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Mechanical and Thermal Stresses in Functionally Graded Cylinders

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Abstract**—**In this study, thermal elastic stress distribution occurred on long hollow cylinders made of functionally graded material (FGM) was analytically defined under thermal, mechanical and thermo mechanical loads. In closed form solutions for elastic stresses and displacements are obtained analytically by using the infinitesimal deformation theory of elasticity. It was assumed that elasticity modulus, thermal expansion coefficient and density of cylinder materials could change in terms of an exponential function as for that Poisson's ratio was constant. A gradient parameter n is chosen between - 1 and 1. When n equals to zero, the disc becomes isotropic. Circumferential, radial and longitudinal stresses in the FGMs cylinders are depicted in the figures. As a result, the gradient parameters have great effects on the stress systems of FGMs cylinders.

Keywords**—**Functionally graded materials, hollow cylinder, thermoelasticity, thermomechanical load.

I. Introduction

ONE way to design hollow cylindrical structures is to suitably vary mechanical, thermal, and physical properties as a function of position within the materials. Materials with continuously varying properties are usually called functionally graded materials (FGMs). Advantages of these materials over laminated composites include the elimination of interfaces between different layers by this means avoiding points of high stress concentration.

Analytical solutions for functionally graded incompressible eccentric and non-axisymmetrically loaded circular cylinders are given in [1]. Elastoplastic stress analysis of functionally graded disc subjected mechanical, thermal and thermomechanical loads have been studied in [2], and obtained elastic, elastoplastic and residual stress distribution along the radius of disc. A functionally graded circular hollow cylinder under arbitrarily non-uniform loads on the inner and outer surfaces has been analyzed in [3], and for the solution by dividing the cylinder into some homogeneous subdomain cylinders, the analytical solutions for the stresses and displacements are derived explicitly. A technique has presented in [4], to tailor materials for functionally graded (FG) linear elastic hollow cylinders and spheres to attain through-the-thickness either a constant circumferential stress or a constant in-plane shear stress. They have found that the required radial variation of the volume fractions of constituents to make a linear combination of the radial and the hoop stresses uniform throughout the thickness. Nie and Batra [5] has studied a plane-strain problem when the cylinder material is polar-orthotropic, material properties vary exponentially in the radial direction, and deformations are independent of the axial coordinate, and the cylinder with elliptical inner and circular outer surfaces. They have solved by expanding the radial and the circumferential displacements in Fourier series in the angular coordinate, and Frobenius series was used to solve ordinary differential equations for coefficients of the Fourier series. Elastic–plastic deformation of a solid functionally graded cylinder with fixed ends, and with uniform internal heat generation has investigated in [6], based on Tresca's yield criterion and its associated flow rule, considering of the material properties to vary radially according to a parabolic form. They obtained that the stress distribution, as well as the development of plastic region radii, is influenced substantially by the material non-homogeneity. Analytical solutions for thick-walled cylinders subjected to internal and external pressure in which the entire wall is made of functionally graded material or only a thin functionally graded coating has presented in [7]. A unified formulation of finite cylindrical prism methods (FCPMs), based on the Reissner mixed variational theorem (RMVT), has developed in [8], for the three-dimensional (3D) analysis of multilayered FGMs circular hollow cylinders.

In this paper, hollow cylinders with non-uniform material properties subjected to thermal, mechanical and thermomechanical loads are solved using the infinitesimal deformation theory of elasticity. Results show that the material property gradation index and loads effects on stresses and displacement.

