International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:5, 2013

Positive solutions of initial value problem for the systems of second order integro-differential equations in Banach space

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Abstract—In this paper, by establishing a new comparison result, we investigate the existence of positive solutions for initial value problems of nonlinear systems of second order integro-differential equations in Banach space.We improve and generalize some results(see[5,6]),and the results is new even in finite dimensional spaces.

Keywords—Systems of integro-differential equations, monotone iterative method, comparison result, cone.

I. Introduction

In recent years, the research about initial value problem for second order integro-differential equation is more and more active (see[1-6]), paper[5] investigates the existence of solution for the following equation:

$$\begin{cases} x^{''}(t) = f(t, x(t), x^{'}, Tx), & t \in [0,1] \\ x(0) = x_0, x^{'}(0) = x_1, \end{cases}$$

paper [6] discusses the existence of solutions of initial value problems for the following systems of second order integro-differential equations:

$$\begin{cases} x^{''}(t) = f(t, x(t), y, Tx), & x(0) = x_0, x^{'}(0) = x_1, \\ y^{''}(t) = g(t, y(t), x, Ty), & y(0) = y_0, y^{'}(0) = y_1, \end{cases}$$

Motivated by the paper [5,6], we consider the existence of solutions of initial value problems for the following systems of second order integro-differential equations:

$$\begin{cases} x^{''}(t) = f(t, x(t), y, x^{'}, Tx), & x(0) = x_0, x^{'}(0) = x_1, \\ y^{''}(t) = g(t, y(t), x, y^{'}, Ty), & y(0) = y_0, y^{'}(0) = y_1, \end{cases} \tag{1}$$

where $t \in I, f, g \in C[I \times E \times E \times E \times E, E]$, $Tx(t) = \int_0^t k(t,s)x(s)ds, k \in C[D, R^+]$, $D = \{(t,s) \in R^2 | 0 \leq s \leq t \leq 1\}$.

Suppose $(E, \|.\|)$ is a real Banach space, P is a normal cone in E, and the normal constant is 1, the partial order induced by P is $\leq$: $x \leq y \Leftrightarrow y - x \in P$. E^* is the dual space of E, $P^* = \{\varphi \in E^* | \varphi(x) \geq 0, \forall x \in P\}$ denote the dual cone of P. Obviously, $x \in P$ if and only if $\varphi(x) \geq 0, \forall \varphi \in P^*$. $(C[I,E], \|.\|_c)$ is also a Banach space, where $\|.\|_c = \max_{t \in I} \|x(t)\|$. Let $I = [0,1], P_c = \{x \in C[I,E] | x(t) \geq \theta, \forall t \in I\}$, then P_c is a normal cone in $C[I,E]$, and normal constant is 1, moreover, it defines the partial order of $C[I,E]$. $\forall u_0, v_0 \in C[I,E]$, and $u_0 < v_0$, we define order interval $[u_0, v_0] = \{x \in C[I,E] \mid u_0 \leq x \leq v_0\}$. For the sake of convenience, we first list some lemmas.

Lemma 1[5] Let E be a real Banach space, P be a cone of E, $\omega \in C^2[I,E]$ such that

$$\omega^{''}(t) \geq a\omega(t) - N\omega^{'}(t) - LT\omega(t), \omega(0) \geq \theta, \omega^{'}(0) \geq \theta, \tag{2}$$

where $a \geq 0, N > 0, L \geq 0$, $k_0 = \max\{k(t,s) \mid (t,s) \in D\}$, which satisfy

$$Lk_0 \leq a \leq N, \tag{3}$$

then $\omega(t) \geq \theta, \omega^{'}(t) \geq \theta$.

