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Minimization problems for generalized reflexive and generalized anti-reflexive matrices

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Abstract—Let $R \in \mathbf{C}^{m\times m}$ and $S \in \mathbf{C}^{n\times n}$ be nontrivial unitary involutions, i.e., $R^{\rm H} = R = R^{-1} \neq \pm I_m$ and $S^{\rm H} = S = S^{-1} \neq \pm I_n$. $A \in \mathbf{C}^{m\times n}$ is said to be a generalized reflexive (anti-reflexive) matrix if $RAS = A$ ($RAS = -A$). Let ρ be the set of $m \times n$ generalized reflexive (anti-reflexive) matrices. Given $X \in \mathbf{C}^{n\times p}, Z \in \mathbf{C}^{m\times p}, Y \in \mathbf{C}^{m\times q}$ and $W \in \mathbf{C}^{n\times q}$, we characterize the matrices A in ρ that minimize $\|AX - Z\|^2 + \|Y^{\rm H}A - W^{\rm H}\|^2$, and, given an arbitrary $\tilde{A} \in \mathbf{C}^{m\times n}$, we find a unique matrix among the minimizers of $\|AX - Z\|^2 + \|Y^{\rm H}A - W^{\rm H}\|^2$ in ρ that minimizes $\|A - \tilde{A}\|$. We also obtain sufficient and necessary conditions for existence of $A \in \rho$ such that $AX = Z, Y^{\rm H}A = W^{\rm H}$, and characterize the set of all such matrices A if the conditions are satisfied. These results are applied to solve a class of left and right inverse eigenproblems for generalized reflexive (anti-reflexive) matrices.

Keywords—approximation, generalized reflexive matrix, generalized anti-reflexive matrix, inverse eigenvalue problem.

I. Introduction

In this paper we shall adopt the following notation. $\mathbf{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices, $\mathbf{UC}^{n\times n}$ denotes the set of all unitary matrices in $\mathbf{C}^{n\times n}$. $A^{\rm H}, A^+$ and $\|A\|$ stand for the conjugate transpose, the Moore-Penrose generalized inverse and the Frobenius norm of a complex matrix A, respectively. For $A, B \in \mathbf{C}^{m\times n}$, an inner product in $\mathbf{C}^{m\times n}$ is defined by $\langle A, B\rangle = \text{trace}(B^{\rm H}A)$, then $\mathbf{C}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. I_n represents the identity matrix of order n. For $A = (a_{ij}), B = (b_{ij}) \in \mathbf{C}^{m\times n}$, $A * B$ represents the Hadamard product of the matrices A and B, i.e., $A * B = (a_{ij}b_{ij}) \in \mathbf{C}^{m\times n}$.

Throughout this paper $R \in \mathbf{C}^{m\times m}$ and $S \in \mathbf{C}^{n\times n}$ are nontrivial unitary involutions, i.e., $R^{\rm H} = R = R^{-1} \neq \pm I_m$ and $S^{\rm H} = S = S^{-1} \neq \pm I_n$. We say that $A \in \mathbf{C}^{m\times n}$ is a generalized reflexive (anti-reflexive) matrix (see [9]) if $RAS = A$ ($RAS = -A$). If $m = n, R = S$, then the generalized reflexive (anti-reflexive) matrices reduce to the reflexive (anti-reflexive) matrices (see, e.g., [8]). Let $J_n = (j_{i,k})$ represent the exchange matrix of order n defined by $j_{i,k} = \delta_{i,n-k+1}$ for $1 \leq i, k \leq n$, where $\delta_{i,k}$ is the Kronecker delta, i.e., J_n is a matrix with ones on the secondary diagonal and zeros elsewhere. By taking $m = n, R = S = J_n$, then the generalized reflexive (anti-reflexive) matrices reduce to the centrosymmetric (centroskew) matrices (see [25]) which play an important role in many areas [10, 13, 14]. Therefore, centrosymmetric (centroskew) matrices, whose special properties have been under extensive study [1, 2, 7, 15, 19, 20, 26], are the special cases of generalized reflexive (anti-reflexive) matrices. Chen [9] discussed applications that give rise to the generalized reflexive (anti-reflexive) matrices and considered least squares problems involving them.

In the following ρ is either the set of $m \times n$ generalized reflexive matrices or the set of $m\times n$ generalized anti-reflexive matrices. We consider the following problems.

Problem I. Given $X \in \mathbf{C}^{n\times p}, Z \in \mathbf{C}^{m\times p}, Y \in \mathbf{C}^{m\times q}$ and $W \in \mathbf{C}^{n\times q}$, find

$$\sigma(X, Z, Y, W) = \min_{A\in\rho}(\|AX - Z\|^2 + \|Y^{\rm H}A - W^{\rm H}\|^2)^{1/2},$$

and characterize the set

$$\begin{aligned}\rho(X, Z, Y, W) &= \{A \in \rho | (\|AX - Z\|^2 \\ &+ \|Y^{\rm H}A - W^{\rm H}\|^2)^{1/2} = \sigma(X, Z, Y, W)\}.\end{aligned}$$

Problem II. Given $\tilde{A} \in \mathbf{C}^{m\times n}$, find

$$\sigma(X, Z, Y, W; \tilde{A}) = \min_{A\in\rho(X,Z,Y,W)}\|A - \tilde{A}\|,$$

and find $\hat{A} \in \rho(X, Z, Y, W)$ such that

$$\|\hat{A} - \tilde{A}\| = \sigma(X, Z, Y, W; \tilde{A}).$$

If $m = n, \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_p), \Omega = \text{diag}(\omega_1, \cdots, \omega_q)$, $X = [x_1, \cdots, x_p]$, $Y = [y_1, \cdots, y_q]$, where $x_i \in \mathbf{C}^n (i = 1, \cdots, p)$, $y_j \in \mathbf{C}^n (j = 1, \cdots, q)$, $Z = X\Lambda, W^{\rm H} = \Omega Y^{\rm H}$, then the set $\rho(X, Z, Y, W)$ in Problem I is determined by partial left and right eigenpairs $(\omega_j, y_j)(j = 1, \cdots, q)$ and $(\lambda_i, x_i)(i = 1, \cdots, p)$, and Problem I is a left and right inverse eigenproblem for generalized reflexive (anti-reflexive) matrices. The left and right inverse eigenproblem is a special inverse eigenvalue problem, indeed, the recursive inverse eigenvalue problem (see [17]). Problem II is an optimal approximation problem under spectral constraint.

There are many publications (see, e.g., [5, 6, 11, 12, 16, 24, 29] and their references) concerning minimization problems for matrices. Recently, Bai and Chan [4] considered inverse eigenproblems related to centrosymmetric and centroskew matrices, where $m = n, R = S = J_n$, and $Y = 0, W = 0$. Zhou et al. [28] discussed the minimization problems for centrosymmetric matrices, where $m = n, R = S = J_n, Y = 0$ and $W = 0$. Peng and Hu [18] studied the existence of $n \times n$ reflexive and anti-reflexive matrices X such that $AX = B$, where A and B are given in $\mathbf{C}^{m\times n}$, and the nearest matrix to a given matrix. Trench [21] considered the minimization problems for hermitian, hermitian reflexive and hermitian anti-reflexive matrices, where $m = n, Y = 0$ and $W = 0$. In

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[22] and [23], Trench has studied inverse eigenproblems for generalized symmetric or skew symmetric matrices and the minimization problems for (R,S)-symmetric and (R,S)-skew symmetric matrices, respectively.

In this paper we obtain explicit formulas for $\sigma(X,Z,Y,W)$, $\sigma(X,Z,Y,W;\tilde{A})$, all matrices in $\rho(X,Z,,Y,W)$, and the solution of Problem II. As a byproduct of our results on Problem I we obtain necessary and sufficient conditions on X,Z,Y and W for existence of $A\in\rho$ such that $AX=Z$ and $Y^{\rm H}A=W^{\rm H}$, and an explicit formula for all such A. These results are applied to solve a class of left and right inverse eigenproblems for generalized reflexive (anti-reflexive) matrices.

If $m=n, R=S, Y=0$ and $W=0$, our results apply to reflexive and anti-reflexive matrices, which are discussed in [18, 27]. In particular, if $m=n, R=S=J_n, Y=0$ and $W=0$, our results apply to centrosymmetric and centroskew matrices, which are considered in [4, 28].