II. Elastic Solutions

For a plane-strain problem, equations of equilibrium in cylindrical coordinates (r,θ), in the absence of body forces, are

$$\begin{aligned} &\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+R=0 \\ &\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{2\tau_{r\theta}}{r}+R=0 \\ &r_i\le r\le r_o \end{aligned} \tag{1}$$

Because of the symmetry, the value of shear stress, $\tau_{r\theta}$, is equal to zero. For a rotating cylinders, stress components are independent of θ, thus equilibrium (1) is reduced to

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$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}+\rho(r)\omega^2 r=0 \tag{2}$$

where $\rho(r)$ is the density of the material and it is assumed to vary radially according to power law function, ω is the angular velocity, the strain-displacement relations are given by

$$\varepsilon_r=\frac{du}{dr},\quad \varepsilon_\theta=\frac{u}{r} \tag{3}$$

where u is the displacement component in the radial direction. The strain compatibility equation is

$$\varepsilon_r=\frac{d}{dr}\left(r\,\varepsilon_\theta\right) \tag{4}$$

using Hooke's law for plane strain case, the strains are given by

$$\begin{aligned}\varepsilon_r&=\frac{1+\nu}{E(r)}\left[(1-\nu)\sigma_r-\nu\sigma_\theta+E(r)\alpha(r)T(r)\right]\\ \varepsilon_\theta&=\frac{1+\nu}{E(r)}\left[(1-\nu)\sigma_\theta-\nu\sigma_r+E(r)\alpha(r)T(r)\right]\end{aligned} \tag{5}$$

The equation equilibrium (2) is satisfied by the stress function F defined as

$$\sigma_r=\frac{F}{r},\qquad \sigma_\theta=\frac{dF}{dr}+\rho\omega^2 r^2 \tag{6}$$

Equation (6) is substituted into (5) then the results are substituted into the compatibility (4), finally differential equation of the stress function is found.

$$\begin{aligned}&r^2\frac{d^2F}{dr^2}+r\left(1-r\frac{E'(r)}{E(r)}\right)\frac{dF}{dr}+\left(\nu r\frac{E'(r)}{E(r)}-1\right)F=\\ &\rho(r)\omega^2 r^3\left(r\frac{E'(r)}{E(r)}-r\frac{\rho'(r)}{\rho(r)}-3-\frac{\nu}{1-\nu}\right)\\ &-(1-\nu)E(r)r^2\left[\alpha'(r)T(r)+\alpha(r)T'(r)\right]\end{aligned} \tag{7}$$

Now suppose that

$$E(r)=E_0\left(\frac{r}{r_o}\right)^n,\alpha(r)=\alpha_0\left(\frac{r}{r_o}\right)^\beta,\rho(r)=\rho_o\left(\frac{r}{r_o}\right)^\gamma \tag{8}$$

here n, β and γ are gradient parameters, and E_0, α_0 and ρ_o are nominal modulus of elasticity, thermal expansion coefficient and density of the material, respectively. If gradient parameters are zero, materials will be homogeneous. Boundary conditions are

$$\sigma_r=0,\ \ r=r_i \text{ and } \sigma_r=0,\ r=r_o$$

A. Solution for Thermal Loading

In this section FGMs cylinders subject to uniformly thermal loads, the results are

$$\begin{aligned}\sigma_r&=C_1 r^{\frac{n+k-2}{2}}+C_2 r^{\frac{n-k-2}{2}}+Br^{(\beta+n)}\\ \sigma_\theta&=\left(\frac{n+k}{2}\right)C_1 r^{\frac{n+k-2}{2}}+\left(\frac{n-k}{2}\right)C_2 r^{\frac{n-k-2}{2}}\\ &+(\beta+n+1)Br^{(\beta+n)}\end{aligned} \tag{9}$$

and the term B and the positive constant k are

$$\begin{aligned}B&=-\frac{(1-\nu)E_o\beta\alpha_o T_o}{r_o^{(\beta+n)}\left((\beta+n+1)^2-n(\beta+n+1)+\nu n-1\right)}\\ k&=\sqrt{n^2-4n\nu+4}\end{aligned} \tag{10}$$

By using boundary conditions, $C_1$and C_2 are determined as

$$\begin{aligned}C_1&=\frac{B\left(-r_i^{(\beta+n)}r_o^{\frac{n-k-2}{2}}+r_o^{(\beta+n)}r_i^{\frac{n-k-2}{2}}\right)}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}\\ C_2&=\frac{B\left(-r_o^{(\beta+n)}r_i^{\frac{n+k-2}{2}}+r_i^{(\beta+n)}r_o^{\frac{n+k-2}{2}}\right)}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}\end{aligned} \tag{11}$$

