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Solving SPDEs by a Least Squares Method

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Abstract—We present in this paper a useful strategy to solve stochastic partial differential equations (SPDEs) involving stochastic coefficients. Using the Wick-product of higher order and the Wiener-Itô chaos expansion, the SPDEs is reformulated as a large system of deterministic partial differential equations. To reduce the computational complexity of this system, we shall use a decomposition-coordination method. To obtain the chaos coefficients in the corresponding deterministic equations, we use a least square formulation. Once this approximation is performed, the statistics of the numerical solution can be easily evaluated.

Keywords—Least squares, Wick product, SPDEs, finite element, Wiener chaos expansion, gradient method.

I. Introduction

The purpose of this paper is to give a finite element approximation of a linear stochastic partial differential equation (SPDE) under the framework of the white noise analysis. We study the following problem: Find $u(x,\omega)$ solution of the linear SPDE

$$\begin{cases} -\nabla\cdot(\kappa(x,\omega)\nabla u(x,\omega)=f(x,\omega) & \text{in } \mathcal{D}\times\Omega \\ u(x,\omega)=0 & \text{on } \partial\mathcal{D}\times\Omega \end{cases} \tag{1}$$

where u is a scalar field, f is the stochastic source term, κ is the diffusion of the medium, Ω is the set of random events, $\mathcal{D}\subset\mathbb{R}^d$ ($d=1,2$ or 3) an open bounded domain with a smooth boundary $\partial\mathcal{D}$.

Equation (1), is a linearized model for the evolution of a scalar field in a random medium. It arises in several physical and mathematical problems like:

- Flow in a porous medium where $u(x,\omega)$ denotes the pressure, κ is the permeability of the medium, f represents the external forces (for example sources or sinks in an oil-reservoir). We allow κ, f to be stochastic processes.
  The pressure equation was introduced in [3] as a stochastic model for single-phase flow in an isotropic porous medium.
- Stochastic heat equation with white noise potential and where $u(x,\omega)$ denotes the temperature, κ is the conductivity of the medium, f is the source term [5].

We will study a formulation of (1) based on the same family of spaces as in [6]. We shall reformulate this stochastic problem as an infinite set of deterministic variational problems, using the properties of the Wick product. Each of these variational problems will give one of the coefficients in the Wiener-Itô chaos expansion of the solution of (1). The method we shall use is based on the ideas of Fourier analysis on Wiener space. In fact, Wiener Chaos expansion represents a stochastic function $u(x,\omega)$x as a Fourier series with respect to an orthonormal basis $\mathcal{H}_\alpha$, i.e., $u(x,\omega)=\sum_{\alpha\in\mathcal{I}}u_\alpha(x)\mathcal{H}_\alpha(\omega)$ where $\mathcal{I}$ denotes the set of multi -indices $\alpha=(\alpha_j)$ where all $\alpha_j\in\mathbb{N}$ and only finitely many $\alpha_j\neq 0$, the u_α's are deterministic coefficients and the $\mathcal{H}_\alpha$'s are the stochastic variables $\mathcal{H}_\alpha(\omega)=\Pi_{j=1}^{\infty}h_{\alpha_j}(\langle\omega,\eta_j(x)\rangle)$, $\omega\in\mathcal{S}'(\mathbb{R}^d)$ where h_n denotes the Hermite polynomial and the family $\{\eta_j\}_{j=1}^{\infty}\subset\mathcal{S}(\mathbb{R}^d)$ forms an orthonormal basis for $L^2(\mathbb{R}^d)$. This decomposition separates the deterministic effects (described by the coefficients u_α) from the randomness (that is covered by the base $\mathcal{H}_\alpha$). The orthogonality of $\mathcal{H}_\alpha$ and the properties of the Wick product enable us to reduce SPDEs like (1) to a system of uncoupled deterministic equations for the coefficients $u_\alpha(x)$. Standard deterministic numerical methods can be applied to solve it sufficiently accurately. The main statistics, such as mean, covariance and higher order statistical moments can be calculated by simple formulas involving only these deterministic coefficients. Moreover, in the procedure de scribed above, there is no randomness directly involved in the simulations. One does not have to deal with the selection of random number generators, and there is no need to solve the SPDE equations realization by realization. Instead, uncoupled coefficient equations are solved once and for all. Moreover, one can reconstruct particular realizations of the solution directly from Wiener chaos expansions once the coefficients are available.