II. Preliminary Considerations

If λ is an eigenvalue of $E\in\mathbf{C}^{m\times m}$, let $V_E(\lambda)$ denote the eigenspace of E corresponding to the eigenvalue λ. A vector $z\in\mathbf{C}^m$ is said to be R-symmetric (R-skew symmetric) if $Rz=z$ ($Rz=-z$); thus, $V_R(1)$ and $V_R(-1)$ are the subspaces of $\mathbf{C}^{m\times m}$ consisting respectively of R-symmetric and R-skew symmetric vectors. Let $r=\dim[V_R(1)], s=\dim[V_R(-1)]$. Since a unitary involution is diagonalizable and $R\neq\pm I_m$, then $r,s\geq1$, and $r+s=m$. Let $\{p_1,\cdots,p_r\}$ and $\{q_1,\cdots,q_s\}$ be the orthonormal bases for $V_R(1)$ and $V_R(-1)$, respectively, and define

$$P=[p_1,\cdots,p_r]\in\mathbf{C}^{m\times r}, Q=[q_1,\cdots,q_s]\in\mathbf{C}^{m\times s},$$

then $[P,Q]$ is a unitary matrix and R has the following spectral decomposition

$$R=[P,Q]\begin{bmatrix} I_r & 0\\ 0 & -I_s\end{bmatrix}\begin{bmatrix} P^{\rm H}\\ Q^{\rm H}\end{bmatrix}. \quad (1)$$

In particular, if $R=J_{2k}$, then $r=s=k$, we can take

$$P=\frac{1}{\sqrt{2}}\begin{bmatrix} I_k\\ J_k\end{bmatrix}, Q=\frac{1}{\sqrt{2}}\begin{bmatrix} I_k\\ -J_k\end{bmatrix}.$$

If $R=J_{2k+1}$, then $r=k+1, s=k$, we can take

$$P=\frac{1}{\sqrt{2}}\begin{bmatrix} I_k & 0\\ 0 & \sqrt{2}\\ J_k & 0\end{bmatrix}, Q=\frac{1}{\sqrt{2}}\begin{bmatrix} I_k\\ 0\\ -J_k\end{bmatrix}.$$

Similarly, there are positive integers k and l such that $k+l=n$, and the matrices $U\in\mathbf{C}^{n\times k}$ and $V\in\mathbf{C}^{n\times l}$ whose column vectors form the orthonormal bases for the eigenspaces $V_S(1)$ and $V_S(-1)$, respectively. Thus, $[U,V]$ is a unitary matrix and S has the spectral decomposition:

$$S=[U,V]\begin{bmatrix} I_k & 0\\ 0 & -I_l\end{bmatrix}\begin{bmatrix} U^{\rm H}\\ V^{\rm H}\end{bmatrix}. \quad (2)$$

In the following P,Q,U,V are always defined by (1) and (2).

(1) and (2) yield the following characterizations of $m\times n$ generalized reflexive or generalized anti-reflexive matrices.

Lemma 1: (a) $A\in\mathbf{C}^{m\times n}$ is a generalized reflexive matrix if and only if there exist $A_{PU}\in\mathbf{C}^{r\times k}$, $A_{QV}\in\mathbf{C}^{s\times l}$ such that

$$A=[P,Q]\begin{bmatrix} A_{PU} & 0\\ 0 & A_{QV}\end{bmatrix}\begin{bmatrix} U^{\rm H}\\ V^{\rm H}\end{bmatrix}. \quad (3)$$

(b) $A\in\mathbf{C}^{m\times n}$ is a generalized anti-reflexive matrix if and only if there exist $A_{PV}\in\mathbf{C}^{r\times l}$, $A_{QU}\in\mathbf{C}^{s\times k}$ such that

$$A=[P,Q]\begin{bmatrix} 0 & A_{PV}\\ A_{QU} & 0\end{bmatrix}\begin{bmatrix} U^{\rm H}\\ V^{\rm H}\end{bmatrix}. \quad (4)$$

Proof. (a) If $A\in\mathbf{C}^{m\times n}$ is a generalized reflexive matrix, then it follows from $RAS=A$, (1) and (2) that $P^{\rm H}AV=0, Q^{\rm H}AU=0$. Let $A_{PU}=P^{\rm H}AU, A_{QV}=Q^{\rm H}AV$, then $A_{PU}\in\mathbf{C}^{r\times k}$, $A_{QV}\in\mathbf{C}^{s\times l}$, and

$$\begin{aligned} A=RAS &= [P,Q]\begin{bmatrix} I_r & 0\\ 0 & -I_s\end{bmatrix}\begin{bmatrix} P^{\rm H}\\ Q^{\rm H}\end{bmatrix}\\ &\quad A[U,V]\begin{bmatrix} I_k & 0\\ 0 & -I_l\end{bmatrix}\begin{bmatrix} U^{\rm H}\\ V^{\rm H}\end{bmatrix}\\ &= [P,Q]\begin{bmatrix} A_{PU} & 0\\ 0 & A_{QV}\end{bmatrix}\begin{bmatrix} U^{\rm H}\\ V^{\rm H}\end{bmatrix}. \end{aligned}$$

The verification of the converse is straightforward. Similarly, (b) is may proved.

In order to solve Problem I, we will need the following lemma.

Lemma 2: Let $\tilde{X}\in\mathbf{C}^{m\times p}, \tilde{W}\in\mathbf{C}^{m\times t}, \tilde{Y}\in\mathbf{C}^{n\times t}$, $\tilde{Z}\in\mathbf{C}^{n\times p}$, and the singular value decompositions(SVDs) of the matrices $\tilde{X}$ and $\tilde{Y}$ be, respectively

$$\tilde{X}=\tilde{U}\begin{bmatrix} \Delta & 0\\ 0 & 0\end{bmatrix}\tilde{V}^{\rm H},\ \ \tilde{Y}=\tilde{P}\begin{bmatrix} \Gamma & 0\\ 0 & 0\end{bmatrix}\tilde{Q}^{\rm H}, \quad (5)$$

where $\tilde{U}=[\tilde{U}_1,\tilde{U}_2]\in\mathbf{UC}^{m\times m}, \tilde{V}=[\tilde{V}_1,\tilde{V}_2]\in\mathbf{UC}^{p\times p}, \tilde{P}=[\tilde{P}_1,\tilde{P}_2]\in\mathbf{UC}^{n\times n}, \tilde{Q}=[\tilde{Q}_1,\tilde{Q}_2]\in\mathbf{UC}^{t\times t}, \Delta=\mathrm{diag}\{\delta_1,\cdots,\delta_e\}>0, \Gamma=\mathrm{diag}\{\gamma_1,\cdots,\gamma_f\}>0, e=\mathrm{rank}(\tilde{X}), f=\mathrm{rank}(\tilde{Y}), \tilde{U}_1\in\mathbf{C}^{m\times e}, \tilde{V}_1\in\mathbf{C}^{p\times e}, \tilde{P}_1\in\mathbf{C}^{n\times f}, \tilde{Q}_1\in\mathbf{C}^{t\times f}$, and let

$$\begin{aligned} \Phi_1 &= [\varphi_{ij}]\in\mathbf{C}^{f\times e},\\ \varphi_{ij} &= \frac{1}{(\gamma_i^2+\delta_j^2)^{1/2}}, 1\leq i\leq f, 1\leq j\leq e, \end{aligned}$$

then

$$\begin{aligned} &\min_{B\in\mathbf{C}^{n\times m}}(\|B\tilde{X}-\tilde{Z}\|^2+\|\tilde{Y}^{\rm H}B-\tilde{W}^{\rm H}\|^2)\\ &=\|\tilde{Z}\tilde{V}_2\|^2+\|\tilde{W}\tilde{Q}_2\|^2\\ &+\|\Phi_1*(\Gamma\tilde{P}_1^{\rm H}\tilde{Z}\tilde{V}_1-\tilde{Q}_1^{\rm H}\tilde{W}^{\rm H}\tilde{U}_1\Delta)\|^2, \end{aligned} \quad (6)$$

and $B\in\mathbf{C}^{n\times m}$ attains this minimum if and only if

$$B=\tilde{P}\begin{bmatrix} \Phi*B_{11} & \Gamma^{-1}\tilde{Q}_1^{\rm H}\tilde{W}^{\rm H}\tilde{U}_2\\ \tilde{P}_2^{\rm H}\tilde{Z}\tilde{V}_1\Delta^{-1} & B_{22}\end{bmatrix}\tilde{U}^{\rm H}, \quad (7)$$