B. Solution for Mechanical Loading

The FGMs cylinders subject to angular velocity, the results are

$$\begin{aligned}\sigma_r&=C_1 r^{\frac{n+k-2}{2}}+C_2 r^{\frac{n-k-2}{2}}+Ar^{(2+\gamma)}\\ \sigma_\theta&=\left(\frac{n+k}{2}\right)C_1 r^{\frac{n+k-2}{2}}+\left(\frac{n-k}{2}\right)C_2 r^{\frac{n-k-2}{2}}\\ &+(3+\gamma)Ar^{(2+\gamma)}+\rho(r)\omega^2 r^2\end{aligned} \tag{12}$$

and the term A

$$A=\frac{\rho_o\omega^2\left(n-\gamma-3-\frac{\nu}{1-\nu}\right)}{r_o^{\gamma}\left((3+\gamma)^2-n(3+\gamma)+\nu n-1\right)} \tag{13}$$

By using boundary conditions, $C_1$and C_2 are determined as

$$C_1=\frac{A\left(-r_i^{(\gamma+2)}r_o^{\frac{n-k-2}{2}}+r_o^{(\gamma+2)}r_i^{\frac{n-k-2}{2}}\right)}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}$$
$$C_2=\frac{A\left(-r_o^{(\gamma+2)}r_i^{\frac{n+k-2}{2}}+r_i^{(\gamma+2)}r_o^{\frac{n+k-2}{2}}\right)}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}} \tag{14}$$

C. Solution for Thermomechanical Loading

The FGMs cylinders subject to uniformly temperature loads and angular velocity, the results are

$$\begin{aligned}\sigma_r&=C_1r^{\frac{n+k-2}{2}}+C_2r^{\frac{n-k-2}{2}}+Hr^{(2+\gamma)}+Lr^{(n+\beta)}\\ \sigma_\theta&=\left(\frac{n+k}{2}\right)C_1r^{\frac{n+k-2}{2}}+\left(\frac{n-k}{2}\right)C_2r^{\frac{n-k-2}{2}}\\ &+(3+\gamma)Hr^{(2+\gamma)}+(n+\beta+1)Lr^{(n+\beta)}+\rho(r)\omega^2r^2\end{aligned} \tag{15}$$

and the term H and L are

$$\begin{aligned}H&=\frac{\rho_o\omega^2\left(n-\gamma-3-\frac{\nu}{1-\nu}\right)}{r_o^{\gamma}\left((3+\gamma)^2-n(3+\gamma)+\nu n-1\right)}\\ L&=-\frac{(1-\nu)E_o\beta\alpha_oT_o}{r_o^{(\beta+n)}\left((\beta+n+1)^2-n(\beta+n+1)+\nu n-1\right)}\end{aligned} \tag{16}$$

By using boundary conditions, C_1 and C_2 are determined as

$$\begin{aligned}C_1&=\frac{\left(-Hr_i^{(\gamma+2)}-Lr_i^{(n+\beta)}\right)r_o^{\frac{n-k-2}{2}}}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}\\ &+\frac{\left(Hr_o^{(\gamma+2)}+Lr_o^{(n+\beta)}\right)r_i^{\frac{n-k-2}{2}}}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}\\ C_2&=\frac{\left(-Hr_o^{(\gamma+2)}-Lr_o^{(n+\beta)}\right)r_i^{\frac{n-k-2}{2}}}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}\\ &+\frac{\left(Hr_i^{(\gamma+2)}+Lr_i^{(n+\beta)}\right)r_o^{\frac{n-k-2}{2}}}{r_i^{\frac{n+k-2}{2}}r_o^{\frac{n-k-2}{2}}-r_o^{\frac{n+k-2}{2}}r_i^{\frac{n-k-2}{2}}}\end{aligned} \tag{17}$$

III. Results and Discussion

In this study, elastic stress analysis are carried out on a hollow cylinders made of functionally graded materials (FGM) by using an analytical solution. Inner and outer radii of cylinder are r_i*=40mm* and r_o*=100mm* with plane strain assumption, respectively. Mechanical properties of the cylinders such as elasticity modulus, density and thermal expansion coefficient suppose that according to (8), and nominal mechanical properties are given in Table I.