An outline of the paper is as follows. In Section II we review notation and introduce some white noise spaces. In Section III, using a least squares approach and a gradient method, we split the equation (1) into a cascade of stochastic partial differential equations of Wick type. Finally, in Section IV we give a finite element approximation of our problem.

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II. Elements of white noise analysis

A. White noise space

Let $\mathbb{R}^d$ the set of spatial parameters equipped with the Lebesgue measure. We shall construct a Wiener process indexed by R^d, i.e. a Gaussian white noise and describe the associated Hilbert space of quadratic integrable random variables w.r.t. this process.

Let $\mathcal{S}=\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of smooth, rapidly decreasing functions on $\mathbb{R}^d$, and let $\mathcal{S}'=\mathcal{S}'(\mathbb{R}^d)$ be the dual space of tempered distributions. By the Bochner-Minlos theorem, cf. [4], there exists a unique probability measure μ, called the white noise porbability measure, on the Borel σ-algebra on $\mathcal{S}'$ with characteristic functional

$$C(\eta)=E[e^{i\langle\cdot,\eta\rangle}]:=\int_{\mathcal{S}'}e^{i\langle\omega,\eta\rangle}d\mu(\omega)=e^{-\frac{1}{2}\|\eta\|^2_{L^2(\mathbb{R}^d)}} \tag{2}$$

The random variable $\langle \cdot, \eta \rangle_{\mathcal{S}'}$ defined on the probability space $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ thus follows a Gaussian distribution with mean zero and variance $\|\eta\|^2_{L^2(\mathbb{R}^d)}$, and can be interpreted as the stochastic integral w.r.t a Brownian sheet $B_{t,x}$ defined on $\mathbb{R}^d$, i.e. $\langle \omega, \eta \rangle_{\mathcal{S}'} = \int_{\mathbb{R}^d} \eta(x) dB_{t,x}(\omega),\ \omega \in \mathcal{S}', \eta \in \mathcal{S}$.

B. Chaos decomposition

A chaos decomposition is an orthonormal expansion in the Hilbert space $L^2(\mathcal{S}')$ of quadratic integrable functions defined on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$. For $n \in \mathbb{N}_0, x \in \mathbb{R}$ define the Hermite polynomial $h_n(x) = (-1)^n e^{x^2/2} \dfrac{d^n}{dx^n}(e^{-x^2/2})$, and for $n \in \mathbb{N}$ define the Hermite functions $\xi_n(x) = \pi^{-1/4}((n-1)!)^{-1/2} e^{-x^2/2} h_{n-1}(\sqrt{2}x)$.. It is well-known that $\xi_n \in \mathcal{S}(\mathbb{R})$, $\|\xi_n\|_\infty \le 1$ $(n \in \mathbb{N})$, and that $\{\xi_n : n \in \mathbb{N}\}$ constitutes an orthonormal basis in $L^2(\mathbb{R}, dx)$. We let $\{\eta_j\}_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ denote the orthonormal basis for $L^2(\mathbb{R}^d, dx)$ constructed by taking tensor-products of Hermite functions [4]:

$$\eta_j(x) = \xi_{\delta_1^{(j)}}(x_1) \xi_{\delta_1^{(j)}}(x_2) \cdots \xi_{\delta_d^{(j)}}(x_d)\ ,\ j = 1, 2, \cdots$$

where $\delta^{(j)} = (\delta_1^{(j)}, \delta_2^{(j)}, \cdots, \delta_d^{(j)})$ is the jth multi-index number in some fixed ordering of all d-dimensional multi-indices $\delta = (\delta_1, \cdots, \delta_d)$.

Let $\mathcal{I}$ denote the set of all matrix multi-indices $\alpha = (\alpha_j)$ with $\alpha_j \in \mathbb{N}_0$ $(j \in \mathbb{N})$ with finite length $l(\alpha) = \max\{j; \alpha_j \neq 0\}$, and as usual we define $\alpha + \beta = (\alpha_j + \beta_j)$, $\alpha! = \prod_j \alpha_j!$, and $|\alpha| := \sum_j \alpha_j$. For each $\alpha \in \mathcal{I}$ we define the stochastic variable

$$H_\alpha(\omega) := \prod_{j=1}^{l(\alpha)} h_{\alpha_j}(\langle \omega, \eta_j \rangle),\ \omega \in \mathcal{S}' \tag{3}$$

The family $\{H_\alpha : \alpha \in \mathcal{I}\}$ constitutes an orthogonal basis for $L^2(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ and it holds $E[H_\alpha H_\beta] = \alpha! \delta_{\alpha\beta}$ [4]. Thus, any f in $L^2(\mu) := L^2(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \mu)$ has a unique representation

$$f = \sum_{\alpha \in \mathcal{I}} f_\alpha H_\alpha \tag{4}$$

where $f_\alpha \in \mathcal{R}$ and $\|f\|^2_{L^2(\mu)} = \sum_{\alpha \in \mathcal{I}} c_\alpha^2 \alpha!$. The expansion in (4) is often referred to as the Wiener-Itô chaos expansion. We will in the following adopt the notation f_α to denote the αth chaos coefficient of a random variable f.