where $B_{11}=\tilde{P}_1^{\rm H}\tilde{Z}\tilde{V}_1\Delta+\Gamma\tilde{Q}_1^{\rm H}\tilde{W}^{\rm H}\tilde{U}_1$, $\Phi=\Phi_1*\Phi_1$, $B_{22}\in\mathbf{C}^{(n-f)\times(m-e)}$ is an arbitrary matrix. Moreover, the equations $B\tilde{X}=\tilde{Z}$ and $\tilde{Y}^{\rm H}B=\tilde{W}^{\rm H}$ have a common solution if and only if

$$\tilde{Z}\tilde{X}^{+}\tilde{X}=\tilde{Z}, \tilde{W}\tilde{Y}^{+}\tilde{Y}=\tilde{W}, \tilde{Y}^{\rm H}\tilde{Z}=\tilde{W}^{\rm H}\tilde{X}, \quad (8)$$

in which case a general solution of the equations

$$B\tilde{X}=\tilde{Z},\ \ \tilde{Y}^{\rm H}B=\tilde{W}^{\rm H}$$

can be expressed as

$$B = \tilde{Z}\tilde{X}^{+} + (\tilde{W}\tilde{Y}^{+})^{\mathrm{H}}(I_m - \tilde{X}\tilde{X}^{+}) + \tilde{P}_2 B_{22}\tilde{U}_2^{\mathrm{H}}, \quad (9)$$

where $B_{22} \in \mathbf{C}^{(n-f)\times(m-e)}$ is an arbitrary matrix.

Proof. From (5), we have

$$\begin{aligned} &\|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^{\mathrm{H}}B - \tilde{W}^{\mathrm{H}}\|^2 \\ = &\left\|B\tilde{U}\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}\tilde{V}^{\mathrm{H}} - \tilde{Z}\right\|^2 \\ &+\left\|\tilde{Q}\begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}\tilde{P}^{\mathrm{H}}B - \tilde{W}^{\mathrm{H}}\right\|^2 \\ = &\left\|\tilde{P}^{\mathrm{H}}B\tilde{U}\begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} - \tilde{P}^{\mathrm{H}}\tilde{Z}\tilde{V}\right\|^2 \\ &+\left\|\begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix}\tilde{P}^{\mathrm{H}}B\tilde{U} - \tilde{Q}^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}\right\|^2. \end{aligned}$$

Let

$$\tilde{P}^{\mathrm{H}}B\tilde{U} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (10)$$

where $B_{11} \in \mathbf{C}^{f\times e}$, then

$$\begin{aligned} &\|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^{\mathrm{H}}B - \tilde{W}^{\mathrm{H}}\|^2 \\ &= \|B_{11}\Delta - \tilde{P}_1^{\mathrm{H}}\tilde{Z}\tilde{V}_1\|^2 + \|B_{21}\Delta - \tilde{P}_2^{\mathrm{H}}\tilde{Z}\tilde{V}_1\|^2 \\ &+\|\tilde{Z}\tilde{V}_2\|^2 + \|\Gamma B_{11} - \tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_1\|^2 \\ &+\|\Gamma B_{12} - \tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_2\|^2 + \|\tilde{Q}_2^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\|^2. \end{aligned} \quad (11)$$

Thus, $\|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^{\mathrm{H}}B - \tilde{W}^{\mathrm{H}}\|^2 = \min$ if and only if

$$\|B_{21}\Delta - \tilde{P}_2^{\mathrm{H}}\tilde{Z}\tilde{V}_1\| = \min, \|\Gamma B_{12} - \tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_2\| = \min \quad (12)$$

and

$$g(B_{11}) := \|B_{11}\Delta - \tilde{P}_1^{\mathrm{H}}\tilde{Z}\tilde{V}_1\|^2 + \|\Gamma B_{11} - \tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_1\|^2 = \min. \quad (13)$$

Clearly, (12) implies that

$$B_{21} = \tilde{P}_2^{\mathrm{H}}\tilde{Z}\tilde{V}_1\Delta^{-1}, \quad B_{12} = \Gamma^{-1}\tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_2. \quad (14)$$

Let $B_{11} = [b_{ij}], \tilde{P}_1^{\mathrm{H}}\tilde{Z}\tilde{V}_1 = [z_{ij}]$, $\tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_1 = [w_{ij}] \in \mathbf{C}^{f\times e}$, then

$$g(B_{11}) = \sum_{i=1}^{f}\sum_{j=1}^{e}(|b_{ij}\delta_j - z_{ij}|^2 + |\gamma_i b_{ij} - w_{ij}|^2). \quad (15)$$

Now we minimize the quantities

$$q_{ij} = |b_{ij}\delta_j - z_{ij}|^2 + |\gamma_i b_{ij} - w_{ij}|^2, \quad 1 \le i \le f, 1 \le j \le e.$$

It is easy to obtain the minimizers

$$b_{ij} = \frac{z_{ij}\delta_j + \gamma_i w_{ij}}{\gamma_i^2 + \delta_j^2}, \quad 1 \le i \le f, 1 \le j \le e, \quad (16)$$

and the minima

$$q_{ij} = \frac{|\gamma_i z_{ij} - w_{ij}\delta_j|^2}{\gamma_i^2 + \delta_j^2}, \quad 1 \le i \le f, 1 \le j \le e. \quad (17)$$

(17), (14) and (11) imply (6). Substituting (14) and (16) into (10) yields (7).

It follows from (6) that $B\tilde{X} = \tilde{Z}$ and $\tilde{Y}^{\mathrm{H}}B = \tilde{W}^{\mathrm{H}}$ have a common solution if and only if

$$\tilde{Z}\tilde{V}_2 = 0, \ \tilde{W}\tilde{Q}_2 = 0, \ \Gamma\tilde{P}_1^{\mathrm{H}}\tilde{Z}\tilde{V}_1 = \tilde{Q}_1^{\mathrm{H}}\tilde{W}^{\mathrm{H}}\tilde{U}_1\Delta,$$

which implies (8). In this case, $B_{11} = \tilde{P}_1^{\mathrm{H}}\tilde{Z}\tilde{V}_1\Delta^{-1}$. Substituting the representation of B_{11} and (14) into (10) yields (9).

III. Minimization Problem for Generalized Reflexive Matrices

In this section ρ denotes the set of $m \times n$ generalized reflexive matrices.

For the given matrices $X \in \mathbf{C}^{n\times p}$, and $Y \in \mathbf{C}^{m\times q}$, let the SVDs of matrices $U^{\mathrm{H}}X, P^{\mathrm{H}}Y$, $V^{\mathrm{H}}X$ and $Q^{\mathrm{H}}Y$ be, respectively

$$U^{\mathrm{H}}X = F_1\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}G_1^{\mathrm{H}}, \ P^{\mathrm{H}}Y = F_2\begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix}G_2^{\mathrm{H}}, \quad (18)$$

$$V^{\mathrm{H}}X = F_3\begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix}G_3^{\mathrm{H}}, \ Q^{\mathrm{H}}Y = F_4\begin{bmatrix} \Sigma_4 & 0 \\ 0 & 0 \end{bmatrix}G_4^{\mathrm{H}}, \quad (19)$$

where all matrices $F_i = [F_{i1}, F_{i2}], G_i = [G_{i1}, G_{i2}] (i = 1,2,3,4)$ are unitary matrices and partitions are compatible with the size of $\Sigma_i = \mathrm{diag}(\alpha_1^{(i)}, \cdots, \alpha_{t_i}^{(i)}) > 0 (i = 1,2,3,4)$, $t_1 = \mathrm{rank}(U^{\mathrm{H}}X)$, $t_2 = \mathrm{rank}(P^{\mathrm{H}}Y)$, $t_3 = \mathrm{rank}(V^{\mathrm{H}}X)$, $t_4 = \mathrm{rank}(Q^{\mathrm{H}}Y)$.

