

# Minimization problems for generalized reflexive and generalized anti-reflexive matrices

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**Abstract**—Let  $R \in \mathbb{C}^{m \times m}$  and  $S \in \mathbb{C}^{n \times n}$  be nontrivial unitary involutions, i.e.,  $R^H = R = R^{-1} \neq \pm I_m$  and  $S^H = S = S^{-1} \neq \pm I_n$ .  $A \in \mathbb{C}^{m \times n}$  is said to be a generalized reflexive (anti-reflexive) matrix if  $RAS = A$  ( $RAS = -A$ ). Let  $\rho$  be the set of  $m \times n$  generalized reflexive (anti-reflexive) matrices. Given  $X \in \mathbb{C}^{n \times p}$ ,  $Z \in \mathbb{C}^{m \times p}$ ,  $Y \in \mathbb{C}^{m \times q}$  and  $W \in \mathbb{C}^{n \times q}$ , we characterize the matrices  $A$  in  $\rho$  that minimize  $\|AX - Z\|^2 + \|Y^H A - W^H\|^2$ , and, given an arbitrary  $\tilde{A} \in \mathbb{C}^{m \times n}$ , we find a unique matrix among the minimizers of  $\|AX - Z\|^2 + \|Y^H A - W^H\|^2$  in  $\rho$  that minimizes  $\|A - \tilde{A}\|$ . We also obtain sufficient and necessary conditions for existence of  $A \in \rho$  such that  $AX = Z$ ,  $Y^H A = W^H$ , and characterize the set of all such matrices  $A$  if the conditions are satisfied. These results are applied to solve a class of left and right inverse eigenproblems for generalized reflexive (anti-reflexive) matrices.

**Keywords**—approximation, generalized reflexive matrix, generalized anti-reflexive matrix, inverse eigenvalue problem.

## I. INTRODUCTION

IN this paper we shall adopt the following notation.  $\mathbb{C}^{m \times n}$  denotes the set of all  $m \times n$  complex matrices,  $\mathbb{U}^{n \times n}$  denotes the set of all unitary matrices in  $\mathbb{C}^{n \times n}$ .  $A^H$ ,  $A^+$  and  $\|A\|$  stand for the conjugate transpose, the Moore-Penrose generalized inverse and the Frobenius norm of a complex matrix  $A$ , respectively. For  $A, B \in \mathbb{C}^{m \times n}$ , an inner product in  $\mathbb{C}^{m \times n}$  is defined by  $\langle A, B \rangle = \text{trace}(B^H A)$ , then  $\mathbb{C}^{m \times n}$  is a Hilbert space. The matrix norm  $\|\cdot\|$  induced by the inner product is the Frobenius norm.  $I_n$  represents the identity matrix of order  $n$ . For  $A = (a_{ij})$ ,  $B = (b_{ij}) \in \mathbb{C}^{m \times n}$ ,  $A * B$  represents the Hadamard product of the matrices  $A$  and  $B$ , i.e.,  $A * B = (a_{ij}b_{ij}) \in \mathbb{C}^{m \times n}$ .

Throughout this paper  $R \in \mathbb{C}^{m \times m}$  and  $S \in \mathbb{C}^{n \times n}$  are nontrivial unitary involutions, i.e.,  $R^H = R = R^{-1} \neq \pm I_m$  and  $S^H = S = S^{-1} \neq \pm I_n$ . We say that  $A \in \mathbb{C}^{m \times n}$  is a generalized reflexive (anti-reflexive) matrix (see [9]) if  $RAS = A$  ( $RAS = -A$ ). If  $m = n$ ,  $R = S$ , then the generalized reflexive (anti-reflexive) matrices reduce to the reflexive (anti-reflexive) matrices (see, e.g., [8]). Let  $J_n = (j_{i,k})$  represent the exchange matrix of order  $n$  defined by  $j_{i,k} = \delta_{i,n-k+1}$  for  $1 \leq i, k \leq n$ , where  $\delta_{i,k}$  is the Kronecker delta, i.e.,  $J_n$  is a matrix with ones on the secondary diagonal and zeros elsewhere. By taking  $m = n$ ,  $R = S = J_n$ , then the generalized reflexive (anti-reflexive) matrices reduce to the centrosymmetric (centroskew) matrices (see [25]) which play an important role in many areas [10, 13, 14]. Therefore, centrosymmetric (centroskew) matrices, whose special properties

have been under extensive study [1, 2, 7, 15, 19, 20, 26], are the special cases of generalized reflexive (anti-reflexive) matrices. Chen [9] discussed applications that give rise to the generalized reflexive (anti-reflexive) matrices and considered least squares problems involving them.

In the following  $\rho$  is either the set of  $m \times n$  generalized reflexive matrices or the set of  $m \times n$  generalized anti-reflexive matrices. We consider the following problems.

**Problem I.** Given  $X \in \mathbb{C}^{n \times p}$ ,  $Z \in \mathbb{C}^{m \times p}$ ,  $Y \in \mathbb{C}^{m \times q}$  and  $W \in \mathbb{C}^{n \times q}$ , find

$$\sigma(X, Z, Y, W) = \min_{A \in \rho} (\|AX - Z\|^2 + \|Y^H A - W^H\|^2)^{1/2},$$

and characterize the set

$$\begin{aligned} \rho(X, Z, Y, W) &= \{A \in \rho \mid (\|AX - Z\|^2 \\ &\quad + \|Y^H A - W^H\|^2)^{1/2} = \sigma(X, Z, Y, W)\}. \end{aligned}$$

**Problem II.** Given  $\tilde{A} \in \mathbb{C}^{m \times n}$ , find

$$\sigma(X, Z, Y, W; \tilde{A}) = \min_{A \in \rho(X, Z, Y, W)} \|A - \tilde{A}\|,$$

and find  $\hat{A} \in \rho(X, Z, Y, W)$  such that

$$\|\hat{A} - \tilde{A}\| = \sigma(X, Z, Y, W; \tilde{A}).$$

If  $m = n$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,  $\Omega = \text{diag}(\omega_1, \dots, \omega_q)$ ,  $X = [x_1, \dots, x_p]$ ,  $Y = [y_1, \dots, y_q]$ , where  $x_i \in \mathbb{C}^n$  ( $i = 1, \dots, p$ ),  $y_j \in \mathbb{C}^n$  ( $j = 1, \dots, q$ ),  $Z = X\Lambda$ ,  $W^H = \Omega Y^H$ , then the set  $\rho(X, Z, Y, W)$  in Problem I is determined by partial left and right eigenpairs  $(\omega_j, y_j)$  ( $j = 1, \dots, q$ ) and  $(\lambda_i, x_i)$  ( $i = 1, \dots, p$ ), and Problem I is a left and right inverse eigenproblem for generalized reflexive (anti-reflexive) matrices. The left and right inverse eigenproblem is a special inverse eigenvalue problem, indeed, the recursive inverse eigenvalue problem (see [17]). Problem II is an optimal approximation problem under spectral constraint.

There are many publications (see, e.g., [5, 6, 11, 12, 16, 24, 29] and their references) concerning minimization problems for matrices. Recently, Bai and Chan [4] considered inverse eigenproblems related to centrosymmetric and centroskew matrices, where  $m = n$ ,  $R = S = J_n$ , and  $Y = 0$ ,  $W = 0$ . Zhou et al. [28] discussed the minimization problems for centrosymmetric matrices, where  $m = n$ ,  $R = S = J_n$ ,  $Y = 0$  and  $W = 0$ . Peng and Hu [18] studied the existence of  $n \times n$  reflexive and anti-reflexive matrices  $X$  such that  $AX = B$ , where  $A$  and  $B$  are given in  $\mathbb{C}^{m \times n}$ , and the nearest matrix to a given matrix. Trench [21] considered the minimization problems for hermitian, hermitian reflexive and hermitian anti-reflexive matrices, where  $m = n$ ,  $Y = 0$  and  $W = 0$ . In

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[22] and [23], Trench has studied inverse eigenproblems for generalized symmetric or skew symmetric matrices and the minimization problems for  $(R, S)$ -symmetric and  $(R, S)$ -skew symmetric matrices, respectively.

In this paper we obtain explicit formulas for  $\sigma(X, Z, Y, W)$ ,  $\sigma(X, Z, Y, W; \tilde{A})$ , all matrices in  $\rho(X, Z, Y, W)$ , and the solution of Problem II. As a byproduct of our results on Problem I we obtain necessary and sufficient conditions on  $X, Z, Y$  and  $W$  for existence of  $A \in \rho$  such that  $AX = Z$  and  $Y^H A = W^H$ , and an explicit formula for all such  $A$ . These results are applied to solve a class of left and right inverse eigenproblems for generalized reflexive (anti-reflexive) matrices.

If  $m = n, R = S, Y = 0$  and  $W = 0$ , our results apply to reflexive and anti-reflexive matrices, which are discussed in [18, 27]. In particular, if  $m = n, R = S = J_n, Y = 0$  and  $W = 0$ , our results apply to centrosymmetric and centroskew matrices, which are considered in [4, 28].

## II. PRELIMINARY CONSIDERATIONS

If  $\lambda$  is an eigenvalue of  $E \in \mathbf{C}^{m \times m}$ , let  $V_E(\lambda)$  denote the eigenspace of  $E$  corresponding to the eigenvalue  $\lambda$ . A vector  $z \in \mathbf{C}^m$  is said to be  $R$ -symmetric ( $R$ -skew symmetric) if  $Rz = z$  ( $Rz = -z$ ); thus,  $V_R(1)$  and  $V_R(-1)$  are the subspaces of  $\mathbf{C}^{m \times m}$  consisting respectively of  $R$ -symmetric and  $R$ -skew symmetric vectors. Let  $r = \dim[V_R(1)], s = \dim[V_R(-1)]$ . Since a unitary involution is diagonalizable and  $R \neq \pm I_m$ , then  $r, s \geq 1$ , and  $r + s = m$ . Let  $\{p_1, \dots, p_r\}$  and  $\{q_1, \dots, q_s\}$  be the orthonormal bases for  $V_R(1)$  and  $V_R(-1)$ , respectively, and define

$$P = [p_1, \dots, p_r] \in \mathbf{C}^{m \times r}, Q = [q_1, \dots, q_s] \in \mathbf{C}^{m \times s},$$

then  $[P, Q]$  is a unitary matrix and  $R$  has the following spectral decomposition

$$R = [P, Q] \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix}. \quad (1)$$

In particular, if  $R = J_{2k}$ , then  $r = s = k$ , we can take

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k \\ J_k \end{bmatrix}, Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k \\ -J_k \end{bmatrix}.$$

If  $R = J_{2k+1}$ , then  $r = k + 1, s = k$ , we can take

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 \\ 0 & \sqrt{2} \\ J_k & 0 \end{bmatrix}, Q = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k \\ 0 \\ -J_k \end{bmatrix}.$$

Similarly, there are positive integers  $k$  and  $l$  such that  $k + l = n$ , and the matrices  $U \in \mathbf{C}^{n \times k}$  and  $V \in \mathbf{C}^{n \times l}$  whose column vectors form the orthonormal bases for the eigenspaces  $V_S(1)$  and  $V_S(-1)$ , respectively. Thus,  $[U, V]$  is a unitary matrix and  $S$  has the spectral decomposition:

$$S = [U, V] \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}. \quad (2)$$

In the following  $P, Q, U, V$  are always defined by (1) and (2).

(1) and (2) yield the following characterizations of  $m \times n$  generalized reflexive or generalized anti-reflexive matrices.

**Lemma 1:** (a)  $A \in \mathbf{C}^{m \times n}$  is a generalized reflexive matrix if and only if there exist  $A_{PU} \in \mathbf{C}^{r \times k}, A_{QV} \in \mathbf{C}^{s \times l}$  such that

$$A = [P, Q] \begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}. \quad (3)$$

(b)  $A \in \mathbf{C}^{m \times n}$  is a generalized anti-reflexive matrix if and only if there exist  $A_{PV} \in \mathbf{C}^{r \times l}, A_{QU} \in \mathbf{C}^{s \times k}$  such that

$$A = [P, Q] \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}. \quad (4)$$

**Proof.** (a) If  $A \in \mathbf{C}^{m \times n}$  is a generalized reflexive matrix, then it follows from  $RAS = A$ , (1) and (2) that  $P^H AV = 0, Q^H AU = 0$ . Let  $A_{PU} = P^H AU, A_{QV} = Q^H AV$ , then  $A_{PU} \in \mathbf{C}^{r \times k}, A_{QV} \in \mathbf{C}^{s \times l}$ , and

$$\begin{aligned} A = RAS &= [P, Q] \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix} \\ &A[U, V] \begin{bmatrix} I_k & 0 \\ 0 & -I_l \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \\ &= [P, Q] \begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}. \end{aligned}$$

The verification of the converse is straightforward. Similarly, (b) is may proved.

In order to solve Problem I, we will need the following lemma.

**Lemma 2:** Let  $\tilde{X} \in \mathbf{C}^{m \times p}, \tilde{W} \in \mathbf{C}^{m \times t}, \tilde{Y} \in \mathbf{C}^{n \times t}, \tilde{Z} \in \mathbf{C}^{n \times p}$ , and the singular value decompositions(SVDs) of the matrices  $\tilde{X}$  and  $\tilde{Y}$  be, respectively

$$\tilde{X} = \tilde{U} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}^H, \tilde{Y} = \tilde{P} \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \tilde{Q}^H, \quad (5)$$

where  $\tilde{U} = [\tilde{U}_1, \tilde{U}_2] \in \mathbf{UC}^{m \times m}, \tilde{V} = [\tilde{V}_1, \tilde{V}_2] \in \mathbf{UC}^{p \times p}, \tilde{P} = [\tilde{P}_1, \tilde{P}_2] \in \mathbf{UC}^{n \times n}, \tilde{Q} = [\tilde{Q}_1, \tilde{Q}_2] \in \mathbf{UC}^{t \times t}, \Delta = \text{diag}\{\delta_1, \dots, \delta_e\} > 0, \Gamma = \text{diag}\{\gamma_1, \dots, \gamma_f\} > 0, e = \text{rank}(\tilde{X}), f = \text{rank}(\tilde{Y}), \tilde{U}_1 \in \mathbf{C}^{m \times e}, \tilde{V}_1 \in \mathbf{C}^{p \times e}, \tilde{P}_1 \in \mathbf{C}^{n \times f}, \tilde{Q}_1 \in \mathbf{C}^{t \times f}$ , and let