Next, we introduce a family of stochastic Banach spaces needed for variational problems. This type of spaces are often used for the Hilbert space treatment of SPDEs of Wick type, other references include [1]–[7]–[8]–[9].

C. Stochastic Sobolev spaces

For m a non-negative integer, let $H^m(\mathcal{D})$ denote the usual Sobolev space of order m defined on $\mathcal{D}$. The norm in the space $H^m(\mathcal{D})$ is defined as $\|v\|_{H^m(\mathcal{D})} = \left(\sum_{|\beta| = m} \|D^\beta v\|^2_{L^2(\mathcal{D})} \right)^{\frac{1}{2}}$.

For $p \in [1, \infty[$, let $L^p(\mathcal{D})$ denote the usual Banach space of order p defined on $\mathcal{D}$. The norm in the space $L^p(\mathcal{D})$ is defined as $\|v\|_{L^p(\mathcal{D})} = \left(\int_{\mathcal{D}} |\ v(x)\ |^p\ dx \right)^{\frac{1}{p}}$.

Let $C_0^\infty(\mathcal{D})$ denote the space of infinitely differentiable functions having compact support and let $H_0^1(\mathcal{D})$ denote the closure of $C_0^\infty(\mathcal{D})$ in $H^1(\mathcal{D})$. We denote the dual of $H_0^1(\mathcal{D})$ by $H^{-1}(\mathcal{D})$.

We shall use the notation $(2\mathbb{N})^\alpha := \prod_{k=1}^{\infty} (2k)^{\alpha_k}$

We have the following result [12]:

Lemma 1: We have that

$$\sum_{\alpha \in \mathcal{I}} (2\mathbb{N})^{-p\alpha} < \infty$$

if and only if $p > 1$.

Definition 1: Let $p \in [1, \infty[$, $-1 \le \rho \le 1$ and $k \in \mathbb{R}$ and let V be a Banach space. We define the stochastic Banach spaces $(\mathcal{S})^{\rho,k,V}$ as the set of all formal sums

$$(\mathcal{S})^{\rho,k,V} := \left\{ v = \sum_{\alpha \in \mathcal{I}} v_\alpha H_\alpha\ :\ v_\alpha \in V \quad \text{and} \quad \|v\|_{\rho,k,V} < \infty \right\}$$

where $\| \cdot \|_{\rho,k,V}$ denote the norm

$$\|u\|_{\rho,k,V} := \left(\sum_{\alpha \in \mathcal{I}} (\alpha!)^{1+\rho} \|u_\alpha\|_V^2\ (2\mathbb{N})^{k\alpha} \right)^{\frac{1}{2}}$$

Theorem 1: If V is a separable Hilbert space, then the space $\mathcal{S}^{\rho,k,V}$ with the inner product

$$(u, v)_{\rho,k,V} : \sum_{\alpha \in \mathcal{I}} (u_\alpha, v_\alpha)_V (\alpha!)^{1+\rho} (2\mathbb{N})^{k\alpha}$$

is a separable Hilbert space. If $k' \le k$ then $\mathcal{S}^{\rho,k,V} \hookrightarrow \mathcal{S}^{\rho,k',V}$. Finally, If H is a Hilbert space such that $V \hookrightarrow H$, then $\mathcal{S}^{\rho,k,V} \hookrightarrow \mathcal{S}^{\rho,k,H}$.

D. Wiener chaos expansion of a log normal process

In this work we focus on equation (1) with random coefficient κ wich satisfy the following condition : there exist two positive constants $C_1, C_2 > 0$ such that

$$0 < C_1 \le \kappa(x, \omega) \le C_2 < \infty,\ \text{a.e and a.s}$$

Using the Lax-Milgarm theorem it can be shown, that there exists a unique solution of (1) in the stochastic Sobolev space $\mathcal{S}^{-1,0,H_0^1(D)} \approx H_0^1(D) \otimes L^2(\mu)$.