Theorem 1: Let $X \in \mathbf{C}^{n\times p}, Z \in \mathbf{C}^{m\times p}, Y \in \mathbf{C}^{m\times q}$, $W \in \mathbf{C}^{n\times q}$, and ρ denote the set of $m \times n$ generalized reflexive matrices. Suppose that the SVDs of the matrices $U^{\mathrm{H}}X, P^{\mathrm{H}}Y, V^{\mathrm{H}}X$ and $Q^{\mathrm{H}}Y$ are given by (18) and (19), then

$$\begin{aligned} &\sigma(X,Z,Y,W) = \min_{A\in\rho}(\|AX - Z\|^2 \\ &\quad +\|Y^{\mathrm{H}}A - W^{\mathrm{H}}\|^2)^{1/2} \\ = \ &(\|P^{\mathrm{H}}ZG_{12}\|^2 + \|U^{\mathrm{H}}WG_{22}\|^2 \\ + \ &\|\Phi_1 * (\Sigma_2 F_{21}^{\mathrm{H}}P^{\mathrm{H}}ZG_{11} - G_{21}^{\mathrm{H}}W^{\mathrm{H}}UF_{11}\Sigma_1)\|^2 \\ + \ &\|Q^{\mathrm{H}}ZG_{32}\|^2 + \|V^{\mathrm{H}}WG_{42}\|^2 \\ + \ &\|\Phi_2 * (\Sigma_4 F_{41}^{\mathrm{H}}Q^{\mathrm{H}}ZG_{31} - G_{41}^{\mathrm{H}}W^{\mathrm{H}}VF_{31}\Sigma_3)\|^2)^{1/2}, \end{aligned} \quad (20)$$

where

$$\Phi_1 = [\varphi_{ij}^{(1)}] \in \mathbf{C}^{t_2\times t_1}, \varphi_{ij}^{(1)} = \frac{1}{((\alpha_i^{(2)})^2 + (\alpha_j^{(1)})^2)^{1/2}}, \quad 1 \le i \le t_2, 1 \le j \le t_1,$$

$$\Phi_2 = [\varphi_{ij}^{(2)}] \in \mathbf{C}^{t_4\times t_3}, \varphi_{ij}^{(2)} = \frac{1}{((\alpha_i^{(4)})^2 + (\alpha_j^{(3)})^2)^{1/2}}, \quad 1 \le i \le t_4, 1 \le j \le t_3,$$

and $A \in \rho$ attains this minimum if and only if

$$A = A_0 + [P, Q]\begin{bmatrix} F_{22}MF_{12}^{\mathrm{H}} & 0 \\ 0 & F_{42}NF_{32}^{\mathrm{H}} \end{bmatrix}\begin{bmatrix} U^{\mathrm{H}} \\ V^{\mathrm{H}} \end{bmatrix} \quad (21)$$

with arbitrary $M \in \mathbf{C}^{(r-t_2)\times(k-t_1)}$, $N \in \mathbf{C}^{(s-t_4)\times(l-t_3)}$ and

$$A_0 = [P, Q]\begin{bmatrix} A_{PU}^{(0)} & 0 \\ 0 & A_{QV}^{(0)} \end{bmatrix}\begin{bmatrix} U^{\mathrm{H}} \\ V^{\mathrm{H}} \end{bmatrix}, \quad (22)$$

$$A_{PU}^{(0)} = F_2\begin{bmatrix} K_1 * L_{11} & \Sigma_2^{-1}G_{21}^{\mathrm{H}}W^{\mathrm{H}}UF_{12} \\ F_{22}^{\mathrm{H}}P^{\mathrm{H}}ZG_{11}\Sigma_1^{-1} & 0 \end{bmatrix}F_1^{\mathrm{H}}, \quad (23)$$

$$A_{QV}^{(0)} = F_4\begin{bmatrix} K_2 * J_{11} & \Sigma_4^{-1}G_{41}^{\mathrm{H}}W^{\mathrm{H}}VF_{32} \\ F_{42}^{\mathrm{H}}Q^{\mathrm{H}}ZG_{31}\Sigma_3^{-1} & 0 \end{bmatrix}F_3^{\mathrm{H}}, \quad (24)$$

where

$$L_{11} = F_{21}^{\mathrm{H}}P^{\mathrm{H}}ZG_{11}\Sigma_1 + \Sigma_2 G_{21}^{\mathrm{H}}W^{\mathrm{H}}UF_{11},$$

$$J_{11} = F_{41}^{\mathrm{H}}Q^{\mathrm{H}}ZG_{31}\Sigma_3 + \Sigma_4 G_{41}^{\mathrm{H}}W^{\mathrm{H}}VF_{31},$$

$K_1=\Phi_1*\Phi_1, K_2=\Phi_2*\Phi_2$. Moreover, the equations $AX=Z$ and $Y^{\rm H}A=W^{\rm H}$ have a common solution $A\in\rho$ if and only if

$$P^{\rm H}Z(U^{\rm H}X)^{+}U^{\rm H}X=P^{\rm H}Z,$$

$$U^{\rm H}W(P^{\rm H}Y)^{+}P^{\rm H}Y=U^{\rm H}W,$$

$$Y^{\rm H}PP^{\rm H}Z=W^{\rm H}UU^{\rm H}X$$

and

$$Q^{\rm H}Z(V^{\rm H}X)^{+}V^{\rm H}X=Q^{\rm H}Z,$$

$$V^{\rm H}W(Q^{\rm H}Y)^{+}Q^{\rm H}Y=V^{\rm H}W,$$

$$Y^{\rm H}QQ^{\rm H}Z=W^{\rm H}VV^{\rm H}X,$$

in which case a general solution of the equations $AX=Z$ and $Y^{\rm H}A=W^{\rm H}$ can be expressed as

$$A=[P,Q]\left[\begin{array}{cc} A_{PU}^{(0)}+F_{22}MF_{12}^{\rm H} & 0\\ 0 & A_{QV}^{(0)}+F_{42}NF_{32}^{\rm H}\end{array}\right]\left[\begin{array}{c}U^{\rm H}\\ V^{\rm H}\end{array}\right],$$

where

$$A_{PU}^{(0)}=P^{\rm H}Z(U^{\rm H}X)^{+}+(Y^{\rm H}P)^{+}W^{\rm H}UF_{12}F_{12}^{\rm H},$$

$$A_{QV}^{(0)}=Q^{\rm H}Z(V^{\rm H}X)^{+}+(Y^{\rm H}Q)^{+}W^{\rm H}VF_{32}F_{32}^{\rm H},$$

and $M\in{\bf C}^{(r-t_2)\times(k-t_1)}$, $N\in{\bf C}^{(s-t_4)\times(l-t_3)}$ are arbitrary matrices.

Proof. If $A\in\rho$, it follows from Lemma 1 that there exist $A_{PU}\in{\bf C}^{r\times k}$, $A_{QV}\in{\bf C}^{s\times l}$ satisfying

$$A=[P,Q]\left[\begin{array}{cc} A_{PU} & 0\\ 0 & A_{QV}\end{array}\right]\left[\begin{array}{c}U^{\rm H}\\ V^{\rm H}\end{array}\right]. \quad (25)$$

Therefore,

$$\begin{aligned}&\min_{A\in\rho}(\|AX-Z\|^2+\|Y^{\rm H}A-W^{\rm H}\|^2)\\ =\;&\min_{A_{PU}\in{\bf C}^{r\times k},A_{QV}\in{\bf C}^{s\times l}}\\ &\left(\left\|[P,Q]\left[\begin{array}{cc} A_{PU} & 0\\ 0 & A_{QV}\end{array}\right]\left[\begin{array}{c}U^{\rm H}\\ V^{\rm H}\end{array}\right]X-Z\right\|^2\right.\\ &\left.+\left\|Y^{\rm H}[P,Q]\left[\begin{array}{cc} A_{PU} & 0\\ 0 & A_{QV}\end{array}\right]\left[\begin{array}{c}U^{\rm H}\\ V^{\rm H}\end{array}\right]-W^{\rm H}\right\|^2\right)\\ =\;&\min_{A_{PU}\in{\bf C}^{r\times k}}(\|A_{PU}(U^{\rm H}X)-P^{\rm H}Z\|^2\\ &+\|(P^{\rm H}Y)^{\rm H}A_{PU}-(U^{\rm H}W)^{\rm H}\|^2)\\ &+\min_{A_{QV}\in{\bf C}^{s\times l}}(\|A_{QV}(V^{\rm H}X)-Q^{\rm H}Z\|^2\\ &+\|(Q^{\rm H}Y)^{\rm H}A_{QV}-(V^{\rm H}W)^{\rm H}\|^2).\end{aligned} \quad (26)$$

It follows from Lemma 2 that

$$\begin{aligned}&\min_{A_{PU}\in{\bf C}^{r\times k}}(\|A_{PU}(U^{\rm H}X)-P^{\rm H}Z\|^2\\ &+\|(P^{\rm H}Y)^{\rm H}A_{PU}-(U^{\rm H}W)^{\rm H}\|^2)\\ &=\|P^{\rm H}ZG_{12}\|^2+\|U^{\rm H}WG_{22}\|^2\\ &+\|\Phi_1*(\Sigma_2F_{21}^{\rm H}P^{\rm H}ZG_{11}-G_{21}^{\rm H}W^{\rm H}UF_{11}\Sigma_1)\|^2,\end{aligned} \quad (27)$$

$$\begin{aligned}&\min_{A_{QV}\in{\bf C}^{s\times l}}(\|A_{QV}(V^{\rm H}X)-Q^{\rm H}Z\|^2\\ &+\|(Q^{\rm H}Y)^{\rm H}A_{QV}-(V^{\rm H}W)^{\rm H}\|^2)\\ &=\|Q^{\rm H}ZG_{32}\|^2+\|V^{\rm H}WG_{42}\|^2\\ &+\|\Phi_2*(\Sigma_4F_{41}^{\rm H}Q^{\rm H}ZG_{31}-G_{41}^{\rm H}W^{\rm H}VF_{31}\Sigma_3)\|^2,\end{aligned} \quad (28)$$

and $A_{PU}\in{\bf C}^{r\times k}$, $A_{QV}\in{\bf C}^{s\times l}$ attain the minima if and only if

$$A_{PU}=F_2\left[\begin{array}{cc} K_1*A_{PU}^{(11)} & \Sigma_2^{-1}G_{21}^{\rm H}W^{\rm H}UF_{12}\\ F_{22}^{\rm H}P^{\rm H}ZG_{11}\Sigma_1^{-1} & M\end{array}\right]F_1^{\rm H}, \quad (29)$$

$$A_{QV}=F_4\left[\begin{array}{cc} K_2*A_{QV}^{(11)} & \Sigma_4^{-1}G_{41}^{\rm H}W^{\rm H}VF_{32}\\ F_{42}^{\rm H}Q^{\rm H}ZG_{31}\Sigma_3^{-1} & N\end{array}\right]F_3^{\rm H}, \quad (30)$$

where $A_{PU}^{(11)}=F_{21}^{\rm H}P^{\rm H}ZG_{11}\Sigma_1+\Sigma_2G_{21}^{\rm H}W^{\rm H}UF_{11}$, $A_{QV}^{(11)}=F_{41}^{\rm H}Q^{\rm H}ZG_{31}\Sigma_3+\Sigma_4G_{41}^{\rm H}W^{\rm H}VF_{31}$, $M\in{\bf C}^{(r-t_2)\times(k-t_1)}$, $N\in{\bf C}^{(s-t_4)\times(l-t_3)}$ are arbitrary matrices. (26), (27) and (28) imply (20). Substituting (29) and (30) into (25) yields (21).

It follows from Lemma 1 that the equations $AX=Z$ and $Y^{\rm H}A=W^{\rm H}$ have a common solution $A\in\rho$ if and only if $A_{PU}(U^{\rm H}X)=P^{\rm H}Z$ and $(P^{\rm H}Y)^{\rm H}A_{PU}=(U^{\rm H}W)^{\rm H}$, $A_{QV}(V^{\rm H}X)=Q^{\rm H}Z$ and $(Q^{\rm H}Y)^{\rm H}A_{QV}=(V^{\rm H}W)^{\rm H}$ have common solutions $A_{PU}\in{\bf C}^{r\times k}$, $A_{QV}\in{\bf C}^{s\times l}$, respectively. Applying Lemma 2, it is easy to obtain the second part of the conclusions.

Let $r_1=r-t_2, k_1=k-t_1, s_1=s-t_4$ and $l_1=l-t_3$. From Theorem 1 we have

$$\begin{aligned}&\rho(X,Z,Y,W)=\\ &\left\{A=A_0+[P,Q]\left[\begin{array}{cc} F_{22}MF_{12}^{\rm H} & 0\\ 0 & F_{42}NF_{32}^{\rm H}\end{array}\right]\left[\begin{array}{c}U^{\rm H}\\ V^{\rm H}\end{array}\right]\right|\\ &\left.M\in{\bf C}^{r_1\times k_1}, N\in{\bf C}^{s_1\times l_1}\right\}.\end{aligned} \quad (31)$$

It is easy to verify that $\rho(X,Z,Y,W)$ is a closed convex set in Hilbert space ${\bf C}^{m\times n}$. Therefore, for given matrix $\tilde{A}\in{\bf C}^{m\times n}$, it follows from the best approximation theorem (see Aubin[3]) that there exists a unique solution $\hat{A}$ in $\rho(X,Z,Y,W)$ such that $\|\hat{A}-\tilde{A}\|=\sigma(X,Z,Y,W;\tilde{A})$.

We shall focus our attention on seeking the unique solution of Problem II. For any matrix $A\in\rho(X,Z,Y,W)$, we have

$$\begin{aligned}&\|A-\tilde{A}\|^2=\\ &\left\|[P,Q]\left[\begin{array}{cc} A_{PU}^{(0)}+F_{22}MF_{12}^{\rm H} & 0\\ 0 & A_{QV}^{(0)}+F_{42}NF_{32}^{\rm H}\end{array}\right]\right.\\ &\left.\left[\begin{array}{c}U^{\rm H}\\ V^{\rm H}\end{array}\right]-\tilde{A}\right\|^2\\ &=\left\|\left[\begin{array}{cc} A_{PU}^{(0)}+F_{22}MF_{12}^{\rm H} & 0\\ 0 & A_{QV}^{(0)}+F_{42}NF_{32}^{\rm H}\end{array}\right]\right.\\ &\left.-\left[\begin{array}{c}P^{\rm H}\\ Q^{\rm H}\end{array}\right]\tilde{A}[U,V]\right\|^2\\ &=\|F_{22}MF_{12}^{\rm H}-(P^{\rm H}\tilde{A}U-A_{PU}^{(0)})\|^2\\ &+\|F_{42}NF_{32}^{\rm H}-(Q^{\rm H}\tilde{A}V-A_{QV}^{(0)})\|^2\\ &+\|Q^{\rm H}\tilde{A}U\|^2+\|P^{\rm H}\tilde{A}V\|^2\\ &=\|M-F_{22}^{\rm H}(P^{\rm H}\tilde{A}U-A_{PU}^{(0)})F_{12}\|^2\\ &+\|N-F_{42}^{\rm H}(Q^{\rm H}\tilde{A}V-A_{QV}^{(0)})F_{32}\|^2\\ &+\|F_{21}^{\rm H}(P^{\rm H}\tilde{A}U-A_{PU}^{(0)})\|^2\\ &+\|F_{22}^{\rm H}(P^{\rm H}\tilde{A}U-A_{PU}^{(0)})F_{11}\|^2+\|Q^{\rm H}\tilde{A}U\|^2\\ &+\|F_{42}^{\rm H}(Q^{\rm H}\tilde{A}V-A_{QV}^{(0)})F_{31}\|^2+\|P^{\rm H}\tilde{A}V\|^2\\ &+\|F_{41}^{\rm H}(Q^{\rm H}\tilde{A}V-A_{QV}^{(0)})\|^2.\end{aligned} \quad (32)$$

Note that $F_{22}^{\rm H}A_{PU}^{(0)}F_{12}=0, F_{42}^{\rm H}A_{QV}^{(0)}F_{32}=0$. It follows from (32) that $\|A-\tilde{A}\|=\min$ if and only if

$$M=F_{22}^{\rm H}P^{\rm H}\tilde{A}UF_{12},\quad N=F_{42}^{\rm H}Q^{\rm H}\tilde{A}VF_{32}.$$

By now, we have proved the following result.

Theorem 2: Let $X\in\mathbf{C}^{n\times p}, Z\in\mathbf{C}^{m\times p}, Y\in\mathbf{C}^{m\times q}, W\in\mathbf{C}^{n\times q}, \tilde{A}\in\mathbf{C}^{m\times n}$ and ρ denote the set of $m\times n$ generalized reflexive matrices. Suppose that the SVDs of the matrices $U^{\rm H}X, P^{\rm H}Y, V^{\rm H}X$ and $Q^{\rm H}Y$ are given by (18) and (19), then Problem II has a unique solution $\hat{A}$ in $\rho(X,Z,Y,W)$. Moreover, $\hat{A}$ can be expressed as

$$\hat{A}=A_0+[P,Q]\begin{bmatrix}\hat{A}_{11} & 0\\ 0 & \hat{A}_{22}\end{bmatrix}\begin{bmatrix}U^{\rm H}\\ V^{\rm H}\end{bmatrix},\qquad(33)$$

and the associated minimum $\sigma(X,Z,Y,W;\tilde{A})$ is

$$\begin{aligned}&\sigma(X,Z,Y,W;\tilde{A})=(\|F_{21}^{\rm H}(P^{\rm H}\tilde{A}U-A_{PU}^{(0)})\|^2\\ &+\|F_{22}^{\rm H}(P^{\rm H}\tilde{A}U-A_{PU}^{(0)})F_{11}\|^2\\ &+\|F_{41}^{\rm H}(Q^{\rm H}\tilde{A}V-A_{QV}^{(0)})\|^2+\|F_{42}^{\rm H}(Q^{\rm H}\tilde{A}V-A_{QV}^{(0)})F_{31}\|^2\\ &+\|P^{\rm H}\tilde{A}V\|^2+\|Q^{\rm H}\tilde{A}U\|^2)^{1/2},\end{aligned}\qquad(34)$$

where

$$\hat{A}_{11}=F_{22}F_{22}^{\rm H}P^{\rm H}\tilde{A}UF_{12}F_{12}^{\rm H},$$
$$\hat{A}_{22}=F_{42}F_{42}^{\rm H}Q^{\rm H}\tilde{A}VF_{32}F_{32}^{\rm H}.$$

Now, we apply Theorem 1 and Theorem 2 to solve the left and right inverse eigenproblem for generalized reflexive matrices and its optimal approximation, i.e., given partial left and right eigenpairs (eigenvalues and corresponding eigenvectors) $(\omega_i,y_i)(i=1,\cdots,q)$, $(\lambda_j,x_j)(j=1,\cdots,p)$ and a matrix $\tilde{A}\in\mathbf{C}^{n\times n}$, find an $n\times n$ generalized reflexive matrix A such that

$$Ax_j=\lambda_jx_j\ (j=1,\cdots,p),\ y_i^{\rm H}A=\omega_iy_i^{\rm H}\ (i=1,\cdots,q),\qquad(35)$$

and a matrix $\hat{A}\in\rho(X,\Lambda,Y,\Omega)$ such that

$$\|\tilde{A}-\hat{A}\|=\min_{A\in\rho(X,\Lambda,Y,\Omega)}\|\tilde{A}-A\|,\qquad(36)$$

where $\rho(X,\Lambda,Y,\Omega)=\{A\in\rho\,|AX=X\Lambda, Y^{\rm H}A=\Omega Y^{\rm H}\}$, ρ denotes the set of $n\times n$ generalized reflexive matrices, $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_p),\Omega=\mathrm{diag}(\omega_1,\cdots,\omega_q)$, $X=[x_1,\cdots,x_p]$, $Y=[y_1,\cdots,y_q]$.

From Theorem 1 and Theorem 2, we obtain the following corollary.

Corollary 3: Let $X\in\mathbf{C}^{n\times p}, Y\in\mathbf{C}^{n\times q}, \tilde{A}\in\mathbf{C}^{n\times n}, \Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_p), \Omega=\mathrm{diag}(\omega_1,\cdots,\omega_q)$ and $\rho(X,\Lambda,Y,\Omega)=\{A\in\rho\,|AX=X\Lambda, Y^{\rm H}A=\Omega Y^{\rm H}\}$, where ρ denotes the set of $n\times n$ generalized reflexive matrices. Suppose that the SVDs of the matrices $U^{\rm H}X, P^{\rm H}Y, V^{\rm H}X$ and $Q^{\rm H}Y$ are given by (18) and (19), then the set $\rho(X,\Lambda,Y,\Omega)$ is nonempty if and only if

$$P^{\rm H}X\Lambda(U^{\rm H}X)^+U^{\rm H}X=P^{\rm H}X\Lambda,$$
$$U^{\rm H}Y\Omega^{\rm H}(P^{\rm H}Y)^+P^{\rm H}Y=U^{\rm H}Y\Omega^{\rm H},$$
$$Y^{\rm H}PP^{\rm H}X\Lambda=\Omega Y^{\rm H}UU^{\rm H}X$$

and

$$Q^{\rm H}X\Lambda(V^{\rm H}X)^+V^{\rm H}X=Q^{\rm H}X\Lambda,$$
$$V^{\rm H}Y\Omega^{\rm H}(Q^{\rm H}Y)^+Q^{\rm H}Y=V^{\rm H}Y\Omega^{\rm H},$$
$$Y^{\rm H}QQ^{\rm H}X\Lambda=\Omega Y^{\rm H}VV^{\rm H}X,$$

in which case, the set $\rho(X,\Lambda,Y,\Omega)$ can be expressed as

$$\rho(X,\Lambda,Y,\Omega)=\{A=[P,Q]\begin{bmatrix}A_{PU}^{(0)}+F_{22}MF_{12}^{\rm H} & 0\\ 0 & A_{QV}^{(0)}+F_{42}NF_{32}^{\rm H}\end{bmatrix}\begin{bmatrix}U^{\rm H}\\ V^{\rm H}\end{bmatrix}\Bigg\},$$

where

$$A_{PU}^{(0)}=P^{\rm H}X\Lambda(U^{\rm H}X)^++(Y^{\rm H}P)^+\Omega Y^{\rm H}UF_{12}F_{12}^{\rm H},$$
$$A_{QV}^{(0)}=Q^{\rm H}X\Lambda(V^{\rm H}X)^++(Y^{\rm H}Q)^+\Omega Y^{\rm H}VF_{32}F_{32}^{\rm H}$$

and $M\in\mathbf{C}^{(r-t_2)\times(k-t_1)}$, $N\in\mathbf{C}^{(s-t_4)\times(l-t_3)}$ are arbitrary matrices, and there is a unique matrix $\hat{A}\in\rho(X,\Lambda,Y,\Omega)$ such that $\|\tilde{A}-\hat{A}\|=\min_{A\in\rho(X,\Lambda,Y,\Omega)}\|\tilde{A}-A\|$. Moreover, $\hat{A}$ can be expressed as

$$\hat{A}=[P,Q]\begin{bmatrix}A_{PU}^{(0)}+\hat{A}_{11} & 0\\ 0 & A_{QV}^{(0)}+\hat{A}_{22}\end{bmatrix}\begin{bmatrix}U^{\rm H}\\ V^{\rm H}\end{bmatrix},$$

where

$$\hat{A}_{11}=F_{22}F_{22}^{\rm H}P^{\rm H}\tilde{A}UF_{12}F_{12}^{\rm H},$$
$$\hat{A}_{22}=F_{42}F_{42}^{\rm H}Q^{\rm H}\tilde{A}VF_{32}F_{32}^{\rm H}.$$

If $m=n, R=S, Y=0$, then $P=U, Q=V$. From Corollary 3, we obtain the following corollary related to the inverse eigenvalue problem for reflexive matrices and its optimal approximation. For more details, see [27].

Corollary 4: Let $X\in\mathbf{C}^{n\times p}, \tilde{A}\in\mathbf{C}^{n\times n}, \Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_p)$, and $\rho(X,\Lambda)=\{A\in\rho\,|AX=X\Lambda\}$, where ρ denotes the set of $n\times n$ reflexive matrices. Suppose that the SVDs of the matrices $U^{\rm H}X, V^{\rm H}X$ are given by (18) and (19), $t_1=\mathrm{rank}(U^{\rm H}X)$, $t_3=\mathrm{rank}(V^{\rm H}X)$, then the set $\rho(X,\Lambda)$ is nonempty if and only if

$$U^{\rm H}X\Lambda(U^{\rm H}X)^+U^{\rm H}X=U^{\rm H}X\Lambda,$$
$$V^{\rm H}X\Lambda(V^{\rm H}X)^+V^{\rm H}X=V^{\rm H}X\Lambda,$$

in which case, the set $\rho(X,\Lambda)$ can be expressed as

$$\rho(X,\Lambda)=\{A=[U,V]\begin{bmatrix}A_U^{(0)}+MF_{12}^{\rm H} & 0\\ 0 & A_V^{(0)}+NF_{32}^{\rm H}\end{bmatrix}\begin{bmatrix}U^{\rm H}\\ V^{\rm H}\end{bmatrix}\Bigg\},$$

where

$$A_U^{(0)}=U^{\rm H}X\Lambda(U^{\rm H}X)^+, A_V^{(0)}=V^{\rm H}X\Lambda(V^{\rm H}X)^+,$$

and $M\in\mathbf{C}^{r\times(r-t_1)}$, $N\in\mathbf{C}^{(n-r)\times(n-r-t_3)}$ are arbitrary matrices, and there is a unique matrix $\hat{A}\in\rho(X,\Lambda)$ such that $\|\tilde{A}-\hat{A}\|=\min_{A\in\rho(X,\Lambda)}\|\tilde{A}-A\|$. Moreover, $\hat{A}$ can be expressed as

$$\hat{A}=[U,V]\begin{bmatrix}A_U^{(0)}+\hat{A}_1 & 0\\ 0 & A_V^{(0)}+\hat{A}_2\end{bmatrix}\begin{bmatrix}U^{\rm H}\\ V^{\rm H}\end{bmatrix}.$$

where $\hat{A}_1=U^{\rm H}\tilde{A}UF_{12}F_{12}^{\rm H}$, $\hat{A}_2=V^{\rm H}\tilde{A}VF_{32}F_{32}^{\rm H}$.