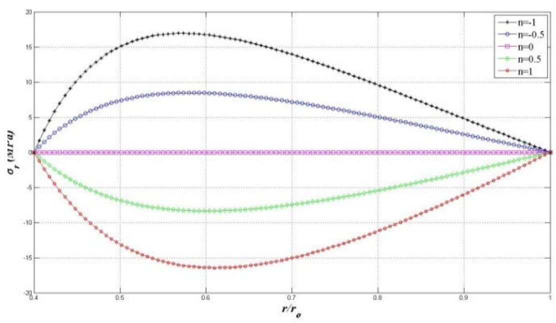
TABLE I
Temperature Independent Mechanical Properties of the Cylinder

E_o (GPa)	ρ_o (kg/m^3)	α_o (1/°C)
200	7850	12x10-6

A. Results for Thermal Loading

In Fig. 1, the variations of radial, circumferential stresses, longitudinal stresses and equivalent stresses along the radial distance of cylinder subjected to uniformly temperature load, T_o=100°C, for different gradient parameters are presented. In Fig. 1 (a), Radial stress is compressive along the radius for positive gradient parameter, and tensile for negative gradient parameter. For gradient parameter is equal to zero both radial and circumferential stresses are equal to zero because of cylinder's materials becomes homogenous material.

The circumferential stresses decrease gradually from inner to outer surface for negative gradient parameters, on the contrary for positive gradient parameters, as seen in Fig. 1 (b). They take maximum values at inner surface as tensile and minimum values at outer surface as compressive for negative gradient parameters, whereas for positive gradient parameters that it is opposite to negative parameters. The equivalent stresses are similar to circumferential stresses distribution along to radius.



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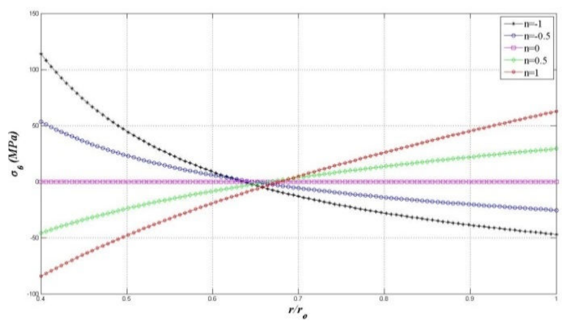
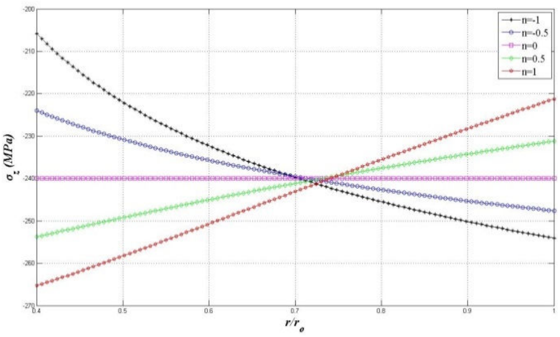


(b)



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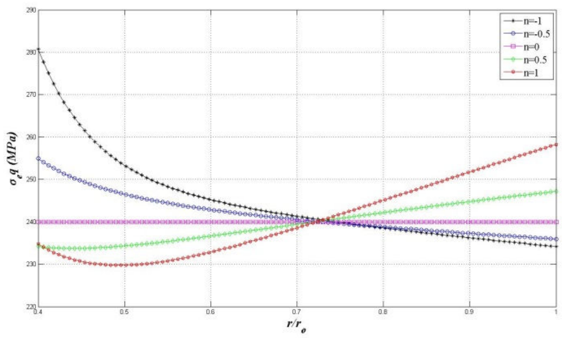
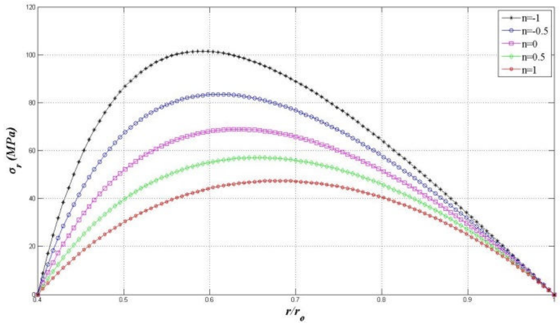