$$\Phi_1 = [\varphi_{ij}] \in \mathbf{C}^{f \times e},$$

$$\varphi_{ij} = \frac{1}{(\gamma_i^2 + \delta_j^2)^{1/2}}, 1 \leq i \leq f, 1 \leq j \leq e,$$

then

$$\begin{aligned} \min_{B \in \mathbf{C}^{n \times m}} (\|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^H B - \tilde{W}^H\|^2) \\ = \|\tilde{Z}\tilde{V}_2\|^2 + \|\tilde{W}\tilde{Q}_2\|^2 \\ + \|\Phi_1 * (\Gamma\tilde{P}_1^H \tilde{Z}\tilde{V}_1 - \tilde{Q}_1^H \tilde{W}^H \tilde{U}_1 \Delta)\|^2, \end{aligned} \quad (6)$$

and  $B \in \mathbf{C}^{n \times m}$  attains this minimum if and only if

$$B = \tilde{P} \begin{bmatrix} \Phi * B_{11} & \Gamma^{-1} \tilde{Q}_1^H \tilde{W}^H \tilde{U}_2 \\ \tilde{P}_2^H \tilde{Z}\tilde{V}_1 \Delta^{-1} & B_{22} \end{bmatrix} \tilde{U}^H, \quad (7)$$

where  $B_{11} = \tilde{P}_1^H \tilde{Z}\tilde{V}_1 \Delta + \Gamma \tilde{Q}_1^H \tilde{W}^H \tilde{U}_1$ ,  $\Phi = \Phi_1 * \Phi_1$ ,  $B_{22} \in \mathbf{C}^{(n-f) \times (m-e)}$  is an arbitrary matrix. Moreover, the equations  $B\tilde{X} = \tilde{Z}$  and  $\tilde{Y}^H B = \tilde{W}^H$  have a common solution if and only if

$$\tilde{Z}\tilde{X}^+ \tilde{X} = \tilde{Z}, \tilde{W}\tilde{Y}^+ \tilde{Y} = \tilde{W}, \tilde{Y}^H \tilde{Z} = \tilde{W}^H \tilde{X}, \quad (8)$$

in which case a general solution of the equations

$$B\tilde{X} = \tilde{Z}, \tilde{Y}^H B = \tilde{W}^H$$

can be expressed as

$$B = \tilde{Z}\tilde{X}^+ + (\tilde{W}\tilde{Y}^+)^H(I_m - \tilde{X}\tilde{X}^+) + \tilde{P}_2 B_{22} \tilde{U}_2^H, \quad (9)$$

where  $B_{22} \in \mathbf{C}^{(n-f) \times (m-e)}$  is an arbitrary matrix.

**Proof.** From (5), we have

$$\begin{aligned} & \|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^H B - \tilde{W}^H\|^2 \\ &= \left\| \tilde{B}\tilde{U} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tilde{V}^H - \tilde{Z} \right\|^2 \\ &+ \left\| \tilde{Q} \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \tilde{P}^H B - \tilde{W}^H \right\|^2 \\ &= \left\| \tilde{P}^H \tilde{B}\tilde{U} \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} - \tilde{P}^H \tilde{Z}\tilde{V} \right\|^2 \\ &+ \left\| \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} \tilde{P}^H \tilde{B}\tilde{U} - \tilde{Q}^H \tilde{W}^H \tilde{U} \right\|^2. \end{aligned}$$

Let

$$\tilde{P}^H \tilde{B}\tilde{U} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad (10)$$

where  $B_{11} \in \mathbf{C}^{f \times e}$ , then

$$\begin{aligned} & \|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^H B - \tilde{W}^H\|^2 \\ &= \|B_{11}\Delta - \tilde{P}_1^H \tilde{Z}\tilde{V}_1\|^2 + \|B_{21}\Delta - \tilde{P}_2^H \tilde{Z}\tilde{V}_1\|^2 \\ &+ \|\tilde{Z}\tilde{V}_2\|^2 + \|\Gamma B_{11} - \tilde{Q}_1^H \tilde{W}^H \tilde{U}_1\|^2 \\ &+ \|\Gamma B_{12} - \tilde{Q}_1^H \tilde{W}^H \tilde{U}_2\|^2 + \|\tilde{Q}_2^H \tilde{W}^H\|^2. \end{aligned} \quad (11)$$

Thus,  $\|B\tilde{X} - \tilde{Z}\|^2 + \|\tilde{Y}^H B - \tilde{W}^H\|^2 = \min$  if and only if

$$\|B_{21}\Delta - \tilde{P}_2^H \tilde{Z}\tilde{V}_1\| = \min, \|\Gamma B_{12} - \tilde{Q}_1^H \tilde{W}^H \tilde{U}_2\| = \min \quad (12)$$

and

$$g(B_{11}) := \|B_{11}\Delta - \tilde{P}_1^H \tilde{Z}\tilde{V}_1\|^2 + \|\Gamma B_{11} - \tilde{Q}_1^H \tilde{W}^H \tilde{U}_1\|^2 = \min. \quad (13)$$

Clearly, (12) implies that

$$B_{21} = \tilde{P}_2^H \tilde{Z}\tilde{V}_1 \Delta^{-1}, \quad B_{12} = \Gamma^{-1} \tilde{Q}_1^H \tilde{W}^H \tilde{U}_2. \quad (14)$$

Let  $B_{11} = [b_{ij}]$ ,  $\tilde{P}_1^H \tilde{Z}\tilde{V}_1 = [z_{ij}]$ ,  $\tilde{Q}_1^H \tilde{W}^H \tilde{U}_1 = [w_{ij}] \in \mathbf{C}^{f \times e}$ , then

$$g(B_{11}) = \sum_{i=1}^f \sum_{j=1}^e (|b_{ij}\delta_j - z_{ij}|^2 + |\gamma_i b_{ij} - w_{ij}|^2). \quad (15)$$

Now we minimize the quantities

$$q_{ij} = |b_{ij}\delta_j - z_{ij}|^2 + |\gamma_i b_{ij} - w_{ij}|^2, \quad 1 \leq i \leq f, 1 \leq j \leq e.$$

It is easy to obtain the minimizers

$$b_{ij} = \frac{z_{ij}\delta_j + \gamma_i w_{ij}}{\gamma_i^2 + \delta_j^2}, \quad 1 \leq i \leq f, 1 \leq j \leq e, \quad (16)$$

and the minima

$$q_{ij} = \frac{|\gamma_i z_{ij} - w_{ij}\delta_j|^2}{\gamma_i^2 + \delta_j^2}, \quad 1 \leq i \leq f, 1 \leq j \leq e. \quad (17)$$

(17), (14) and (11) imply (6). Substituting (14) and (16) into (10) yields (7).

It follows from (6) that  $B\tilde{X} = \tilde{Z}$  and  $\tilde{Y}^H B = \tilde{W}^H$  have a common solution if and only if

$$\tilde{Z}\tilde{V}_2 = 0, \quad \tilde{W}\tilde{Q}_2 = 0, \quad \Gamma \tilde{P}_1^H \tilde{Z}\tilde{V}_1 = \tilde{Q}_1^H \tilde{W}^H \tilde{U}_1 \Delta,$$

which implies (8). In this case,  $B_{11} = \tilde{P}_1^H \tilde{Z}\tilde{V}_1 \Delta^{-1}$ . Substituting the representation of  $B_{11}$  and (14) into (10) yields (9).

### III. MINIMIZATION PROBLEM FOR GENERALIZED REFLEXIVE MATRICES

In this section  $\rho$  denotes the set of  $m \times n$  generalized reflexive matrices.

For the given matrices  $X \in \mathbf{C}^{n \times p}$ , and  $Y \in \mathbf{C}^{m \times q}$ , let the SVDs of matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  be, respectively

$$U^H X = F_1 \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} G_1^H, \quad P^H Y = F_2 \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} G_2^H, \quad (18)$$

$$V^H X = F_3 \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix} G_3^H, \quad Q^H Y = F_4 \begin{bmatrix} \Sigma_4 & 0 \\ 0 & 0 \end{bmatrix} G_4^H, \quad (19)$$

where all matrices  $F_i = [F_{i1}, F_{i2}]$ ,  $G_i = [G_{i1}, G_{i2}]$  ( $i = 1, 2, 3, 4$ ) are unitary matrices and partitions are compatible with the size of  $\Sigma_i = \text{diag}(\alpha_1^{(i)}, \dots, \alpha_{t_i}^{(i)}) > 0$  ( $i = 1, 2, 3, 4$ ),  $t_1 = \text{rank}(U^H X)$ ,  $t_2 = \text{rank}(P^H Y)$ ,  $t_3 = \text{rank}(V^H X)$ ,  $t_4 = \text{rank}(Q^H Y)$ .

**Theorem 1:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $Z \in \mathbf{C}^{m \times p}$ ,  $Y \in \mathbf{C}^{m \times q}$ ,  $W \in \mathbf{C}^{n \times q}$ , and  $\rho$  denote the set of  $m \times n$  generalized reflexive matrices. Suppose that the SVDs of the matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  are given by (18) and (19), then

$$\begin{aligned} \sigma(X, Z, Y, W) &= \min_{A \in \rho} (\|AX - Z\|^2 \\ &+ \|Y^H A - W^H\|^2)^{1/2} \\ &= (\|P^H Z G_{12}\|^2 + \|U^H W G_{22}\|^2 \\ &+ \|\Phi_1 * (\Sigma_2 F_{21}^H P^H Z G_{11} - G_{21}^H W^H U F_{11} \Sigma_1)\|^2 \\ &+ \|\tilde{Q}^H Z G_{32}\|^2 + \|V^H W G_{42}\|^2 \\ &+ \|\Phi_2 * (\Sigma_4 F_{41}^H Q^H Z G_{31} - G_{41}^H W^H V F_{31} \Sigma_3)\|^2)^{1/2}, \end{aligned} \quad (20)$$

where

$$\Phi_1 = [\varphi_{ij}^{(1)}] \in \mathbf{C}^{t_2 \times t_1}, \quad \varphi_{ij}^{(1)} = \frac{1}{((\alpha_i^{(2)})^2 + (\alpha_j^{(1)})^2)^{1/2}}, \quad 1 \leq i \leq t_2, 1 \leq j \leq t_1,$$

$$\Phi_2 = [\varphi_{ij}^{(2)}] \in \mathbf{C}^{t_4 \times t_3}, \quad \varphi_{ij}^{(2)} = \frac{1}{((\alpha_i^{(4)})^2 + (\alpha_j^{(3)})^2)^{1/2}}, \quad 1 \leq i \leq t_4, 1 \leq j \leq t_3,$$

and  $A \in \rho$  attains this minimum if and only if

$$A = A_0 + [P, Q] \begin{bmatrix} F_{22} M F_{12}^H & 0 \\ 0 & F_{42} N F_{32}^H \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \quad (21)$$

with arbitrary  $M \in \mathbf{C}^{(r-t_2) \times (k-t_1)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (l-t_3)}$  and

$$A_0 = [P, Q] \begin{bmatrix} A_{PU}^{(0)} & 0 \\ 0 & A_{QV}^{(0)} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}, \quad (22)$$

$$A_{PU}^{(0)} = F_2 \begin{bmatrix} K_1 * L_{11} & \Sigma_2^{-1} G_{21}^H W^H U F_{12} \\ F_{22}^H P^H Z G_{11} \Sigma_1^{-1} & 0 \end{bmatrix} F_1^H, \quad (23)$$

$$A_{QV}^{(0)} = F_4 \begin{bmatrix} K_2 * J_{11} & \Sigma_4^{-1} G_{41}^H W^H V F_{32} \\ F_{42}^H Q^H Z G_{31} \Sigma_3^{-1} & 0 \end{bmatrix} F_3^H, \quad (24)$$

where

$$L_{11} = F_{21}^H P^H Z G_{11} \Sigma_1 + \Sigma_2 G_{21}^H W^H U F_{11},$$

$$J_{11} = F_{41}^H Q^H Z G_{31} \Sigma_3 + \Sigma_4 G_{41}^H W^H V F_{31},$$

$K_1 = \Phi_1 * \Phi_1, K_2 = \Phi_2 * \Phi_2$ . Moreover, the equations  $AX = Z$  and  $Y^H A = W^H$  have a common solution  $A \in \rho$  if and only if

$$P^H Z (U^H X)^+ + U^H X = P^H Z,$$

$$U^H W (P^H Y)^+ + P^H Y = U^H W,$$

$$Y^H P P^H Z = W^H U U^H X$$

and

$$Q^H Z (V^H X)^+ + V^H X = Q^H Z,$$

$$V^H W (Q^H Y)^+ + Q^H Y = V^H W,$$

$$Y^H Q Q^H Z = W^H V V^H X,$$

in which case a general solution of the equations  $AX = Z$  and  $Y^H A = W^H$  can be expressed as

$$A = [P, Q] \begin{bmatrix} A_{PU}^{(0)} + F_{22} M F_{12}^H & 0 \\ 0 & A_{QV}^{(0)} + F_{42} N F_{32}^H \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix},$$

where

$$A_{PU}^{(0)} = P^H Z (U^H X)^+ + (Y^H P)^+ W^H U F_{12} F_{12}^H,$$

$$A_{QV}^{(0)} = Q^H Z (V^H X)^+ + (Y^H Q)^+ W^H V F_{32} F_{32}^H,$$

and  $M \in \mathbf{C}^{(r-t_2) \times (k-t_1)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (l-t_3)}$  are arbitrary matrices.

**Proof.** If  $A \in \rho$ , it follows from Lemma 1 that there exist  $A_{PU} \in \mathbf{C}^{r \times k}$ ,  $A_{QV} \in \mathbf{C}^{s \times l}$  satisfying

$$A = [P, Q] \begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}. \quad (25)$$

Therefore,

$$\begin{aligned} & \min_{A \in \rho} (\|AX - Z\|^2 + \|Y^H A - W^H\|^2) \\ &= \min_{A_{PU} \in \mathbf{C}^{r \times k}, A_{QV} \in \mathbf{C}^{s \times l}} \left( \left\| [P, Q] \begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} X - Z \right\|^2 \right. \\ & \quad \left. + \left\| Y^H [P, Q] \begin{bmatrix} A_{PU} & 0 \\ 0 & A_{QV} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} - W^H \right\|^2 \right) \\ &= \min_{A_{PU} \in \mathbf{C}^{r \times k}} (\|A_{PU} (U^H X) - P^H Z\|^2 \\ & \quad + \|(P^H Y)^H A_{PU} - (U^H W)^H\|^2) \\ & \quad + \min_{A_{QV} \in \mathbf{C}^{s \times l}} (\|A_{QV} (V^H X) - Q^H Z\|^2 \\ & \quad + \|(Q^H Y)^H A_{QV} - (V^H W)^H\|^2). \end{aligned} \quad (26)$$

It follows from Lemma 2 that

$$\begin{aligned} & \min_{A_{PU} \in \mathbf{C}^{r \times k}} (\|A_{PU} (U^H X) - P^H Z\|^2 \\ & \quad + \|(P^H Y)^H A_{PU} - (U^H W)^H\|^2) \\ &= \|P^H Z G_{12}\|^2 + \|U^H W G_{22}\|^2 \\ & \quad + \|\Phi_1 * (\Sigma_2 F_{21}^H P^H Z G_{11} - G_{21}^H W^H U F_{11} \Sigma_1)\|^2, \end{aligned} \quad (27)$$

$$\begin{aligned} & \min_{A_{QV} \in \mathbf{C}^{s \times l}} (\|A_{QV} (V^H X) - Q^H Z\|^2 \\ & \quad + \|(Q^H Y)^H A_{QV} - (V^H W)^H\|^2) \\ &= \|Q^H Z G_{32}\|^2 + \|V^H W G_{42}\|^2 \\ & \quad + \|\Phi_2 * (\Sigma_4 F_{41}^H Q^H Z G_{31} - G_{41}^H W^H V F_{31} \Sigma_3)\|^2, \end{aligned} \quad (28)$$

and  $A_{PU} \in \mathbf{C}^{r \times k}$ ,  $A_{QV} \in \mathbf{C}^{s \times l}$  attain the minima if and only if

$$A_{PU} = F_2 \begin{bmatrix} K_1 * A_{PU}^{(11)} & \Sigma_2^{-1} G_{21}^H W^H U F_{12} \\ F_{22}^H P^H Z G_{11} \Sigma_1^{-1} & M \end{bmatrix} F_1^H, \quad (29)$$

$$A_{QV} = F_4 \begin{bmatrix} K_2 * A_{QV}^{(11)} & \Sigma_4^{-1} G_{41}^H W^H V F_{32} \\ F_{42}^H Q^H Z G_{31} \Sigma_3^{-1} & N \end{bmatrix} F_3^H, \quad (30)$$

where  $A_{PU}^{(11)} = F_{21}^H P^H Z G_{11} \Sigma_1 + \Sigma_2 G_{21}^H W^H U F_{11}$ ,  $A_{QV}^{(11)} = F_{41}^H Q^H Z G_{31} \Sigma_3 + \Sigma_4 G_{41}^H W^H V F_{31}$ ,  $M \in \mathbf{C}^{(r-t_2) \times (k-t_1)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (l-t_3)}$  are arbitrary matrices. (26), (27) and (28) imply (20). Substituting (29) and (30) into (25) yields (21).

It follows from Lemma 1 that the equations  $AX = Z$  and  $Y^H A = W^H$  have a common solution  $A \in \rho$  if and only if  $A_{PU} (U^H X) = P^H Z$  and  $(P^H Y)^H A_{PU} = (U^H W)^H$ ,  $A_{QV} (V^H X) = Q^H Z$  and  $(Q^H Y)^H A_{QV} = (V^H W)^H$  have common solutions  $A_{PU} \in \mathbf{C}^{r \times k}$ ,  $A_{QV} \in \mathbf{C}^{s \times l}$ , respectively. Applying Lemma 2, it is easy to obtain the second part of the conclusions.

Let  $r_1 = r - t_2, k_1 = k - t_1, s_1 = s - t_4$  and  $l_1 = l - t_3$ . From Theorem 1 we have

$$\begin{aligned} \rho(X, Z, Y, W) = & \left\{ A = A_0 + [P, Q] \begin{bmatrix} F_{22} M F_{12}^H & 0 \\ 0 & F_{42} N F_{32}^H \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \mid \right. \\ & \left. M \in \mathbf{C}^{r_1 \times k_1}, N \in \mathbf{C}^{s_1 \times l_1} \right\}. \end{aligned} \quad (31)$$

It is easy to verify that  $\rho(X, Z, Y, W)$  is a closed convex set in Hilbert space  $\mathbf{C}^{m \times n}$ . Therefore, for given matrix  $\tilde{A} \in \mathbf{C}^{m \times n}$ , it follows from the best approximation theorem (see Aubin[3]) that there exists a unique solution  $\hat{A}$  in  $\rho(X, Z, Y, W)$  such that  $\|\hat{A} - \tilde{A}\| = \sigma(X, Z, Y, W; \tilde{A})$ .

We shall focus our attention on seeking the unique solution of Problem II. For any matrix  $A \in \rho(X, Z, Y, W)$ , we have

$$\begin{aligned} \|A - \tilde{A}\|^2 = & \left\| [P, Q] \begin{bmatrix} A_{PU}^{(0)} + F_{22} M F_{12}^H & 0 \\ 0 & A_{QV}^{(0)} + F_{42} N F_{32}^H \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} - \tilde{A} \right\|^2 \\ & = \left\| \begin{bmatrix} A_{PU}^{(0)} + F_{22} M F_{12}^H & 0 \\ 0 & A_{QV}^{(0)} + F_{42} N F_{32}^H \end{bmatrix} \begin{bmatrix} P^H \\ Q^H \end{bmatrix} \tilde{A} [U, V] \right\|^2 \\ & = \|F_{22} M F_{12}^H - (P^H \tilde{A} U - A_{PU}^{(0)})\|^2 \\ & \quad + \|F_{42} N F_{32}^H - (Q^H \tilde{A} V - A_{QV}^{(0)})\|^2 \\ & \quad + \|Q^H \tilde{A} U\|^2 + \|P^H \tilde{A} V\|^2 \\ & = \|M - F_{22}^H (P^H \tilde{A} U - A_{PU}^{(0)}) F_{12}\|^2 \\ & \quad + \|N - F_{42}^H (Q^H \tilde{A} V - A_{QV}^{(0)}) F_{32}\|^2 \\ & \quad + \|F_{21}^H (P^H \tilde{A} U - A_{PU}^{(0)})\|^2 \\ & \quad + \|F_{22}^H (P^H \tilde{A} U - A_{PU}^{(0)}) F_{11}\|^2 + \|Q^H \tilde{A} U\|^2 \\ & \quad + \|F_{42}^H (Q^H \tilde{A} V - A_{QV}^{(0)}) F_{31}\|^2 + \|P^H \tilde{A} V\|^2 \\ & \quad + \|F_{41}^H (Q^H \tilde{A} V - A_{QV}^{(0)})\|^2. \end{aligned} \quad (32)$$

Note that  $F_{22}^H A_{PU}^{(0)} F_{12} = 0$ ,  $F_{42}^H A_{QV}^{(0)} F_{32} = 0$ . It follows from (32) that  $\|A - \hat{A}\| = \min$  if and only if

$$M = F_{22}^H P^H \hat{A} U F_{12}, \quad N = F_{42}^H Q^H \hat{A} V F_{32}.$$

By now, we have proved the following result.

**Theorem 2:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $Z \in \mathbf{C}^{m \times p}$ ,  $Y \in \mathbf{C}^{m \times q}$ ,  $W \in \mathbf{C}^{n \times q}$ ,  $\tilde{A} \in \mathbf{C}^{m \times n}$  and  $\rho$  denote the set of  $m \times n$  generalized reflexive matrices. Suppose that the SVDs of the matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  are given by (18) and (19), then Problem II has a unique solution  $\hat{A}$  in  $\rho(X, Z, Y, W)$ . Moreover,  $\hat{A}$  can be expressed as

$$\hat{A} = A_0 + [P, Q] \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}, \quad (33)$$

and the associated minimum  $\sigma(X, Z, Y, W; \tilde{A})$  is

$$\begin{aligned} \sigma(X, Z, Y, W; \tilde{A}) = & (\|F_{21}^H (P^H \tilde{A} U - A_{PU}^{(0)})\|^2 \\ & + \|F_{22}^H (P^H \tilde{A} U - A_{PU}^{(0)}) F_{11}\|^2 \\ & + \|F_{41}^H (Q^H \tilde{A} V - A_{QV}^{(0)})\|^2 + \|F_{42}^H (Q^H \tilde{A} V - A_{QV}^{(0)}) F_{31}\|^2 \\ & + \|P^H \tilde{A} V\|^2 + \|Q^H \tilde{A} U\|^2)^{1/2}, \end{aligned} \quad (34)$$

where

$$\begin{aligned} \hat{A}_{11} &= F_{22} F_{22}^H P^H \tilde{A} U F_{12} F_{12}^H, \\ \hat{A}_{22} &= F_{42} F_{42}^H Q^H \tilde{A} V F_{32} F_{32}^H. \end{aligned}$$

Now, we apply Theorem 1 and Theorem 2 to solve the left and right inverse eigenproblem for generalized reflexive matrices and its optimal approximation, i.e., given partial left and right eigenpairs (eigenvalues and corresponding eigenvectors)  $(\omega_i, y_i) (i = 1, \dots, q)$ ,  $(\lambda_j, x_j) (j = 1, \dots, p)$  and a matrix  $A \in \mathbf{C}^{n \times n}$ , find an  $n \times n$  generalized reflexive matrix  $\hat{A}$  such that

$$A x_j = \lambda_j x_j \quad (j = 1, \dots, p), \quad y_i^H A = \omega_i y_i^H \quad (i = 1, \dots, q), \quad (35)$$

and a matrix  $\hat{A} \in \rho(X, \Lambda, Y, \Omega)$  such that

$$\|\tilde{A} - \hat{A}\| = \min_{A \in \rho(X, \Lambda, Y, \Omega)} \|\tilde{A} - A\|, \quad (36)$$

where  $\rho(X, \Lambda, Y, \Omega) = \{A \in \rho \mid AX = X\Lambda, Y^H A = \Omega Y^H\}$ ,  $\rho$  denotes the set of  $n \times n$  generalized reflexive matrices,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,  $\Omega = \text{diag}(\omega_1, \dots, \omega_q)$ ,  $X = [x_1, \dots, x_p]$ ,  $Y = [y_1, \dots, y_q]$ .

From Theorem 1 and Theorem 2, we obtain the following corollary.

**Corollary 3:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $Y \in \mathbf{C}^{n \times q}$ ,  $\tilde{A} \in \mathbf{C}^{n \times n}$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,  $\Omega = \text{diag}(\omega_1, \dots, \omega_q)$  and  $\rho(X, \Lambda, Y, \Omega) = \{A \in \rho \mid AX = X\Lambda, Y^H A = \Omega Y^H\}$ , where  $\rho$  denotes the set of  $n \times n$  generalized reflexive matrices. Suppose that the SVDs of the matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  are given by (18) and (19), then the set  $\rho(X, \Lambda, Y, \Omega)$  is nonempty if and only if

$$\begin{aligned} P^H X \Lambda (U^H X)^+ U^H X &= P^H X \Lambda, \\ U^H Y \Omega^H (P^H Y)^+ P^H Y &= U^H Y \Omega^H, \\ Y^H P P^H X \Lambda &= \Omega Y^H U U^H X \end{aligned}$$

and

$$Q^H X \Lambda (V^H X)^+ V^H X = Q^H X \Lambda,$$

$$V^H Y \Omega^H (Q^H Y)^+ Q^H Y = V^H Y \Omega^H,$$

$$Y^H Q Q^H X \Lambda = \Omega Y^H V V^H X,$$

in which case, the set  $\rho(X, \Lambda, Y, \Omega)$  can be expressed as

$$\rho(X, \Lambda, Y, \Omega) = \{A = [P, Q] \begin{bmatrix} A_{PU}^{(0)} + F_{22} M F_{12}^H & 0 \\ 0 & A_{QV}^{(0)} + F_{42} N F_{32}^H \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}\},$$

where

$$\begin{aligned} A_{PU}^{(0)} &= P^H X \Lambda (U^H X)^+ + (Y^H P)^+ \Omega Y^H U F_{12} F_{12}^H, \\ A_{QV}^{(0)} &= Q^H X \Lambda (V^H X)^+ + (Y^H Q)^+ \Omega Y^H V F_{32} F_{32}^H \end{aligned}$$

and  $M \in \mathbf{C}^{(r-t_2) \times (k-t_1)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (l-t_3)}$  are arbitrary matrices, and there is a unique matrix  $\hat{A} \in \rho(X, \Lambda, Y, \Omega)$  such that  $\|\tilde{A} - \hat{A}\| = \min_{A \in \rho(X, \Lambda, Y, \Omega)} \|\tilde{A} - A\|$ . Moreover,  $\hat{A}$  can be expressed as

$$\hat{A} = [P, Q] \begin{bmatrix} A_{PU}^{(0)} + \hat{A}_{11} & 0 \\ 0 & A_{QV}^{(0)} + \hat{A}_{22} \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix},$$

where

$$\begin{aligned} \hat{A}_{11} &= F_{22} F_{22}^H P^H \tilde{A} U F_{12} F_{12}^H, \\ \hat{A}_{22} &= F_{42} F_{42}^H Q^H \tilde{A} V F_{32} F_{32}^H. \end{aligned}$$

If  $m = n$ ,  $R = S$ ,  $Y = 0$ , then  $P = U$ ,  $Q = V$ . From Corollary 3, we obtain the following corollary related to the inverse eigenvalue problem for reflexive matrices and its optimal approximation. For more details, see [27].

**Corollary 4:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $\tilde{A} \in \mathbf{C}^{n \times n}$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ , and  $\rho(X, \Lambda) = \{A \in \rho \mid AX = X\Lambda\}$ , where  $\rho$  denotes the set of  $n \times n$  reflexive matrices. Suppose that the SVDs of the matrices  $U^H X$ ,  $V^H X$  are given by (18) and (19),  $t_1 = \text{rank}(U^H X)$ ,  $t_3 = \text{rank}(V^H X)$ , then the set  $\rho(X, \Lambda)$  is nonempty if and only if

$$U^H X \Lambda (U^H X)^+ U^H X = U^H X \Lambda,$$

$$V^H X \Lambda (V^H X)^+ V^H X = V^H X \Lambda,$$

in which case, the set  $\rho(X, \Lambda)$  can be expressed as

$$\rho(X, \Lambda) = \{A = [U, V] \begin{bmatrix} A_U^{(0)} + M F_{12}^H & 0 \\ 0 & A_V^{(0)} + N F_{32}^H \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}\},$$

where

$$A_U^{(0)} = U^H X \Lambda (U^H X)^+, \quad A_V^{(0)} = V^H X \Lambda (V^H X)^+,$$

and  $M \in \mathbf{C}^{r \times (r-t_1)}$ ,  $N \in \mathbf{C}^{(n-r) \times (n-r-t_3)}$  are arbitrary matrices, and there is a unique matrix  $\hat{A} \in \rho(X, \Lambda)$  such that  $\|\tilde{A} - \hat{A}\| = \min_{A \in \rho(X, \Lambda)} \|\tilde{A} - A\|$ . Moreover,  $\hat{A}$  can be expressed as

$$\hat{A} = [U, V] \begin{bmatrix} A_U^{(0)} + \hat{A}_1 & 0 \\ 0 & A_V^{(0)} + \hat{A}_2 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}.$$

where  $\hat{A}_1 = U^H \tilde{A} U F_{12} F_{12}^H$ ,  $\hat{A}_2 = V^H \tilde{A} V F_{32} F_{32}^H$ .

#### IV. MINIMIZATION PROBLEM FOR GENERALIZED ANTI-REFLEXIVE MATRICES

In this section  $\rho$  denotes the set of  $m \times n$  generalized anti-reflexive matrices.

For the given matrices  $X \in \mathbf{C}^{n \times p}$ ,  $Z \in \mathbf{C}^{m \times p}$ ,  $Y \in \mathbf{C}^{m \times q}$  and  $W \in \mathbf{C}^{n \times q}$ , if  $A \in \rho$ , it follows from Lemma 1 that

$$\begin{aligned} & \min_{A \in \rho} (\|AX - Z\|^2 + \|Y^H A - W^H\|^2) \\ &= \min_{A_{PV} \in \mathbf{C}^{r \times l}, A_{QU} \in \mathbf{C}^{s \times k}} \left( \left\| \begin{bmatrix} P, Q \end{bmatrix} \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} X - Z \right\|^2 \right. \\ & \quad \left. + \left\| Y^H \begin{bmatrix} P, Q \end{bmatrix} \begin{bmatrix} 0 & A_{PV} \\ A_{QU} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} - W^H \right\|^2 \right) \\ &= \min_{A_{PV} \in \mathbf{C}^{r \times l}} (\|A_{PV}(V^H X) - (P^H Z)\|^2 \\ & \quad + \|(P^H Y)^H A_{PV} - (V^H W)^H\|^2) \\ & \quad + \min_{A_{QU} \in \mathbf{C}^{s \times k}} (\|A_{QU}(U^H X) - (Q^H Z)\|^2 \\ & \quad + \|(Q^H Y)^H A_{QU} - (U^H W)^H\|^2). \end{aligned} \quad (37)$$

Applying Lemma 2 to (37), we obtain the following theorem.

**Theorem 5:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $Z \in \mathbf{C}^{m \times p}$ ,  $Y \in \mathbf{C}^{m \times q}$  and  $W \in \mathbf{C}^{n \times q}$ ,  $\rho$  be the set of  $m \times n$  generalized anti-reflexive matrices. Suppose that the SVDs of matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  are given by (18) and (19), then

$$\begin{aligned} \sigma(X, Z, Y, W) &= \min_{A \in \rho} (\|AX - Z\|^2 \\ & \quad + \|Y^H A - W^H\|^2)^{1/2} \\ &= (\|P^H Z G_{32}\|^2 + \|V^H W G_{22}\|^2 \\ & \quad + \|\Psi_1 * (\Sigma_2 F_{21}^H P^H Z G_{31} - G_{21}^H W^H V F_{31} \Sigma_3)\|^2 \\ & \quad + \|Q^H Z G_{12}\|^2 + \|U^H W G_{42}\|^2 \\ & \quad + \|\Psi_2 * (\Sigma_4 F_{41}^H Q^H Z G_{11} - G_{41}^H W^H U F_{11} \Sigma_1)\|^2)^{1/2}, \end{aligned}$$

where

$$\Psi_1 = [\psi_{ij}^{(1)}] \in \mathbf{C}^{t_2 \times t_3}, \psi_{ij}^{(1)} = \frac{1}{((\alpha_i^{(2)})^2 + (\alpha_j^{(3)})^2)^{1/2}},$$

$$1 \leq i \leq t_2, 1 \leq j \leq t_3;$$

$$\Psi_2 = [\psi_{ij}^{(2)}] \in \mathbf{C}^{t_4 \times t_1}, \psi_{ij}^{(2)} = \frac{1}{((\alpha_i^{(4)})^2 + (\alpha_j^{(1)})^2)^{1/2}},$$

$$1 \leq i \leq t_4, 1 \leq j \leq t_1,$$

and  $A \in \rho$  attains this minimum if and only if

$$A = A_0 + [P, Q] \begin{bmatrix} 0 & F_{22} M F_{32}^H \\ F_{42} N F_{12}^H & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \quad (38)$$

with arbitrary  $M \in \mathbf{C}^{(r-t_2) \times (l-t_3)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (k-t_1)}$  and

$$A_0 = [P, Q] \begin{bmatrix} 0 & A_{PV}^{(0)} \\ A_{QU}^{(0)} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix},$$

$$A_{PV}^{(0)} = F_2 \begin{bmatrix} K_3 * C_{11} & \Sigma_2^{-1} G_{21}^H W^H V F_{32} \\ F_{22}^H P^H Z G_{31} \Sigma_3^{-1} & 0 \end{bmatrix} F_3^H, \quad (39)$$

$$A_{QU}^{(0)} = F_4 \begin{bmatrix} K_4 * D_{11} & \Sigma_4^{-1} G_{41}^H W^H U F_{12} \\ F_{42}^H Q^H Z G_{11} \Sigma_1^{-1} & 0 \end{bmatrix} F_1^H, \quad (40)$$

where

$$C_{11} = F_{21}^H P^H Z G_{31} \Sigma_3 + \Sigma_2 G_{21}^H W^H V F_{31},$$

$$D_{11} = F_{41}^H Q^H Z G_{11} \Sigma_1 + \Sigma_4 G_{41}^H W^H U F_{11},$$

$K_3 = \Psi_1 * \Psi_1$ ,  $K_4 = \Psi_2 * \Psi_2$ . Moreover, the equations  $AX = Z$  and  $Y^H A = W^H$  have a common solution  $A \in \rho$  if and only if

$$P^H Z (V^H X)^+ V^H X = P^H Z,$$

$$V^H W (P^H Y)^+ P^H Y = V^H W,$$

$$Y^H P P^H Z = W^H V V^H X,$$

and

$$Q^H Z (U^H X)^+ U^H X = Q^H Z,$$

$$U^H W (Q^H Y)^+ Q^H Y = U^H W,$$

$$Y^H Q Q^H Z = W^H U U^H X,$$

in which case a general solution of the equations  $AX = Z$  and  $Y^H A = W^H$  can be expressed as

$$A = [P, Q] \begin{bmatrix} 0 & A_{PV}^{(0)} + F_{22} M F_{32}^H \\ A_{QU}^{(0)} + F_{42} N F_{12}^H & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix}$$

with arbitrary  $M \in \mathbf{C}^{(r-t_2) \times (l-t_3)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (k-t_1)}$  and

$$A_{PV}^{(0)} = P^H Z (V^H X)^+ + (Y^H P)^+ W^H V F_{32} F_{32}^H,$$

$$A_{QU}^{(0)} = Q^H Z (U^H X)^+ + (Y^H Q)^+ W^H U F_{12} F_{12}^H.$$

Let  $r_1 = r - t_2$ ,  $k_1 = k - t_1$ ,  $s_1 = s - t_4$  and  $l_1 = l - t_3$ .

From Theorem 5 we know

$$\begin{aligned} \rho(X, Z, Y, W) &= \\ & \left\{ A = A_0 + [P, Q] \begin{bmatrix} 0 & F_{22} M F_{32}^H \\ F_{42} N F_{12}^H & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \mid \right. \\ & \quad \left. M \in \mathbf{C}^{r_1 \times l_1}, N \in \mathbf{C}^{s_1 \times k_1} \right\}, \end{aligned} \quad (41)$$

Similar to Theorem 2, we have the following result.

**Theorem 6:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $Z \in \mathbf{C}^{m \times p}$ ,  $Y \in \mathbf{C}^{m \times q}$ ,  $W \in \mathbf{C}^{n \times q}$ ,  $\tilde{A} \in \mathbf{C}^{m \times n}$ , and let  $\rho$  be the set of  $m \times n$  generalized anti-reflexive matrices. Suppose that the SVDs of matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  are given by (18) and (19), then Problem II has a unique solution  $\hat{A}$  in  $\rho(X, Z, Y, W)$ . Moreover,  $\hat{A}$  can be expressed as

$$\hat{A} = A_0 + [P, Q] \begin{bmatrix} 0 & \hat{A}_{12} \\ \hat{A}_{21} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix},$$

where

$$\hat{A}_{12} = F_{22} F_{22}^H P^H \tilde{A} V F_{32} F_{32}^H,$$

$$\hat{A}_{21} = F_{42} F_{42}^H Q^H \tilde{A} U F_{12} F_{12}^H.$$

In this case, the associated minimum  $\sigma(X, Z, Y, W; \tilde{A})$  is

$$\begin{aligned} \sigma(X, Z, Y, W; \tilde{A}) &= (\|F_{21}^H (P^H \tilde{A} V - A_{PV}^{(0)})\|^2 \\ & \quad + \|F_{22}^H (P^H \tilde{A} V - A_{PV}^{(0)}) F_{31}\|^2 \\ & \quad + \|F_{41}^H (Q^H \tilde{A} U - A_{QU}^{(0)})\|^2 \\ & \quad + \|F_{42}^H (Q^H \tilde{A} U - A_{QU}^{(0)}) F_{11}\|^2 \\ & \quad + \|P^H \tilde{A} U\|^2 + \|Q^H \tilde{A} V\|^2)^{1/2}. \end{aligned}$$

From Theorem 5 and Theorem 6, we get the following result related to the left and right inverse eigenproblem for generalized anti-reflexive matrices and its optimal approximation.

**Corollary 7:** Let  $X \in \mathbf{C}^{n \times p}$ ,  $Y \in \mathbf{C}^{n \times q}$ ,  $\tilde{A} \in \mathbf{C}^{n \times n}$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ ,  $\Omega = \text{diag}(\omega_1, \dots, \omega_q)$  and  $\rho(X, \Lambda, Y, \Omega) = \{A \in \rho \mid AX = X\Lambda, Y^H A = \Omega Y^H\}$ , where  $\rho$  denotes the set of  $n \times n$  generalized anti-reflexive matrices. Suppose that the SVDs of the matrices  $U^H X$ ,  $P^H Y$ ,  $V^H X$  and  $Q^H Y$  are given by (18) and (19), then the set  $\rho(X, \Lambda, Y, \Omega)$  is nonempty if and only if

$$\begin{aligned} P^H X \Lambda (V^H X)^+ + V^H X &= P^H X \Lambda, \\ V^H Y \Omega^H (P^H Y)^+ + P^H Y &= V^H Y \Omega^H, \\ Y^H P P^H X \Lambda &= \Omega Y^H V V^H X, \end{aligned}$$

and

$$\begin{aligned} Q^H X \Lambda (U^H X)^+ + U^H X &= Q^H X \Lambda, \\ U^H Y \Omega^H (Q^H Y)^+ + Q^H Y &= U^H Y \Omega^H, \\ Y^H Q Q^H X \Lambda &= \Omega Y^H U U^H X, \end{aligned}$$

in which case, the set  $\rho(X, \Lambda, Y, \Omega)$  can be expressed as

$$\rho(X, \Lambda, Y, \Omega) = \left\{ A = \begin{bmatrix} P & Q \\ \begin{bmatrix} 0 & A_{PV}^{(0)} + F_{22} M F_{32}^H \\ A_{QU}^{(0)} + F_{42} N F_{12}^H & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \end{bmatrix} \right\},$$

where  $M \in \mathbf{C}^{(r-t_2) \times (l-t_3)}$ ,  $N \in \mathbf{C}^{(s-t_4) \times (k-t_1)}$  are arbitrary matrices and

$$\begin{aligned} A_{PV}^{(0)} &= P^H X \Lambda (V^H X)^+ + (Y^H P)^+ \Omega Y^H V F_{32} F_{32}^H, \\ A_{QU}^{(0)} &= Q^H X \Lambda (U^H X)^+ + (Y^H Q)^+ \Omega Y^H U F_{12} F_{12}^H, \end{aligned}$$

and there exists a unique matrix  $\hat{A} \in \rho(X, \Lambda, Y, \Omega)$  such that  $\|\hat{A} - \hat{A}\| = \min_{A \in \rho(X, \Lambda, Y, \Omega)} \|\hat{A} - A\|$ . Moreover,  $\hat{A}$  can be expressed as

$$\hat{A} = \begin{bmatrix} P & Q \\ \begin{bmatrix} 0 & A_{PV}^{(0)} + \hat{A}_{12} \\ A_{QU}^{(0)} + \hat{A}_{21} & 0 \end{bmatrix} \begin{bmatrix} U^H \\ V^H \end{bmatrix} \end{bmatrix},$$

where

$$\begin{aligned} \hat{A}_{12} &= F_{22} F_{22}^H P^H \tilde{A} V F_{32} F_{32}^H, \\ \hat{A}_{21} &= F_{42} F_{42}^H Q^H \tilde{A} U F_{12} F_{12}^H. \end{aligned}$$

In addition, if  $m = n$ ,  $R = S$ ,  $Y = 0$ , similar to Corollary 4, we can obtain the result related to the inverse eigenvalue problem for anti-reflexive matrices and its optimal approximation from Corollary 7.

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