IV. Minimization Problem for Generalized Anti-reflexive Matrices

In this section ρ denotes the set of $m\times n$ generalized anti-reflexive matrices.

For the given matrices $X\in \mathbf{C}^{n\times p}, Z\in \mathbf{C}^{m\times p}, Y\in \mathbf{C}^{m\times q}$ and $W\in \mathbf{C}^{n\times q}$, if $A\in\rho$, it follows from Lemma 1 that

$$\begin{aligned} & \min_{A\in\rho}(\|AX-Z\|^2+\|Y^{\mathrm{H}}A-W^{\mathrm{H}}\|^2)\\ = & \min_{A_{PV}\in\mathbf{C}^{r\times l},A_{QU}\in\mathbf{C}^{s\times k}}\\ & \left(\left\|[P,Q]\begin{bmatrix}0 & A_{PV}\\ A_{QU} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix}X-Z\right\|^2\right.\\ & \left.+\left\|Y^{\mathrm{H}}[P,Q]\begin{bmatrix}0 & A_{PV}\\ A_{QU} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix}-W^{\mathrm{H}}\right\|^2\right)\\ = & \min_{A_{PV}\in\mathbf{C}^{r\times l}}(\|A_{PV}(V^{\mathrm{H}}X)-(P^{\mathrm{H}}Z)\|^2\\ & +\|(P^{\mathrm{H}}Y)^{\mathrm{H}}A_{PV}-(V^{\mathrm{H}}W)^{\mathrm{H}}\|^2)\\ & +\min_{A_{QU}\in\mathbf{C}^{s\times k}}(\|A_{QU}(U^{\mathrm{H}}X)-(Q^{\mathrm{H}}Z)\|^2\\ & +\|(Q^{\mathrm{H}}Y)^{\mathrm{H}}A_{QU}-(U^{\mathrm{H}}W)^{\mathrm{H}}\|^2).\end{aligned} \tag{37}$$

Applying Lemma 2 to (37), we obtain the following theorem.

Theorem 5: Let $X\in\mathbf{C}^{n\times p}, Z\in\mathbf{C}^{m\times p}, Y\in\mathbf{C}^{m\times q}$ and $W\in\mathbf{C}^{n\times q}$, ρ be the set of $m\times n$ generalized anti-reflexive matrices. Suppose that the SVDs of matrices $U^{\mathrm{H}}X, P^{\mathrm{H}}Y, V^{\mathrm{H}}X$ and $Q^{\mathrm{H}}Y$ are given by (18) and (19), then

$$\begin{aligned} & \sigma(X,Z,Y,W)=\min_{A\in\rho}(\|AX-Z\|^2\\ & +\|Y^{\mathrm{H}}A-W^{\mathrm{H}}\|^2)^{1/2}\\ = & (\|P^{\mathrm{H}}ZG_{32}\|^2+\|V^{\mathrm{H}}WG_{22}\|^2\\ & +\|\Psi_1*(\Sigma_2F_{21}^{\mathrm{H}}P^{\mathrm{H}}ZG_{31}-G_{21}^{\mathrm{H}}W^{\mathrm{H}}VF_{31}\Sigma_3)\|^2\\ & +\|Q^{\mathrm{H}}ZG_{12}\|^2+\|U^{\mathrm{H}}WG_{42}\|^2\\ & +\|\Psi_2*(\Sigma_4F_{41}^{\mathrm{H}}Q^{\mathrm{H}}ZG_{11}-G_{41}^{\mathrm{H}}W^{\mathrm{H}}UF_{11}\Sigma_1)\|^2)^{1/2},\end{aligned}$$

where

$$\Psi_1=[\psi_{ij}^{(1)}]\in\mathbf{C}^{t_2\times t_3}, \psi_{ij}^{(1)}=\frac{1}{((\alpha_i^{(2)})^2+(\alpha_j^{(3)})^2)^{1/2}},\quad 1\le i\le t_2, 1\le j\le t_3;$$

$$\Psi_2=[\psi_{ij}^{(2)}]\in\mathbf{C}^{t_4\times t_1}, \psi_{ij}^{(2)}=\frac{1}{((\alpha_i^{(4)})^2+(\alpha_j^{(1)})^2)^{1/2}},\quad 1\le i\le t_4, 1\le j\le t_1,$$

and $A\in\rho$ attains this minimum if and only if

$$A=A_0+[P,Q]\begin{bmatrix}0 & F_{22}MF_{32}^{\mathrm{H}}\\ F_{42}NF_{12}^{\mathrm{H}} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix} \tag{38}$$

with arbitrary $M\in\mathbf{C}^{(r-t_2)\times(l-t_3)}, N\in\mathbf{C}^{(s-t_4)\times(k-t_1)}$ and

$$A_0=[P,Q]\begin{bmatrix}0 & A_{PV}^{(0)}\\ A_{QU}^{(0)} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix},$$

$$A_{PV}^{(0)}=F_2\begin{bmatrix}K_3*C_{11} & \Sigma_2^{-1}G_{21}^{\mathrm{H}}W^{\mathrm{H}}VF_{32}\\ F_{22}^{\mathrm{H}}P^{\mathrm{H}}ZG_{31}\Sigma_3^{-1} & 0\end{bmatrix}F_3^{\mathrm{H}}, \tag{39}$$

$$A_{QU}^{(0)}=F_4\begin{bmatrix}K_4*D_{11} & \Sigma_4^{-1}G_{41}^{\mathrm{H}}W^{\mathrm{H}}UF_{12}\\ F_{42}^{\mathrm{H}}Q^{\mathrm{H}}ZG_{11}\Sigma_1^{-1} & 0\end{bmatrix}F_1^{\mathrm{H}}, \tag{40}$$

where

$$C_{11}=F_{21}^{\mathrm{H}}P^{\mathrm{H}}ZG_{31}\Sigma_3+\Sigma_2G_{21}^{\mathrm{H}}W^{\mathrm{H}}VF_{31},$$

$$D_{11}=F_{41}^{\mathrm{H}}Q^{\mathrm{H}}ZG_{11}\Sigma_1+\Sigma_4G_{41}^{\mathrm{H}}W^{\mathrm{H}}UF_{11},$$

$K_3=\Psi_1*\Psi_1, K_4=\Psi_2*\Psi_2$. Moreover, the equations $AX=Z$ and $Y^{\mathrm{H}}A=W^{\mathrm{H}}$ have a common solution $A\in\rho$ if and only if

$$P^{\mathrm{H}}Z(V^{\mathrm{H}}X)^+V^{\mathrm{H}}X=P^{\mathrm{H}}Z,$$
$$V^{\mathrm{H}}W(P^{\mathrm{H}}Y)^+P^{\mathrm{H}}Y=V^{\mathrm{H}}W,$$
$$Y^{\mathrm{H}}PP^{\mathrm{H}}Z=W^{\mathrm{H}}VV^{\mathrm{H}}X,$$

and

$$Q^{\mathrm{H}}Z(U^{\mathrm{H}}X)^+U^{\mathrm{H}}X=Q^{\mathrm{H}}Z,$$
$$U^{\mathrm{H}}W(Q^{\mathrm{H}}Y)^+Q^{\mathrm{H}}Y=U^{\mathrm{H}}W,$$
$$Y^{\mathrm{H}}QQ^{\mathrm{H}}Z=W^{\mathrm{H}}UU^{\mathrm{H}}X,$$

in which case a general solution of the equations $AX=Z$ and $Y^{\mathrm{H}}A=W^{\mathrm{H}}$ can be expressed as

$$A=[P,Q]\begin{bmatrix}0 & A_{PV}^{(0)}+F_{22}MF_{32}^{\mathrm{H}}\\ A_{QU}^{(0)}+F_{42}NF_{12}^{\mathrm{H}} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix}$$

with arbitrary $M\in\mathbf{C}^{(r-t_2)\times(l-t_3)}, N\in\mathbf{C}^{(s-t_4)\times(k-t_1)}$ and

$$A_{PV}^{(0)}=P^{\mathrm{H}}Z(V^{\mathrm{H}}X)^++(Y^{\mathrm{H}}P)^+W^{\mathrm{H}}VF_{32}F_{32}^{\mathrm{H}},$$
$$A_{QU}^{(0)}=Q^{\mathrm{H}}Z(U^{\mathrm{H}}X)^++(Y^{\mathrm{H}}Q)^+W^{\mathrm{H}}UF_{12}F_{12}^{\mathrm{H}}.$$

Let $r_1=r-t_2, k_1=k-t_1, s_1=s-t_4$ and $l_1=l-t_3$. From Theorem 5 we know

$$\begin{aligned}&\rho(X,Z,Y,W)=\\ &\left\{A=A_0+[P,Q]\begin{bmatrix}0 & F_{22}MF_{32}^{\mathrm{H}}\\ F_{42}NF_{12}^{\mathrm{H}} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix}\right|\\ & \left. M\in C^{r_1\times l_1}, N\in C^{s_1\times k_1}\right\},\end{aligned} \tag{41}$$

Similar to Theorem 2, we have the following result.