(d)

Fig. 1 Variations of the radial stresses (a), circumferential stresses (b), longitudinal stresses (c) and equivalent stresses along the radial distance of cylinder subjected to uniformly temperature load, T_o=100°C, for different gradient parameters

B. Results for Mechanical Loading

In this section, the FGMs hollow cylinders subject to the inertia force due to the rotating cylinder. In Fig. 2, the variations of radial, circumferential, longitudinal and equivalent stresses for FGM cylinder subjected to an inertia force due to rotating of disc, $\omega=75$ *rad/s*, are presented. It is clear from this Fig. 2 (a) that, the radial stress is tensile along the radius, and it is the highest value for minimum gradient parameter, the lowest value for maximum gradient parameter, and for different FGM cylinders this stresses occur in between. The circumferential stresses decrease gradually from inner to outer surface by means of tensile, as seen in Fig. 2 (b), and it is the highest value at inner surface for minimum gradient parameter. The equivalent and longitudinal stresses are similar to circumferential stresses distribution along to radius. Fig. 2 suggests that the stresses are the lowest value, and homogenous stress distribution along to radius for minimum gradient parameter.



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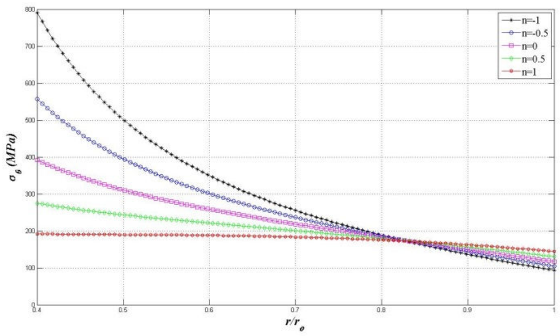

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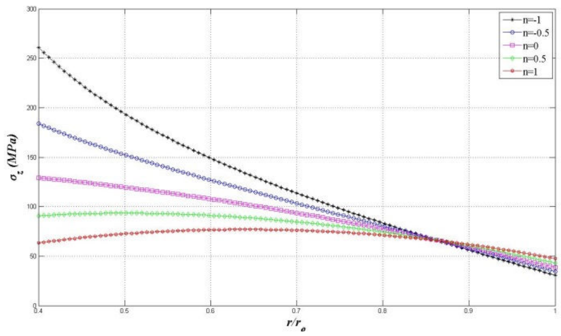

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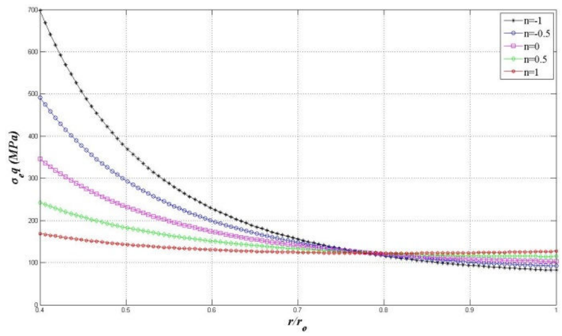
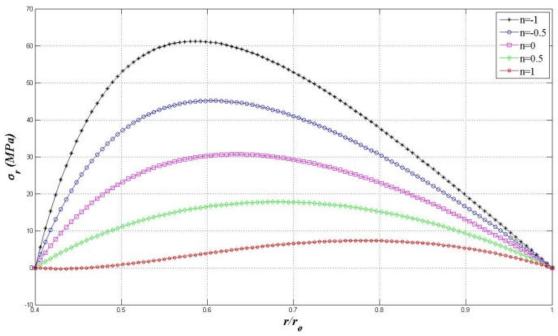


