 1258-1267.
[4] H.J.Cho,J.H.Park,Novel delay-dependent robust stability criterion of delayed cellular neural networks,Chaos Solitons Fractals 32(5)(2007) 1194-1200.
[5] K.Gu,V.L.Kharitonov,J.Chen,Stability of Time-Delay Systems,Birkhauser,Boston,2003.

[6] H.Gu,H.Jiang,Z.Teng,Existence and global exponential stability of equilibrium of competitive neural networks with different time-scales and multiple delays,J.Franklin Inst.347(2010)719-731.

[7] P.Gupta,N.K.Sinha,An improved approach for nonlinear system identification using neural networks,J.Franklin Inst.336(1999)721C734.

[8] Y.He,M.Wu,J.H.She,Delay-dependent exponential stability of delayed neural networks with time-varyingdelays, IEEE Trans. Circuits Syst.II Exp.Briefs 53 (7)(2006)553-557.

[9] Y.He,G.P.Liu,D.Rees,New delay-dependent stability criteria for neutral networks with time-varying delays, IEEE Trans. Neural Netw.18(1)(2007)310-314.

[10] C.C.Hua,C.N.Long,X.P.Guan,New results on stability analysis of neural networks with time-varying delays, Phys.Lett.A 352(2006)335-340.

[11] Phat,V.N.Trinh, H.(2010).Exponential stabilization of neural networks with various activation functions and mixed time-varying delays.IEEE Transactions on Neural Networks,21(7),1180-1184.

[12] Z.Zuo,C.Yang,Y Wang,A new method for stability analysis of recurrent neural networks with interval time-varying delay,IEEE Trans.Neural Networks 21(2)(2010)339-344.

[13] Y.Liu,Z.Wang,X.Liu,Global exponential stability of generalized recurrent neural networks with discrete and distributed delays, Neural Networks 19 (2006)667-675.

[14] K.Gu,An integral inequality in the stability problem of time delay systems,in: Proceedings of 39th IEEE Conference Decision Control, (2000) 2805-2810.

[15] X.Zhu,Y.Wang,Delay-dependent exponential stability for neural networks with time-varying delays,Phys.LETTE.A 373(2009)4066-4072.

[16] P.Park,J.W.Ko,C.Jeong,Reciprocally convex approach to stability of systems with time-varying delays,Automatica 47(1)(2011)235-238.

[17] Yuanhua D,S.M.Zhong,Exponential passivity of BAM neural networks with time-varying delays.Applied Mathematics and Computation 221(2013)727-740.

[18] J.H.Park,Further note on global exponential stability of uncertain recurrent neural networks with variable delays.Applied Mathematics and Computations 188 (2007)850-854.

[19] R.S.Gau,C.H.Lien,J.G.Hsish,Global exponential stability of uncertain cellular neural networks with multiple time-varying delays via LMI approach,Chaos,Solitons and Fractals 32 (2007)1258-1267.

[20] S.Mou,H.Gao,W.Qiang,K.Chen,New delay-dependent exponential stability for neural networks with time delays,IEEE Transactions on Systems,Man,and Cybernetics B 38 (2008)571-576.

[21] O.M.Kwon,J.H.Park,Exponential stability of uncertain cellular neural networks with discrete and distributed time-varying delays.Applied Mathematics and Computations 203 (2008)813-823.

[22] O.M.Kwon,J.H.Park,Exponential stability of uncertain neural networks with interval time-varying delays.Applied Mathematics and Computations 212 (2009)530-541.

Miaomiao Yang was born in Anhui Province, China,in 1989. She received the B.S. degree from Huaibei Normal University in 2012. She is currently pursuing the M.S.degree from University of Electronic Science and Technology of China. Her research interests include stability of neural networks, switch and delay dynamic systems.

Shouming Zhong was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the Department of Mathematics in Sun Yatsen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the Department of Mathematics in University of Waterloo,Waterloo,Canada.Heis currently as a full professor with School of Applied Mathematics, UESTC. His current research interests include differential equations,neural networks, biomathematics and robustcontrol.He has authored more than 80 papers in reputed journals such as the International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics.