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Continuous Adaptive Robust Control for Nonlinear Uncertain Systems

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***Abstract*—**We consider nonlinear uncertain systems such that a priori information of the uncertainties is not available. For such systems, we assume that the upper bound of the uncertainties is represented as a Fredholm integral equation of the first kind and we propose an adaptation law that is capable of estimating the upper bound and design a continuous robust control which renders nonlinear uncertain systems ultimately bounded.

***Keywords*—**Adaptive Control, Estimation, Fredholm Integral, Uncertain System.

I. Introduction

MUCH attention has been paid to the problem of designing feedback controllers for uncertain dynamic systems containing uncertain elements due to model-parameter uncertainty, extraneous disturbance and measurement error. To design feedback controllers of such systems, if a priori probabilistic information of the uncertainties is unavailable but bounds on the uncertainties are known, then a deterministic approach is possible.

Many researchers [1]-[4] within the deterministic framework have been designed feedback controls based on Lyapunov min-max approach. Roughly speaking, a Lyapunov function of a stable nominal system in the absence of the uncertainties is employed as the Lyapunov function candidate for the actual uncertain dynamic system, and a control law is determined such that the Lyapunov function decreases along every possible trajectory of the uncertain dynamic system at least outside a neighborhood of the zero state. Thus, uniform ultimate boundedness is obtained for all possible uncertainties.

In the deterministic designs, the uncertainties are bounded and their upper bounds are available to the designer. And the upper bounds of the uncertainties are very important to guarantee the asymptotic stability or uniform ultimate boundedness of uncertain dynamic systems. However, sometimes they may not be easily obtained because of the complexity of the structure of uncertainties. A parameter adaptation method provides a good tool to solve this problem.

Recently, several authors [5]-[8] have proposed new control laws with parameter adaptation for upper bounds of the uncertainties. For estimating upper bounds, Chen [5] and Yoo and Chung [6] assumed that the matched uncertainty is cone-bounded about the state vector norm. And Brogliato and Trofino Neto [7] and Wu [8] assumed that upper bounds of the uncertainties linearly depend on some unknown constant parameters. In these studies, the structure of the uncertainties is at least partially known. However, since in many cases it may be difficult to get information of the structure of the uncertainties. Integral equations [9] provide attractive mathematical basis to describe upper bounds of the uncertainties without a priori information except that the uncertainties are bounded. In an interesting paper, assuming that a nonlinear disturbance function is represented as an integral equation, Messner et al. [10] proposed an adaptive learning rule for a class of nonlinear systems.

In this paper, unlike in Brogliato and Trofino Neto [7] and Wu [8], we consider the problem of robust stabilization of uncertain dynamic systems such that any information of the uncertainties is not available except the fact that they are bounded. For such uncertain dynamic systems, we assume that upper bounds of the uncertainties are represented as a Fredholm integral equation of the first kind, *i.e.*, an integral of the product of a predefine kernel with an unknown influence function, and we provide a sufficient condition for the existence of such a representation. Based on such an assumption, we introduce a class of continuous adaptive robust controllers which can guarantee the uniform boundedness of the resulting closed-loop dynamic systems in the presence of the uncertainties.

II. Continuous Adaptive Robust Control

Consider a class of uncertain nonlinear systems, as used by Corless and Leitman [2] such that

$$\begin{aligned} \dot{x}(t) &= f(x,t) + B(x,t)(u(t) + e(x,t)) \\ x(t_0) &= x_0 \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control input, and $e(x,t)$ represents the lumped uncertainty. The known functions $f(\cdot): R^n \times R \to R^n$ and $B(\cdot): R^n \times R \to R^{n\times m}$ as well as the unknown function $e(\cdot): R^n \times R \to R^m$ are Caratheodory ones, and $f(0, t) = 0$ for all $t \in R$. Also, the origin, $x = 0$, is uniformly asymptotically stable for the unforced nominal system $\dot{x}(t) = f(x,t)$. That is, there is a continuous and positive definite Lyapunov function $V(\cdot): R^n \times R \to R_+$ such that for all $(x,t) \in R^n \times R$

$$\begin{aligned} \gamma_1(\|x\|) \le V(x,t) \le \gamma_2(\|x\|) \\ \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) f(x,t) \le -\gamma_3(\|x\|) \end{aligned} \tag{2}$$

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where the scalar functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ are of Class K_∞, and $\gamma_3(\cdot)$ of Class *K*.

Based only on the knowledge of the upper bound of the uncertainty, the following assumption is presumed to be valid:

Assumption 1: There exists a continuous function $\rho(\cdot): R \to R_+$ such that $\|e(t,x)\| \le \rho(t)$ for all $(x,t) \in R^n \times R$.

In the design of a class of feedback controls, finding of the continuous function $\rho(\cdot)$ satisfying Assumption 1 is very important for the asymptotic stability or uniform ultimate boundedness for uncertain dynamic systems. To manipulate the upper bound of the uncertainties easily, Chen [5] and Yoo and Chung [6] assumed that $\rho(\cdot)$ is cone-bounded. And Brogliato and Trofino Neto [7] and Wu [8] assumed that it linearly depends on some unknown constant parameters.

Integral equations are equations in which an unknown function appears under an integral sign. As Arfken and Weber [9], a Fredholm integral equation of the first kind is defined as follows:

$$g(x) = \int_a^b \psi(x,t)\varphi(t)dt$$

where $\varphi(\cdot)$ is an unknown function, $g(\cdot)$ is a known function, and $\psi(\cdot,\cdot)$ is another known function of two variables, often called the kernel function. The limits of integration, that is, *a* and *b* are constant.

In our problems, the upper bound $\rho(\cdot)$ may have dynamic characteristic, and if $\rho(\cdot)$ is known, it may be helpful to formulate the uncertainties using integral equations. Thus, we will state the following assumption using a Fredholm integral equation of the first kind.

Assumption 2: The function $\rho(\cdot)$ can be represented as a Fredholm integral equation of the first kind, *i.e.*, there exist a predefined kernel function $\psi(\cdot): R \times R \to R^l$ and an unknown influence function $c(\cdot): R \to R^l$ such that, for real constants $\tau_1, \tau_2 \in R$,

$$\rho(t) = \int_{\tau_1}^{\tau_2} \psi^T(t,\tau)\, c(\tau) d\tau \,. \tag{3}$$

In general, the purpose of such an integral equation is to determine the unknown function $c(\cdot)$ for known functions $\rho(\cdot)$ and $\psi(\cdot,\cdot)$. In reverse, given a predefined kernel function $\psi(\cdot,\cdot)$, if we can estimate the unknown function $c(\cdot)$ using an appropriate method, we can also estimate $\rho(\cdot)$. Moreover, assumption 2 allows the designers to choose any arbitrary kernel function regardless of the structure of the uncertainties. That is, even though the designers do not have any information of the uncertainties except that the uncertainties are bounded (*i.e.* assumption 1), they can describe the upper bound of uncertainties using a designer-chosen kernel function and an unknown influence function.

Based on the assumptions, we now propose the following adaptation law for estimating the upper bound of the uncertainties:

$$\frac{\partial}{\partial t}\bar{c}(t,\tau) = \|\mu(x,t)\| \Phi \psi(t,\tau) - \Phi\Gamma\bar{c}(t,\tau) \tag{4}$$

where $\mu(x,t) = B^T(x,t)\nabla_x V(x,t)$ and $\bar{c}(\cdot): R \times R \to R^l$ is the estimate of the unknown influence function $c(\cdot)$ in (3) and $\Phi \in R^{l\times l}$ and $\Gamma \in R^{l\times l}$ are positive definite diagonal matrices. Then $\bar{\rho}(t)$, the function estimate of $\rho(t)$, is defined by

$$\bar{\rho}(t) = \int_{\tau_1}^{\tau_2} \psi^T(t,\tau)\bar{c}(t,\tau)d\tau \tag{5}$$

Defining the influence function error by $\tilde{c}(t,\tau) = \bar{c}(t,\tau) - c(\tau)$ and using (3), (4) and (5), we obtain

$$\frac{\partial}{\partial t}\tilde{c}(t,\tau) = \|\mu(x,t)\| \Phi \psi(t,\tau) - \Phi\Gamma\bar{c}(t,\tau) \tag{6}$$

and

$$\tilde{\rho}(t) = \int_{\tau_1}^{\tau_2} \psi^T(t,\tau)\tilde{c}(t,\tau)d\tau = \bar{\rho}(t) - \rho(t)\,. \tag{7}$$

Now, consider the class of feedback controls described by

$$u(t) = p(x,t) = \begin{cases} -\dfrac{\mu(x,t)\bar{\rho}(t)}{\|\mu(x,t)\|\bar{\rho}(t)}\bar{\rho}(t) & \text{if } \|\mu(x,t)\|\bar{\rho}(t) \ge \varepsilon \\ -\dfrac{\mu(x,t)\bar{\rho}(t)}{\varepsilon}\bar{\rho}(t) & \text{if } \|\mu(x,t)\|\bar{\rho}(t) < \varepsilon \end{cases} \tag{8}$$

where ε is a positive constant.

Applying (8) to (1) yields a closed-loop dynamic system of the form

$$\begin{aligned} \dot{x}(t) &= f(x,t) + B(x,t)(p(x,t) + e(x,t)) \\ x(t_0) &= x_0 \end{aligned} \tag{9}$$

With regard to the uniform ultimate boundedness of uncertain dynamic systems, we may state the following theorem.

Theorem 1: Given system (1), if Assumptions 1 and 2 are valid, and if in addition the following condition is satisfied:

$$\sup_{t\in[t_0\ \infty]} \int_{\tau_1}^{\tau_2} \|\psi(t,\tau)\|^2 d\tau = \kappa < \infty \tag{10}$$

where κ is a positive constant and τ_1 and τ_2 are real constants, $x(t)$ and $\tilde{\rho}(t)$ are uniformly ultimately bounded by employing the continuous robust control (8) and the continuous adaptation law (4) and (5).

Proof: First, we introduce an augmented Lyapunov function such that

$$V_a(x,\tilde{c}) = V(x,t) + \tfrac{1}{2}\int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Phi^{-1}\tilde{c}(t,\tau)\,d\tau \tag{11}$$

where $V(x,t)$ is the Lyapunov function for the unforced nominal system and Φ is defined in (4). Then, taking the derivative of $V_a(x,\tilde{c})$ along the trajectories of the closed-loop system in (9), we obtain

$$\begin{aligned}
\dot{V}_a(x,\tilde{c}) &= \frac{\partial V(x,t)}{\partial t} + \nabla_x^T V(x,t) f(x,t) \\
&\quad + \nabla_x^T V(x,t) B(x,t)(p(x,t) \\
&\quad + e(x,t)) + \int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Phi^{-1}\frac{\partial}{\partial t}\tilde{c}(t,\tau)\,d\tau \\
&\le -\gamma_3(\|x\|) + \mu^T(x,t)p(x,t) + \|\mu(x,t)\|\rho(t) \\
&\quad + \|\mu(x,t)\|\int_{\tau_1}^{\tau_2} \psi^T(t,\tau)\tilde{c}(t,\tau)\,d\tau \\
&\quad - \int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\bar{c}(t,\tau)\,d\tau \\
&= -\gamma_3(\|x\|) + \mu^T(x,t)p(x,t) + \|\mu(x,t)\|\rho(t) \\
&\quad + \|\mu(x,t)\|\tilde{\rho}(t) - \int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\bar{c}(t,\tau)\,d\tau
\end{aligned} \tag{12}$$