Lemma 2 Let $x, y \in C^2[I,E]$, and

$$\begin{cases} x^{''}(t) \geq bx(t) + cy(t) - Nx^{'}(t) - L\int_0^t k(t,s)x(s)ds, \\ x(0) \geq \theta, x^{'}(0) \geq \theta, \\ y^{''}(t) \geq by(t) + cx(t) - Ny^{'}(t) - L\int_0^t k(t,s)y(s)ds, \\ y(0) \geq \theta, y^{'}(0) \geq \theta, \end{cases} \tag{4}$$

where $b \geq c \geq 0$ such that

$$N \geq b + c \geq b - c \geq Lk_0, \tag{5}$$

then $x(t) \geq \theta, y(t) \geq \theta, x^{'}(t) \geq \theta, y^{'}(t) \geq \theta$.

Proof Let $\omega(t) = x(t) + y(t), t \in I$, by(4), we have $\omega^{''}(t) \geq (b+c)\omega(t) - N\omega^{'}(t) - L\int_0^t k(t,s)\omega(s)ds$, $\omega(0) \geq \theta, \omega^{'}(0) \geq \theta$, by (5) and lemma 1, we can get

$$\omega(t) \geq \theta, \omega^{'}(t) \geq \theta,$$

i.e.

$$x(t) + y(t) \geq \theta, x^{'}(t) + y^{'}(t) \geq \theta. \tag{6}$$

Next, we prove

$$x(t) \geq \theta, y(t) \geq \theta, x^{'}(t) \geq \theta, y^{'}(t) \geq \theta, \forall t \in I.$$

Actually, by (4) and (6), we can get

$$\begin{aligned} & x^{''} \geq (b-c)x - Nx^{'} - L\int_0^t k(t,s)x(s)ds, \\ & x(0) \geq \theta, x^{'}(0) \geq \theta. \end{aligned} \tag{7}$$

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$$y'' \geq (b-c)y - Ny' - L\int_0^t k(t,s)y(s)ds, \quad y(0) \geq \theta, y'(0) \geq \theta. \tag{8}$$

In the same way, by(7) and (8), we can get

$$x(t) \geq \theta, y(t) \geq \theta, x'(t) \geq \theta, y'(t) \geq \theta.$$

Let $\alpha(.)$ be $Kuratowski$ noncompact measure, we have the following lemmas.

Lemma 3[7] If $B \subset C[I,E]$ is a countable bounded set, then $\alpha(B(t)) \in L[I,R^+]$, and $\alpha(\{\int_I x(t)dt | x \in B\}) \leq 2\int_I \alpha(B(t))dt$.

Lemma 4[7] If $B \subset C[I,E]$ is bounded and equicontinuous, let $m(t) = \alpha(B(t)), t \in I$, then $m(t)$ is continuous on I, and $\alpha(\int_I B(t)dt) \leq \int_I \alpha(B(t))dt$.

II. Conclusions

In this paper,we suppose that the following conditions hold:

($\mathbf{H_1}$) There exist $x_0, y_0 \in C^2[I,E]$, such that $x_0(t) \leq y_0(t), t \in I$, and

$$\begin{array}{l} x_0'' \leq f(t, x_0, y_0, x_0', Tx_0), \forall t \in I, \\ x_0(0) \leq x_0, x_0'(0) \leq x_1, \\ y_0'' \leq f(t, y_0, x_0, y_0', Ty_0), \forall t \in I, \\ y_0(0) \geq y_0, y_0'(0) \geq y_1. \end{array}$$

($\mathbf{H_2}$) There exist non-negative constants b, c, N, L satisfying inequality (5), such that

$$\begin{array}{l} \quad f(t, x_{n+1}, y_{n+1}, x_{n+1}', Tx_{n+1}) - f(t, x_n, y_n, x_n', Tx_n) \\ \geq b(x_{n+1} - x_n) + c(y_{n+1} - y_n) - N(x_{n+1}' - x_n') \\ \quad -LT(x_{n+1} - x_n), \\ \quad g(t, y_{n+1}, x_{n+1}, y_{n+1}', Ty_{n+1}) - g(t, y_n, x_n, y_n', Ty_n) \\ \leq b(y_{n+1} - y_n) + c(x_{n+1} - x_n) - N(y_{n+1}' - y_n') \\ \quad -LT(y_{n+1} - y_n), \\ \quad g(t, y_{n+1}, x_{n+1}, y_{n+1}', Ty_{n+1}) - f(t, x_n, y_n, x_n', Tx_n) \\ \geq b(y_{n+1} - x_n) + c(x_{n+1} - y_n) - N(y_{n+1}' - x_n') \\ \quad -LT(y_{n+1} - x_n), \end{array}$$