Theorem 6: Let $X\in\mathbf{C}^{n\times p}, Z\in\mathbf{C}^{m\times p}, Y\in\mathbf{C}^{m\times q}, W\in\mathbf{C}^{n\times q}$, $\tilde{A}\in\mathbf{C}^{m\times n}$, and let ρ be the set of $m\times n$ generalized anti-reflexive matrices. Suppose that the SVDs of matrices $U^{\mathrm{H}}X, P^{\mathrm{H}}Y, V^{\mathrm{H}}X$ and $Q^{\mathrm{H}}Y$ are given by (18) and (19), then Problem II has a unique solution $\hat{A}$ in $\rho(X,Z,Y,W)$. Moreover, $\hat{A}$ can be expressed as

$$\hat{A}=A_0+[P,Q]\begin{bmatrix}0 & \hat{A}_{12}\\ \hat{A}_{21} & 0\end{bmatrix}\begin{bmatrix}U^{\mathrm{H}}\\ V^{\mathrm{H}}\end{bmatrix},$$

where

$$\hat{A}_{12}=F_{22}F_{22}^{\mathrm{H}}P^{\mathrm{H}}\tilde{A}VF_{32}F_{32}^{\mathrm{H}},$$
$$\hat{A}_{21}=F_{42}F_{42}^{\mathrm{H}}Q^{\mathrm{H}}\tilde{A}UF_{12}F_{12}^{\mathrm{H}}.$$

In this case, the associated minimum $\sigma(X,Z,Y,W;\tilde{A})$ is

$$\begin{aligned}&\sigma(X,Z,Y,W;\tilde{A})=(\|F_{21}^{\mathrm{H}}(P^{\mathrm{H}}\tilde{A}V-A_{PV}^{(0)})\|^2\\ &+\|F_{22}^{\mathrm{H}}(P^{\mathrm{H}}\tilde{A}V-A_{PV}^{(0)})F_{31}\|^2\\ &+\|F_{41}^{\mathrm{H}}(Q^{\mathrm{H}}\tilde{A}U-A_{QU}^{(0)})\|^2\\ &+\|F_{42}^{\mathrm{H}}(Q^{\mathrm{H}}\tilde{A}U-A_{QU}^{(0)})F_{11}\|^2\\ &+\|P^{\mathrm{H}}\tilde{A}U\|^2+\|Q^{\mathrm{H}}\tilde{A}V\|^2)^{1/2}.\end{aligned}$$

From Theorem 5 and Theorem 6, we get the following result related to the left and right inverse eigenproblem for generalized anti-reflexive matrices and its optimal approximation.

Corollary 7: Let $X \in \mathbf{C}^{n\times p}, Y \in \mathbf{C}^{n\times q}, \tilde{A} \in \mathbf{C}^{n\times n}, \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_p), \Omega = \mathrm{diag}(\omega_1, \cdots, \omega_q)$ and $\rho(X, \Lambda, Y, \Omega) = \{A \in \rho \,\big|\, AX = X\Lambda, Y^{\mathrm{H}}A = \Omega Y^{\mathrm{H}}\}$, where ρ denotes the set of $n \times n$ generalized anti-reflexive matrices. Suppose that the SVDs of the matrices $U^{\mathrm{H}}X, P^{\mathrm{H}}Y, V^{\mathrm{H}}X$ and $Q^{\mathrm{H}}Y$ are given by (18) and (19), then the set $\rho(X, \Lambda, Y, \Omega)$ is nonempty if and only if

$$P^{\mathrm{H}}X\Lambda(V^{\mathrm{H}}X)^{+}V^{\mathrm{H}}X = P^{\mathrm{H}}X\Lambda,$$
$$V^{\mathrm{H}}Y\Omega^{\mathrm{H}}(P^{\mathrm{H}}Y)^{+}P^{\mathrm{H}}Y = V^{\mathrm{H}}Y\Omega^{\mathrm{H}},$$
$$Y^{\mathrm{H}}PP^{\mathrm{H}}X\Lambda = \Omega Y^{\mathrm{H}}VV^{\mathrm{H}}X,$$

and

$$Q^{\mathrm{H}}X\Lambda(U^{\mathrm{H}}X)^{+}U^{\mathrm{H}}X = Q^{\mathrm{H}}X\Lambda,$$
$$U^{\mathrm{H}}Y\Omega^{\mathrm{H}}(Q^{\mathrm{H}}Y)^{+}Q^{\mathrm{H}}Y = U^{\mathrm{H}}Y\Omega^{\mathrm{H}},$$
$$Y^{\mathrm{H}}QQ^{\mathrm{H}}X\Lambda = \Omega Y^{\mathrm{H}}UU^{\mathrm{H}}X,$$

in which case, the set $\rho(X, \Lambda, Y, \Omega)$ can be expressed as

$$\rho(X, \Lambda, Y, \Omega) = \{A = [P, Q] \left[\begin{array}{cc} 0 & A_{PV}^{(0)} + F_{22}MF_{32}^{\mathrm{H}} \\ A_{QU}^{(0)} + F_{42}NF_{12}^{\mathrm{H}} & 0 \end{array}\right] \left[\begin{array}{c} U^{\mathrm{H}} \\ V^{\mathrm{H}} \end{array}\right] \right\},$$

where $M \in \mathbf{C}^{(r-t_2)\times(l-t_3)}, N \in \mathbf{C}^{(s-t_4)\times(k-t_1)}$ are arbitrary matrices and

$$A_{PV}^{(0)} = P^{\mathrm{H}}X\Lambda(V^{\mathrm{H}}X)^{+} + (Y^{\mathrm{H}}P)^{+}\Omega Y^{\mathrm{H}}VF_{32}F_{32}^{\mathrm{H}},$$
$$A_{QU}^{(0)} = Q^{\mathrm{H}}X\Lambda(U^{\mathrm{H}}X)^{+} + (Y^{\mathrm{H}}Q)^{+}\Omega Y^{\mathrm{H}}UF_{12}F_{12}^{\mathrm{H}},$$

and there exists a unique matrix $\hat{A} \in \rho(X, \Lambda, Y, \Omega)$ such that $\|\tilde{A} - \hat{A}\| = \min_{A\in\rho(X,\Lambda,Y,\Omega)} \|\tilde{A} - A\|$. Moreover, $\hat{A}$ can be expressed as

$$\hat{A} = [P, Q] \left[\begin{array}{cc} 0 & A_{PV}^{(0)} + \hat{A}_{12} \\ A_{QU}^{(0)} + \hat{A}_{21} & 0 \end{array}\right] \left[\begin{array}{c} U^{\mathrm{H}} \\ V^{\mathrm{H}} \end{array}\right],$$

where

$$\hat{A}_{12} = F_{22}F_{22}^{\mathrm{H}}P^{\mathrm{H}}\tilde{A}VF_{32}F_{32}^{\mathrm{H}},$$
$$\hat{A}_{21} = F_{42}F_{42}^{\mathrm{H}}Q^{\mathrm{H}}\tilde{A}UF_{12}F_{12}^{\mathrm{H}}.$$

In addition, if $m = n, R = S, Y = 0$, similar to Corollary 4, we can obtain the result related to the inverse eigenvalue problem for anti-reflexive matrices and its optimal approximation from Corollary 7.

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