If $\|\mu(x,t)\|\bar{\rho}(t) \ge \varepsilon$, (12) can be rewritten as

$$\dot{V}_a(x,\tilde{c}) \le -\gamma_3(\|x\|) - \int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\bar{c}(t,\tau)\,d\tau \tag{13}$$

and if $\|\mu(x,t)\|\bar{\rho}(t) < \varepsilon$, (12) can be rewritten as

$$\begin{aligned}
\dot{V}_a(x,\tilde{c}) \le -\gamma_3(\|x\|) - \|\mu(x,t)\|^2\bar{\rho}^2(t)/\varepsilon + \|\mu(x,t)\|\bar{\rho}(t) \\
- \int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\bar{c}(t,\tau)\,d\tau
\end{aligned} \tag{14}$$

when $\|\mu(x,t)\|\bar{\rho}(t) = \varepsilon/2$, the term

$$[-\|\mu(x,t)\|^2\bar{\rho}^2(t)/\varepsilon + \|\mu(x,t)\|\bar{\rho}(t)]$$

reaches the maximum value of $\varepsilon/2$. Thus we may write as follows:

$$\dot{V}_a(x,\tilde{c}) \le -\gamma_3(\|x\|) - \int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\bar{c}(t,\tau)\,d\tau + \varepsilon/2 \tag{15}$$

On the other hand, by matrix manipulation we have

$$\begin{aligned}
-\int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\bar{c}(t,\tau)\,d\tau &= -\tfrac{1}{2}\int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\tilde{c}(t,\tau)\,d\tau \\
&\quad - \tfrac{1}{2}\int_{\tau_1}^{\tau_2} (\tilde{c}(t,\tau)+c(\tau))^T\Gamma(\tilde{c}(t,\tau)+c(\tau))\,d\tau \\
&\quad + \tfrac{1}{2}\int_{\tau_1}^{\tau_2} c^T(\tau)\Gamma c(\tau)\,d\tau \\
&\le -\tfrac{1}{2}\int_{\tau_1}^{\tau_2} \tilde{c}^T(t,\tau)\Gamma\tilde{c}(t,\tau)\,d\tau + \tfrac{1}{2}\int_{\tau_1}^{\tau_2} c^T(\tau)\Gamma c(\tau)\,d\tau
\end{aligned} \tag{16}$$

Consequently, from results of (15) and (16), we obtain

$$\begin{aligned}
\dot{V}_a(x,\tilde{c}) &\le -\gamma_3(\|x\|) - \tfrac{1}{2}\lambda_{\min}(\Gamma)\int_{\tau_1}^{\tau_2}\|\tilde{c}(t,\tau)\|^2\,d\tau + \varepsilon_1 \\
&= -\gamma_3(\|z\|) + \varepsilon_1
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
z &= [x^T \ \ \omega]^T, \quad \omega(t) = \int_{\tau_1}^{\tau_2}\|\tilde{c}(t,\tau)\|^2\,d\tau \\
\gamma_4(\|z\|) &= \gamma_3(\|x\|) + \tfrac{1}{2}\lambda_{\min}(\Gamma)\omega(t) \\
\varepsilon_1 &= \tfrac{1}{2}\int_{\tau_1}^{\tau_2} c^T(\tau)\Gamma c(\tau)\,d\tau + \varepsilon/2
\end{aligned}$$

and $\lambda_{\min}(\cdot)$ is the smallest eigenvalue of a given matrix. Using the result and terminology in [2], $x(t)$ and $\int_{\tau_1}^{\tau_2}\tilde{c}^T(t,\tau)\tilde{c}(t,\tau)\,d\tau$ are uniformly ultimately bounded.

We will now show that $\tilde{\rho}(t)$ is uniformly ultimately bounded. Taking absolute value of (7) and using the Schwarz inequality, we obtain

$$\begin{aligned}
|\tilde{\rho}(t)| &= \left|\int_{\tau_1}^{\tau_2}\psi^T(t,\tau)\tilde{c}(t,\tau)\,d\tau\right| \\
&\le \left[\int_{\tau_1}^{\tau_2}\|\psi(t,\tau)\|^2\,d\tau\right]^{1/2}\cdot\left[\int_{\tau_1}^{\tau_2}\|\tilde{c}(t,\tau)\|^2\,d\tau\right]^{1/2} \\
&\le \sqrt{\kappa}\cdot\left[\int_{\tau_1}^{\tau_2}\|\tilde{c}(t,\tau)\|^2\,d\tau\right]^{1/2} < \infty
\end{aligned} \tag{18}$$

From (18), since $\int_{\tau_1}^{\tau_2}\tilde{c}^T(t,\tau)\tilde{c}(t,\tau)\,d\tau$ is uniformly ultimately bounded, $\tilde{\rho}(t)$ is also uniformly ultimately bounded.