where $x_n, y_n, x_{n+1}, y_{n+1} \in [x_0, y_0]$, and $x_{n+1} \geq x_n, y_n \geq y_{n+1}$, $x_n', y_n', x_{n+1}', y_{n+1}' \in [x_1, y_1]$, and $x_{n+1}' \geq x_n', y_n' \geq y_{n+1}', n = 1, 2, 3, ...$

($\mathbf{H_3}$) There exists constant $d > 0$, for any bounded equicontinuous set $B_i (i = 1, 2, 3, 4)$ in $[x_0, y_0]$ and $[x_1, y_1]$, we have

$$\begin{array}{rl} & \alpha(f(t, B_1(t), B_2(t), B_3(t), TB_1(t))) \\ \leq & d[\alpha(B_1(t)) + \alpha(B_2(t)) + \alpha(B_3(t)) + \alpha(T(B_1(t)))], \\ & \alpha(g(t, B_2(t), B_1(t), B_4(t), TB_2(t))) \\ \leq & d[\alpha(B_2(t)) + \alpha(B_1(t)) + \alpha(B_4(t)) + \alpha(T(B_2(t)))]. \end{array}$$

Theorem 1 Suppose $P \subset E$ is a normal cone, and conditions $(H_1) - (H_3)$ hold, then initial value problem (1) has solutions $x^*, y^* \in [x_0, y_0]$. Moreover, there exist monotone iterative sequences $\{x_n(t)\}, \{y_n(t)\} \subset [x_0, y_0]$ and $\{x_n'(t)\}, \{y_n'(t)\} \subset [x_0', y_0']$, which converge uniformly to x^*,y^* and $(x^*)'$,$(y^*)'$ on I, where $x_n(t), y_n(t)$ and $x_n'(t), y_n'(t)$ satisfy

$$\begin{array}{l} x_n(t) = x_0 + tx_1 \\ + \int_0^t (t-s)[f(s, x_{n-1}(s), y_{n-1}(s), x_{n-1}'(s), Tx_{n-1}(s)) \\ +b(x_n(s) - x_{n-1}(s)) + c(y_n(s) - y_{n-1}(s)) \\ -N(x_n'(s) - x_{n-1}'(s)) - LT(x_n(s) - x_{n-1}(s))]ds, \end{array} \tag{9}$$

$$\begin{array}{l} y_n(t) = y_0 + ty_1 \\ + \int_0^t (t-s)[g(s, y_{n-1}(s), x_{n-1}(s), y_{n-1}'(s), Ty_{n-1}(s)) \\ +b(y_n(s) - y_{n-1}(s)) + c(x_n(s) - x_{n-1}(s)) \\ -N(y_n'(s) - y_{n-1}'(s)) - LT(y_n(s) - u_{n-1}(s))]ds, \end{array} \tag{10}$$

$$\begin{array}{l} x_n'(t) = x_1 \\ + \int_0^t [f(s, x_{n-1}(s), y_{n-1}(s), x_{n-1}'(s), Tx_{n-1}(s)) \\ +b(x_n(s) - x_{n-1}(s)) + c(y_n(s) - y_{n-1}(s)) \\ -N(x_n'(s) - x_{n-1}'(s)) - LT(x_n(s) - x_{n-1}(s))]ds, \end{array} \tag{11}$$

$$\begin{array}{l} y_n'(t) = y_1 \\ + \int_0^t [g(s, y_{n-1}(s), x_{n-1}(s), y_{n-1}'(s), Ty_{n-1}(s)) \\ +b(y_n(s) - y_{n-1}(s)) \\ +c(x_n(s) - x_{n-1}(s)) - N(y_n'(s) - y_{n-1}'(s)) \\ -LT(y_n(s) - u_{n-1}(s))]ds, \end{array} \tag{12}$$