(d)

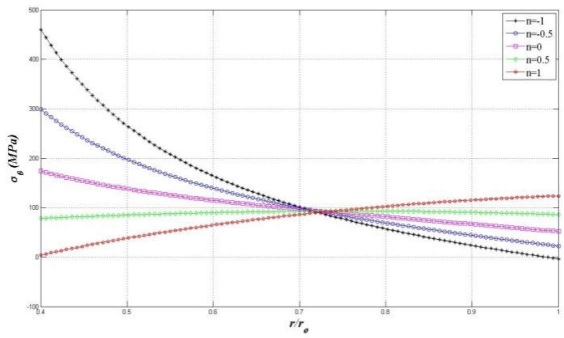
Fig. 2 Variations of the radial stresses (a), circumferential stresses (b), longitudinal stresses (c) and equivalent stresses along the radial distance of cylinder subjected to inertia force due to rotating, $\omega=75rad/sn$, for different gradient parameters

C. Results for Thermomechanical Loading

The FGMs cylinders expose to both inertia force due to the rotating and uniform temperature loads. The radial stresses decrease gradually when gradient parameters increase, in Fig. 3 (a). The maximum radial stress occurs for minimum gradient parameter, it looks like thermal loading radial stresses distribution along to radius, but smaller than thermal loading stresses, and the radial stress is tensile along the radius for all gradient parameters. As it can be seen Figs. 3 (b)-(d) that the stresses are the most homogeneous distribution along to radius for *n=0.5*. Fig. 3 (c) shows that longitudinal stresses are the maximum value at inner surface for the minimum gradient parameter by means of compressive.



(a)



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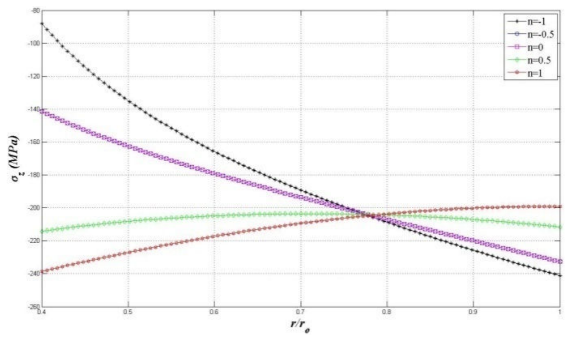

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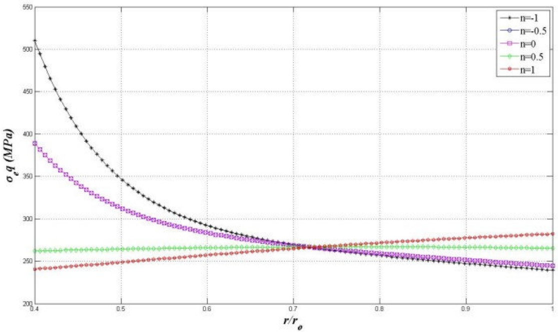

(d)

Fig. 3 Variations of the radial stresses (a), circumferential stresses (b), longitudinal stresses (c) and equivalent stresses along the radial distance of cylinder subjected to inertia force due to rotating, ω=75 rad/sn, and uniformly temperature load, To=100^{o}C, for different gradient parameters

IV. Conclusions

In this study, stress analysis of functionally graded cylinders subjected to thermal, mechanical and thermomechanical loads are performed in solutions. The following conclusions can be derived from this study:

- Gradient parameters and loads play an important role distribution of stresses along the radius of the disc.
- The circumferential stresses take maximum values at inner surface as tensile and minimum values at outer surface as compressive for negative gradient parameters, whereas for positive gradient parameters that it is opposite to negative parameters for thermal loading. However the stresses take the maximum value at inner surface by means of tensile for mechanical loading.
- The circumferential, longitudinal and equivalent stresses are the most homogeneous distribution along to radius for gradient parameter equals to *0.5,* and for mechanical and thermomechanical loads.

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