Remark 1: The constraint on the predefined kernel function given by (10) is the sufficient condition that guarantees that $\tilde{\rho}(t)$ is bound. There exist many kernel functions that satisfy the constraint (10). An example of such a kernel function is the cosine function used in the illustrative example.

III. An Illustrative Example

In order to illustrate the effectiveness of the proposed method, we consider a pendulum with mass center *G*, mass *m*, moment of inertia *I*, and gravitational acceleration *g* as shown in Brogliato and Trofino Neto [7] and Wu [8].

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{mgl}{I}\cos(x_1(t)) + \frac{\tau(t)}{I}
\end{aligned} \tag{19}$$

where $x_1(t) \equiv q(t)$ is the angular position. As in Brogliato and Trofino Neto [7], letting

$$\begin{aligned}
\tilde{x}_1(t) &= x_1(t) - x_{1d}(t) \\
\tilde{x}_2(t) &= x_2(t) - x_{2d}(t) \\
y(t) &= \tilde{x}_2(t) + \lambda\tilde{x}_1(t), \qquad \lambda > 0
\end{aligned} \tag{20}$$

where the subscript '*d*' denotes the desired value, we can choose a control input of the form

$$\tau(t) = a_0(\dot{x}_{2d}(t) - \lambda\tilde{x}_2(t)) + b_0\cos(x_1(t)) - ky(t) + u(t) + d(t) \quad (21)$$

where a_0, b_0 are positive constants, k is strictly positive, $u(t)$ is an auxiliary control input, and $d(t)$ is an exogenous time-varying bounded input disturbance. Then, we obtain an error system of the form

$$\dot{y}(t) = -\frac{k}{I}y(t) + \frac{1}{I}(u(t) + e(y,t)) \quad (22)$$

where

$$e(y,t) = (a_0 - I)(\dot{x}_{2d} - \lambda\tilde{x}_2) + (b_0 - mgl)\cos(x_1) + d(t) \quad (23)$$

If $V(y,t) = \frac{1}{2}Iy^2$ is chosen, we can obtain

$$\mu(y,t) = B^T(y,t)\nabla_y V(y,t) = y(t) = \tilde{x}_2(t) + \lambda\tilde{x}_1(t) \quad (24)$$

Hence, the adaptive robust control and the adaptation law are described as follows:

$$u(t) = p(y,t) = \begin{cases} -\dfrac{y(t)\bar{\rho}^2(t)}{|y(t)|\bar{\rho}(t)} & \text{if } |y(t)|\bar{\rho}(t) \geq \varepsilon \\ -\dfrac{y(t)\bar{\rho}^2(t)}{\varepsilon} & \text{if } |y(t)|\bar{\rho}(t) < \varepsilon \end{cases} \quad (25)$$

$$\frac{\partial}{\partial t}\bar{c}(t,\tau) = |y(t)|\Phi\psi(t,\tau) - \Phi\Gamma\bar{c}(t,\tau) \quad (26)$$

We choose the same numerical values as those given by Brogliato and Trofino Neto [7] as in Table I.

TABLE I
PARAMETERS FOR SIMULATIONS

Parameters	Description	Values[Unit]
m	mass	10[kg]
l	length	0.5[m]
I	moment of inertia	0.2[kgm^2]
g	gravitational acceleration	9.81[m/s^2]

The disturbance $d(t) = 2\sin(15t)$, the desired angle trajectory $x_{1d}(t) \equiv q_d(t) = \cos(2t)$, and the control parameters $\lambda = 5$, $k = 10$, $a_0 = 0.5$ and $b_0 = 50$ are chosen.In addition, for the controller in (25) and the adaptation law in (26), we also choose

$$\varepsilon = 0.2,\ \Phi = diag[50\ 100],\ \Gamma = diag[0.1\ 0.1]$$
$$\psi(t,\tau) = [1\ \ 1 + \cos(t-\tau)]^T,\ \tau_1 = 0,\ \tau_2 = 0.5.$$

The results of the simulation are shown in Figs. 1 and 2. As seen in figures, the system described in (22) is indeed uniformly ultimately bounded in the presence of the uncertainties.