and

$$\begin{array}{l} x_0 \leq x_1 \leq ... \leq x_n \leq ... \leq x^* \leq y^* \leq ... \\ \leq ... \leq y_n \leq ... \leq y_1 \leq y_0, \end{array} \tag{13}$$

$$\begin{array}{l} x_0' \leq x_1' \leq ... \leq x_n' \leq ... \leq (x^*)' \leq (y^*)' \leq ... \\ \leq ... \leq y_n' \leq ... \leq y_1' \leq y_0'. \end{array} \tag{14}$$

Proof Firstly, by mathematical induction, we can prove $\{x_n(t)\}, \{y_n(t)\}$ satisfy

$$x_{n-1} \leq x_n \leq y_n \leq y_{n-1}, n = 1, 2, 3, ..., \tag{15}$$

$$x_{n-1}' \leq x_n' \leq y_n' \leq y_{n-1}', n = 1, 2, 3, ... \tag{16}$$

Obviously, $\forall x_{n-1}, y_{n-1} \in C[I,E] (n = 1, 2, 3...)$, it is easy to see equations (9) and (10) have only a couple of solutions x_n, y_n in $C[I,E]$, by(9)(10),we have

$$\begin{array}{l} x_n''(t) = f(t, x_{n-1}(t), y_{n-1}(t), x_{n-1}'(t), Tx_{n-1}(t)) \\ +b(x_n(t) - x_{n-1}(t)) + c(y_n(t) - y_{n-1}(t)) \\ -N(x_n'(t) - x_{n-1}'(t)) - LT(x_n(t) - x_{n-1}(t)), \\ x_n(0) = x_0, x_n'(0) = x_1, n = 1, 2, 3... \end{array} \tag{17}$$

$$\begin{array}{l} y_n''(t) = f(t, y_{n-1}(t), x_{n-1}(t), y_{n-1}'(t), Ty_{n-1}(t)) \\ +b(y_n(t) - y_{n-1}(t)) + c(x_n(t) - x_{n-1}(t)) \\ -N(y_n'(t) - y_{n-1}'(t)) - LT(y_n(t) - y_{n-1}(t)), \\ y_n(0) = y_0, y_n'(0) = y_1, n = 1, 2, 3... \end{array} \tag{18}$$

By(17),(18), $(H_1), (H_2)$, we have

$$\begin{array}{rl} (x_1 - x_0)''(t) & \geq b(x_1 - x_0)(t) + c(y_1(t) - y_0(t) \\ & -N(x_1 - x_0)'(t) - LT(x_1 - x_0)(t), \end{array}$$

$$(x_1 - x_0)(0) = x_0 - x_0 = \theta,$$
$$x_1^{'} - x_0^{'})(0) = x_1 - x_1 = \theta,$$

$$\begin{aligned} (y_1 - y_0)^{''}(t) &\leq b(y_1 - y_0)(t) + c(x_1(t) - x_0(t) \\ &\quad -N(y_1 - y_0)^{'}(t) - LT(y_1 - y_0)(t), \end{aligned}$$
$$(y_1 - y_0)(0) = y_0 - y_0 = \theta,$$
$$(y_1^{'} - y_0^{'})(0) = y_1 - y_1 = \theta,$$

$$\begin{aligned} (y_1 - x_1)^{''}(t) &\geq (b-c)(y_1 - x_1)(t) \\ &\quad -N(y_1 - x_1)^{'}(t) - LT(y_1 - x_1)(t), \end{aligned}$$
$$(y_1 - x_1)(0) = y_0 - x_0 \geq \theta,$$
$$(y_1^{'} - x_1^{'})(0) = y_1 - x_1 \geq \theta.$$

By lemma 1 and lemma 2, we can get

$$x_0 \leq x_1 \leq y_1 \leq y_0, x_0^{'} \leq x_1^{'} \leq y_1^{'} \leq y_0^{'}.$$

Now, suppose that for $k > 1$, (15),(16) hold, i.e.

$$x_{k-1} \leq x_k \leq y_k \leq y_{k-1}, x_{k-1}^{'} \leq x_k^{'} \leq y_k^{'} \leq y_{k-1}^{'}.$$

Next, we will show it also hold for $k+1$. Actually, by(15),(16) and (H_2), we have

$$\begin{aligned} (x_{k+1} - x_k)^{''}(t) &\geq b(x_{k+1} - x_k)(t) + c(y_{k+1}(t) - y_k(t) \\ &\quad -N(x_{k+1} - x_k)^{'}(t) - LT(x_{k+1} - x_k)(t), \end{aligned}$$
$$(x_{k+1} - x_k)(0) = x_0 - x_0 = \theta,$$
$$(x_{k+1}^{'} - x_k^{'})(0) = x_1 - x_1 = \theta,$$

$$\begin{aligned} (y_{k+1} - y_k)^{''}(t) &\leq b(y_{k+1} - y_k)(t) + c(x_{k+1}(t) - x_k(t) \\ &\quad -N(y_{k+1} - y_k)^{'}(t) - LT(y_{k+1} - y_k)(t), \end{aligned}$$
$$(y_{k+1} - y_k)(0) = y_0 - y_0 = \theta,$$
$$(y_{k+1}^{'} - y_k^{'})(0) = y_1 - y_1 = \theta,$$

$$\begin{aligned} (y_{k+1} - x_{k+1})^{''}(t) &\geq (b-c)(y_{k+1} - x_{k+1})(t) \\ &\quad -N(y_{k+1} - x_{k+1})^{'}(t) \\ &\quad -LT(y_{k+1} - x_{k+1})(t), \end{aligned}$$
$$(y_{k+1} - x_{k+1})(0) = y_0 - x_0 \geq \theta$$
$$(y_{k+1}^{'} - x_{k+1}^{'})(0) = y_1 - x_1 \geq \theta.$$

Therefore, by lemma 1 and lemma 2, we have

$$x_k \leq x_{k+1} \leq y_{k+1} \leq y_k, x_k^{'} \leq x_{k+1}^{'} \leq y_{k+1}^{'} \leq y_k^{'},$$

so, $\forall n \in N$, (15)and (16) hold. Consequently

$$x_0 \leq x_1 \leq ... \leq x_n \leq ... \leq y_n \leq ... \leq y_1 \leq y_0, \qquad (19)$$

and

$$x_0^{'} \leq x_1^{'} \leq ... \leq x_n^{'} \leq ... \leq y_n^{'} \leq ... \leq y_1^{'} \leq y_0^{'}. \qquad (20)$$

Let $B_1(t) = \{x_n(t)\}$, $B_2(t) = \{y_n(t)\}$, $B_3(t) = \{x_n^{'}(t)\}$, $B_4(t) = \{y_n^{'}(t)\}$, where $n \in N$, $m_i(t) = \alpha(B_i(t))(i = 1,2,3,4)$. P_c is normal cone, by the normality of cone P, $B_i(i = 1,2,3,4)$ is the bounded set in $C[I,E]$, obviously, it is equicontinuous on I, by (9),(10),(H_3) and lemma 3, we have

$$\begin{aligned} m_1(t) &\leq 2\int_0^t \alpha((t-s)[f(s, B_1(s)B_2(s)B_3(s), TB_1(s)) \\ &\quad +2bB_1(s) + 2cB_2(s) + 2NB_3(s) + 2LTB_1(s)])ds \\ &\leq 2\int_0^t [(2b+d)m_1(s) + (2c+d)m_2(s) \\ &\quad +(2N+d)m_3(s) + (2L+d)\alpha(TB_1(s))]ds, \end{aligned} \qquad (21)$$