We assume also that κ has the following Karhunen-Loeve (K-L) expansion

$$\kappa(x, \omega) = \sum_{k=0}^{\infty} \sqrt{\lambda_k} \beta_k(\omega) \phi_k(x) \tag{5}$$

where $\beta_k(\omega)$ are the uncorrelated zero mean and unite variance random variables, (λ_i, ϕ_i) is the pair of eigenvalues and eigenfunctions of the covariance function. Since each $\beta_i \in L^2(\mu)$, it my be expanded in its Wiener chaos exansion

$$\beta_k(\omega) = \sum_{\alpha \in \mathcal{I}} \beta_{k,\alpha} H_\alpha(\omega) \tag{6}$$

By substituting (7) in (6) we obtain

$$\begin{aligned}\kappa(x,\omega) &= \sum_{\alpha\in\mathcal{I}}\left(\sum_{k=0}^{\infty}\sqrt{\lambda}_k\beta_{k,\alpha}\phi_k(x)\right)H_\alpha(\omega) \\ &:= \sum_{\alpha\in\mathcal{I}}\kappa_\alpha(x)H_\alpha(\omega)\end{aligned} \tag{7}$$

E. Ordinary and Wick products

Definition 2: The Wick product $f\diamond g$ of two formal series $f=\sum_\alpha f_\alpha H_\alpha$, $g=\sum_\alpha g_\alpha H_\alpha$ is defined as

$$f\diamond g := \sum_{\alpha,\beta\in\mathcal{I}} f_\alpha g_\beta H_{\alpha+\beta}. \tag{8}$$

From [11] we have the following result:

Theorem 2: Suppose $u=\sum_{\alpha\in\mathcal{I}}u_\alpha H_\alpha, v=\sum_{\alpha\in\mathcal{I}}v_\beta H_\beta$. If $E_\mu \mid uv\mid^2)<\infty$, then the product uv has the Wiener chaos expansion

$$uv=\sum_{\theta\in\mathcal{I}}\left(\sum_{p\in\mathcal{I}}\sum_{0\le\beta\le\theta}C(\theta,\beta,p)u_{\theta-\beta+p}v_{\beta+p}\right)H_\theta$$

where

$$C(\theta,\beta,p)=\frac{(\theta-\beta+p)!(\beta+p)!}{\beta!p!(\theta-\beta)!}$$

Since

$$\begin{aligned}u\diamond v &= \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in\mathcal{I}}u_\alpha v_\beta H_{\alpha+\beta} \\ &= \sum_{\theta\in\mathcal{I}}\sum_{0\le\beta\le\theta}u_{\theta-\beta}v_\beta H_\theta\end{aligned}$$

we have

Theorem 3:

$$uv=u\diamond v+\sum_{\theta\in\mathcal{I}}\left(\sum_{p\in\mathcal{I},p\neq0}\sum_{0\le\beta\le\alpha}C(\theta,\beta,p)u_{\theta-\beta+p}v_{\beta+p}\right)H_\theta$$

III. Least squares formulation

For any $k\in\mathbb{N}$ we introduce the Hilbert spaces $X=\mathcal{S}^{-1,-k,H_0^1(\mathcal{D})}$, $X'=\mathcal{S}^{-1,-k,H^{-1}(\mathcal{D})}$, $Y=\mathcal{S}^{-1,-k,L^2(\mathcal{D})}$.

Using Theorem 3, we can now formulate problem (1) as : Find $u(x,\omega)\in X$ solution of the linear SPDE

$$\begin{cases}-\nabla\cdot(\kappa(x,\omega)\diamond\nabla u(x,\omega)=f(x,\omega)+T(u(x,\omega)) \\ \\ u(x,\omega)=0\end{cases} \tag{9}$$

where we have put

$$T(u(x,\omega))=\sum_{\theta\in\mathcal{I}}\left(\sum_{p\in\mathcal{I},p\neq0}\sum_{0\le\beta\le\alpha}C(\theta,\beta,p)\nabla\cdot(\kappa_{\theta-\beta+p}\nabla u_{\beta+p})\right)$$

Solving problem (9) is a non-trivial task for the following reasons

- numerical approximation of (9) results in solving large and dense system of equations
- strong coupling structure related to the chaos coefficients of the solution

We shall split problem (9) into the solution of a cascade of SPDEs of Wick type using a least square approach. The later systems can be approximated efficiently because Wick SPDEs when discretized have a lower-triangular system structure.

Let $v\in X$. To v we associate the solution $y=y(v)\in X$ of

$$\begin{cases}-\nabla\cdot(\kappa\diamond\nabla y(v))=f+\nabla\cdot(\kappa\diamond\nabla v)+T(v) \\ \\ y(v)=0\end{cases} \tag{10}$$

Using standard results on Wick SPDEs theory [6]–[7]–[10], we have the following result:

Theorem 4: Let $k\in\mathbb{N}$ such that $\kappa\in\mathcal{S}^{-1,-(k+1),L^\infty(\mathcal{D})}$ and $f\in\mathcal{S}^{-1,-k,L^2(\mathcal{D})}$. Then there exists a unique solution $u\in\mathcal{S}^{-1,-k,H_0^1(\mathcal{D})}$ of (10).

Suppose now that v is a solution of (9); the corresponding y obtained from (10) is clearly $y=0$. As a consequence, we introduce the following least-squares formulation of (9): find $u\in X$ such that

$$J(u)\le J(v),\ \forall v\in X \tag{11}$$

where the functional $J:X\longrightarrow\mathbb{R}$ is defined by

$$J(v)=\frac{1}{2}\left(\kappa\diamond\nabla y(v),\nabla y(v)\right)_{-1,-k,L^2(D)}$$

where $y(v)$ is defined from v by (10). To solve (11) we shall use a conjugate gradient algorithme:

- Step 0: Initialisation
$$u^0 given$$
Then compute g^0 from
$$-\nabla\cdot(\kappa\diamond\nabla g^0)=J'(u^0) \tag{12}$$
and set $z^0=g^0$ by: Then for $n\ge0$, assuming u^n,g^n,z^n known, compute u^{n+1},g^{n+1},z^{n+1}
- Step 1: Descent
$$u^{n+1}=u^n-\lambda_n z^n$$
where λ_n is the solution of the one-dimensional minimisation problem
$$\lambda_n\in\mathbb{R},\ J(u^n-\lambda_n z^n)\le J(u^n-\lambda z^n)\,\forall\lambda\in\mathbb{R}$$
- Step 2: Construction of the new descent direction
Define g^{n+1} by
$$-\nabla(\kappa\diamond\nabla g^{n+1})=J'(u^{n+1}) \tag{13}$$
Then
$$\gamma_n=(\kappa\diamond\nabla g^{n+1},\nabla(g^{n+1}-g^n))/(\kappa\diamond\nabla g^n,\nabla g^n)$$
$$z^{n+1}=g^{n+1}+\gamma_n z^n$$
$$n=n+1 \text{ and go to Step 1}$$

For the convergence of this algorithm see [2].

The calculation of g^{n+1} from u^{n+1} requires the solution of two linear Wick equations (10) with $v=u^{n+1}$ and (13).

Since the operator T is linear, the calculation of $J'(v)$ is straightforward. Let $w\in X$, then

$$\langle J'(v),w\rangle_{X',X}=\lim_{t\longrightarrow0}\frac{J(v+tw)-J(v)}{t}$$

For $v\in X$ and $w\in X$ we have

$$-\nabla\cdot(\kappa\diamond\nabla y(v+tw))=f+\nabla\cdot(\kappa\diamond\nabla(v+tw))+T(v+tw) \tag{14}$$

and

$$-\nabla\cdot(\kappa\diamond\nabla y(v))=f+\nabla\cdot(\kappa\diamond\nabla(v))+T(v) \tag{15}$$

and we have $y(v+tw)=y(v)+t\delta$ where δ is the solution of

$$-\nabla\cdot(\kappa\diamond\nabla\delta)=T(w) \tag{16}$$

Hence we have

$$\begin{aligned} J(v+tw)-J(v) &= \tfrac{1}{2}\left(\kappa\diamond\nabla(y(v)+t\delta),\nabla(y(v)+t\delta)\right)\\ &-\tfrac{1}{2}\left(\kappa\diamond\nabla(y(v)),\nabla(y(v))\right)\\ &=\tfrac{t}{2}((\kappa\diamond\nabla y,\nabla\delta)+(\kappa\diamond\nabla\delta,\nabla y))\\ &+\tfrac{t^2}{2}(\kappa\diamond\nabla\delta,\nabla\delta)\end{aligned}$$