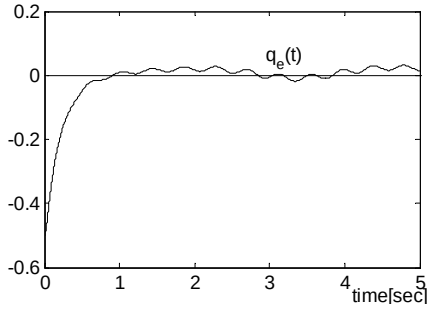

Fig. 1 Position tracking error

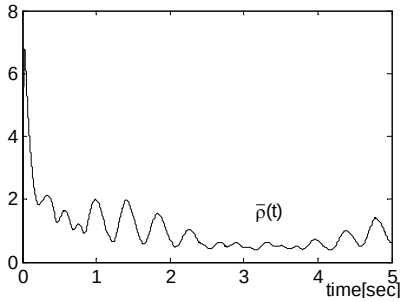

Fig. 2 The progress of parameter adaptation

IV. CONCLUSION

In this paper, we have considered the problem of robust stabilization of uncertain dynamic systems such that any information of the uncertainties is not available except that they are bounded. For such uncertain dynamic systems, we have assumed that the upper bound of the uncertainties is represented as a Fredholm integral equation of the first kind and we have provided a sufficient condition for the existence of such a representation. Based on such an assumption, we have introduced the continuous robust controller with adaptation law for the upper bound of the uncertainties. We proved that the resulting closed-loop dynamic systems are uniformly ultimately bounded in the presence of the uncertainties. The simulation results have shown that the proposed adaptive robust control method controls effectively a class of uncertain dynamic systems.

REFERENCES

[1] Gutman, S., "Uncertain dynamical systems - A Lyapunov min-max approach," IEEE Trans. on Automatic Control, Vol. 24, No. 3, pp.437-443(1979).
[2] Corless, M. J., and Leitman, G., "Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems," IEEE Trans. on Automatic Control, Vol. 26, No. 5, pp.1139-1144(1981).
[3] Barmish, B. R., Corless, M. J., and Leitmann, G., "A new class of stabilizing controllers for uncertain dynamical systems," SIAM J. on Control and Optimization, Vol. 21, No. 2, pp.246-255(1983).
[4] Chen, Y. H., "Design of robust controllers for uncertain dynamical systems," IEEE Trans. on Automatic Control, Vol. 33, No. 5, pp.487-491(1988).
[5] Chen, Y. H., "Robust control system design: non-adaptive and adaptive," Int. J. of Control, Vol. 51, No. 6, pp.1457-1477(1990).
[6] Yoo, D. S., and Chung, M. J., "A Variable structure control with simple adaptation laws for upper bounds on the norm of the uncertainties," IEEE Trans. on Automatic Control, Vol. 37, No. 6, pp.860-865(1992).
[7] Brogliato, B., and Trofino Neto, A., "Practical stabilization of a class of nonlinear systems with partially known uncertainties," Automatica, Vol. 31, No. 1, pp.145-150(1995).
[8] Wu, H., "Continuous adaptive robust controllers guaranteeing uniform ultimate boundedness for uncertain nonlinear system," Int. J. of Control, Vol. 72, No. 2, pp.115-122(1999).
[9] Arfken, G., and Weber, H., Mathematical methods for physicists, New York: Academic Press, N.Y., pp.725- 748 (1970).
[10] Messner, W., Horowitz, R., Kao, W.-W., and Boals, M., "A New Adaptive Learning Rule," IEEE Trans. on Automatic Control, Vol. 36, No. 2, pp.188-197(1991).