$$\begin{aligned} m_2(t) &\leq 2\int_0^t \alpha((t-s)[g(s, B_2(s)B_1(s)B_4(s), TB_2(s)) \\ &\quad +2bB_2(s) + 2cB_1(s) + 2NB_4(s) + 2LTB_2(s)])ds \\ &\leq 2\int_0^t [(2b+d)m_2(s) + (2c+d)m_1(s) \\ &\quad +(2N+d)m_4(s) + (2L+d)\alpha(TB_2(s))]ds, \end{aligned} \qquad (22)$$

$$\begin{aligned} m_3(t) &\leq 2\int_0^t \alpha([f(s, B_1(s)B_2(s)B_3(s), TB_1(s)) \\ &\quad +2bB_1(s) + 2cB_2(s) + 2NB_3(s) + 2LTB_1(s)])ds \\ &\leq 2\int_0^t [(2b+d)m_1(s) + (2c+d)m_2(s) \\ &\quad +(2N+d)m_3(s) + (2L+d)\alpha(TB_1(s))]ds, \end{aligned} \qquad (23)$$

$$\begin{aligned} m_4(t) &\leq 2\int_0^t \alpha([g(s, B_2(s)B_1(s)B_4(s), TB_2(s)) \\ &\quad +2bB_2(s) + 2cB_1(s) + 2NB_4(s) + 2LTB_2(s)])ds \\ &\leq 2\int_0^t [(2b+d)m_2(s) + (2c+d)m_1(s) \\ &\quad +(2N+d)m_4(s) + (2L+d)\alpha(TB_2(s))]ds. \end{aligned} \qquad (24)$$

By lemma 4, we have

$$\begin{aligned} \alpha(TB_1(s)) &= \alpha(\int_0^s k(s,\tau)B_1(\tau)d\tau) \\ &\leq k_0 \int_0^s \alpha(B_1(\tau))d\tau \\ &= k_0 \int_0^s m_1(\tau)d\tau, \end{aligned} \qquad (25)$$

$$\begin{aligned} \alpha(TB_2(s)) &= \alpha(\int_0^s k(s,\tau)B_2(\tau)d\tau) \\ &\leq k_0 \int_0^s \alpha(B_2(\tau))d\tau \\ &= k_0 \int_0^s m_2(\tau)d\tau. \end{aligned} \qquad (26)$$

Let $p(t) = \max\{m_1(t), m_2(t), m_3(t), m_4(t)\}$, by (21)-(26), we have

$$\begin{aligned} p(t) &\leq 2\int_0^t [(2b+2c+2N+3d)p(s) \\ &\quad +(2L+d)k_0 \int_0^s p(\tau)d\tau]ds \\ &= 2\int_0^t [2b+2c+2N+3d \\ &\quad +(2L+b)k_0(t-s)]p(s)ds \\ &\leq 2[2b+2c+2N+3d \\ &\quad +(2L+b)k_0] \int_0^t p(s)ds, \end{aligned}$$

similarly to the proof of paper [7], we can get $p(t) = 0, \forall t \in I$, therefore $m_i(t)$=0(i=1,2,3,4), so $\{x_n(t)\}$, $\{y_n(t)\}$, $\{x_n^{'}(t)\}$, $\{y_n^{'}(t)\}$ is relatively compact set in $C[I, E]$. By the Arzela-Ascoli theorem there exist subsequences $\{x_{n_k}(t)\}$,$\{y_{n_k}(t)\}$, $\{x_{n_k}^{'}(t)\}$, $\{y_{n_k}^{'}(t)\}$, which converge uniformly to x^*,y^*,$(x^*)^{'}$ and $(y^*)^{'}$ on I. By the normality of P_c and the monotonicity of $\{x_n(t)\}$, $\{y_n(t)\}$, $\{x_n^{'}(t)\}$,$\{y_n^{'}(t)\}$ we know $\{x_n(t)\}$, $\{y_n(t)\}$, $\{x_n^{'}(t)\}$,$\{y_n^{'}(t)\}$ converge to x^*,y^*,$(x^*)^{'}$ and $(y^*)^{'}$ on I. Taking limits in (9),(10), we have

$$\begin{aligned} x^*(t) &= x_0 + tx_1 \\ &+ \int_0^t (t-s)[f(s, x^*(s), y^*(s), (x^*)^{'}(s), Tx^*(s))]ds, \end{aligned} \tag{27}$$

$$\begin{aligned} y^*(t) &= y_0 + ty_1 \\ &+ \int_0^t (t-s)[g(s, y^*(s), x^*(s), (y^*)^{'}(s), Ty^*(s))]ds, \end{aligned} \tag{28}$$

it is easy to see that $x^*(t), y^*(t)$are solutions of systems of integro-differential equations (1), and (13)(14) hold evidently.

Theorem 2 Suppose $P \subset E$ is a regular cone, and conditions $(H_1) - (H_2)$ hold, then there exist the same result as theorem 1 .

Proof Similarly to the the proof of theorem 1, the only difference is that we get the result $m_i(t) = \alpha(B_i(t)) = 0(i = 1, 2, 3, 4)$ from (H_3) in theorem 1, but we can get it by (19) (20) and the regularity of cone in theorem 2.

III. Example

As an application of theorem 1 and theorem 2, we give an example:

Example using the result of this paper, we study the initial problem for the following integro-differential equation in Banach space E:

$$\omega^{''} = H(t, \omega, \omega, \omega^{'}, T\omega), \omega(0) = \omega_0, \omega^{'}(0) = \omega_1, \tag{29}$$

Suppose $H, \omega, \omega_0, \omega_1$ have the following decomposition:
$H(t, \omega, \omega, \omega^{'}, T\omega) = f(t, x, y, x^{'}, Tx) + g(t, y, x, y^{'}, Ty)$,
$\omega = x+y, \omega_0 = x_0+y_0, \omega_1 = x_1+y_1$, where $f, g \in C[I\times E\times E \times E \times E, E], Tx(t) = \int_0^t k(t, s)x(s)ds$, $t \in I, \omega_0, \omega_1 \in E$, $x(0) = x_0 \le y_0 = y(0), x^{'}(0) = x_1 \le y_1 = y^{'}(0)$.
As the direct result of theorem 1 and theorem 2, we get the following conclusions:

Conclusion 1 Let $P \subset E$ be a normal cone. f, g satisfy conditions $(H_1) - (H_3)$, then there exists solutions ω^* $\in$ $[2x_0, 2y_0]$, and iterative sequence $\{\omega_n\} = \{x_n + y_n\}$ converges uniformly to ω^* on I, where x_n, y_n are defined by (9) and (10).

Conclusion 2 Let $P \subset E$ be a regular cone. f, g satisfy conditions $(H_1) - (H_2)$, then the same result as conclusion 1 holds.

References

[1] Guo Dajun. Initial value problem for second-order integro-differential equation in Banach space[J].Nonlinear Analysis,1999,37:289-300.
[2] Chen F.Q, Chen Y.S, On monotone iterative method for initial value problems of nonlinear second order integro-differential equations in Banach spaces , Applied Mathematics and Mechanics,2000,21(5):459-467(in Chinese).
[3] Liu Lishan.Iterative method for solutiona and coupled quasi-solutions of nonlinear integro-differential equations of mixed type in Banach spaces .Nonlinear Analysis,2000,42:583-598.
[4] Guo Lin.Boundary value problems for second-order integro-differential equations in Bannach space[J]. Acta Analysis Functionalis Applicata,2002,4(3):204-210.
[5] Song Guangxing.The solutions of initial value problems for second order integro-differential equations in Banach space[J]. Acta Mathematatica Sinica, 2004, 47(1) 71-78.
[6] Song Guangxing.Initial value problems for the systems of second order integro-differential equations in Banach space[J]. Acta Mathematatica Scientia, 2008, 28A(6) 1119-1127.
[7] Dajun Guo,V.Lakshmikantham, Xinzhi Liu. Nolinear Integral Equations in Abstract Spaces. Dordrecht:Kluwer Academic Publishers,1996.