We obtain

$$\begin{aligned}\langle J'(v),w\rangle &= \tfrac{1}{2}((\kappa\diamond\nabla y,\nabla\delta)+(\kappa\diamond\nabla\delta,\nabla y))\\ &=\tfrac{1}{2}((\kappa\diamond\nabla y,\nabla\delta)+\tfrac{1}{2}(T(w),y)\end{aligned} \tag{17}$$

IV. The finite element approximation

A. The model problem

At each iteration of the conjugate gradient algorithm, we have to solve many Wick SPDEs such as

$$-\nabla\cdot(\kappa\diamond\nabla g)=F \tag{18}$$

where F is known from the previous iteration of the algorithm. Let us now introduce a discrete version of this model problem using the finite element method. Let $\{X_h\}_{h>0}$ be a family of finite dimensional vector spaces such that $X_h\subset X$. Let M_h a finite element sub-space of $H_0^1(\mathcal{D})$. For $N,K\in\mathbb{N}$, we define the subset

$$\mathcal{I}_{N,K}=\{0\}\cup\bigcup_{n=1}^{N}\bigcup_{k=1}^{K}\left\{\alpha\in\mathbb{N}_0^k:\quad |\alpha|=n\quad\text{and}\quad\alpha_k\neq 0\right\}\subset\mathcal{I}$$

Next, for each $h\in]0,1[$ and $N,K\in\mathbb{N}$, we define the finite-dimensional space

$$X_h:=\{v=\sum_{\alpha\in\mathcal{I}_{N,K}}v_\alpha H_\alpha\in X:\ v_\alpha\in M_h,\ \alpha\in\mathcal{I}_{N,K}\}$$

We approximate (18) as follows: seek $g_h\in X_h$ such that:

$$(\kappa\diamond\nabla g_h,\nabla v_h)_Y=(F,v_h)_Y,\quad\forall v_h\in X_h \tag{19}$$

Thus, if g_h solves the problem (19), then the chaos coefficients of the solution $g_{h,\gamma}:\gamma\in\mathcal{I}_{N,K}$ must solve the following cascade of variational problems: for each $\gamma\in\mathcal{I}_{N,K}$, find $g_{h,\gamma}\in M_h$ such that

$$A_0\left(g_{h,\gamma},w_h\right)=(F,w_h)_{L^2(\mathcal{D})}-\sum_{\alpha<\gamma}A_{\gamma-\alpha}\left(g_{h,\alpha},w_h\right) \tag{20}$$

where

$$A_\beta(g_{h,\alpha},w_h)=\left(\kappa_\beta\nabla g_{h,\alpha},\nabla w_h\right)_{L^2(\mathcal{D})}$$

We assume that the set $\mathcal{I}_{N,K}$ is ordered in such a way that $\{g_{h,\alpha},\ \alpha\prec\gamma\}$ has been calculated when the γ-th equation in (13) is considered. This enable us to solve (18) as a sequence of $(N+K)!/(N!K!)$ problems. Moreover, since the matrix associated to the left hand side of (13) is the same for all problems which leads to a considerable reduction in the required work-load. For the implementation of the method described above [6]–[8].

Once we have calculated the chaos coefficients $\{(g_{h,\gamma}):\gamma\in\mathcal{I}_{N,K}\}$ using (20), we may do stochastic simulations of the solution as follows: first, generate M independent standard Gaussian variables $X(\omega)=(X_i(\omega))$ $(i=1,\dots,M)$ using some random number generator, and then form the sums

$$g_h(x,\omega):=\sum_{\alpha\in\mathcal{I}_{N,K}}g_{h,\alpha}(x)H_\alpha(X(\omega)) \tag{21}$$

where $H_\alpha(X(\omega)):=\prod_{j=1}^{M}h_{\alpha_j}(X_j(\omega))$

The advantage of this approach is that it enables us to generate random samples easy and fast. For example, in situations where one is interested in repeated simulations of g, one may compute the chaos coefficients in advance, store them, and produce the simulations whenever they are needed. The resulting stochastic variable can be viewed as the best approximation of the stochastic object $g(x,\omega)$ we can achieve by including only the chaos-coefficients corresponding to multi-indices in $\mathcal{I}_{N,K}$.

V. Conclusion

We presented in this paper a new method to solve stochastic partial differential equations with random data. This method is based on a decomposition-coordination approach by using the least squares methods. We have shown that the computational cost of the original equation can be drastically reduced using a gradient method. This iterative method decompose the original equation by solving only linear deterministic partial differential equations of Wick type.

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