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Constructive proof of Tychonoff's fixed point theorem for sequentially locally non-constant functions

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Abstract—We present a constructive proof of Tychonoff's fixed point theorem in a locally convex space for uniformly continuous and sequentially locally non-constant functions.

Keywords—sequentially locally non-constant functions, Tychonoff's fixed point theorem, constructive mathematics.

I. Introduction

It is well known that Brouwer's fixed point theorem can not be constructively proved[1]. Thus, Tychonoff's fixed point theorem also can not be constructively proved. Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. See [9] and [10]. Thus, Brouwer's fixed point theorem is constructively, in the sense of constructive mathematics à la Bishop, proved in its approximate version[2].

Also Dalen[9] states a conjecture that a uniformly continuous function f from a simplex into itself, with property that each open set contains a point x such that $x \neq f(x)$, which means $|x - f(x)| > 0$, and also at every point x on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. Recently [2] showed that the following theorem is equivalent to Brouwer's fan theorem.

> Each uniformly continuous function f from a compact metric space X into itself with at most one fixed point and approximate fixed points has a fixed point.

By reference to the notion of *sequentially at most one maximum* in [1] we require a more general and somewhat stronger condition of *sequential local non-constancy* for functions, and in [7] we have shown the following result.

> If each uniformly continuous function from a compact metric space into itself is *sequentially locally non-constant*, then it has a fixed point,

without the fan theorem. It is a partial answer to Dalen's conjecture.

In the next section we present a constructive proof of Tychonoff's fixed point theorem in a locally convex space[3].

II. Tychonoff's fixed point theorems for sequentially locally non-constant functions in a locally convex space

In constructive mathematics a nonempty set is called an *inhabited* set. A set S is inhabited if there exists an element of S.

> Note that in order to show that S is inhabited, we cannot just prove that it is impossible for S to be empty: we must actually construct an element of S (see page 12 of [4]).

Also in constructive mathematics compactness of a set means *total boundedness with completeness*. A set S is *finitely enumerable* if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S. An ε-approximation to S is a subset of S such that for each $\mathbf{p} \in S$ there exists $\mathbf{q}$ in that ε-approximation with $|\mathbf{p}-\mathbf{q}| < \varepsilon$($|\mathbf{p}-\mathbf{q}|$ is the distance between $\mathbf{p}$ and $\mathbf{q}$). S is totally bounded if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to S. Completeness of a set, of course, means that every Cauchy sequence in the set converges.

A locally convex space consists of a vector space E and a family $(p_i)_{i \in I}$ of seminorms on E. I is an index set, for example, the set of positive integers. According to [4] we define, constructively, total boundedness of a set in a locally convex space as follows;

Definition 1: (Total boundedness of a set in a locally convex space) Let X be a subset of E, F be a finitely enumerable subset of I[4], and $\varepsilon > 0$. By an ε-approximation to X relative to F we mean a subset T of X such that for each $x \in X$ there exists $y \in T$ with $\sum_{i \in F} p_i(x - y) < \varepsilon$. X is totally bounded relative to F if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to X relative to F. It is totally bounded if it is totally bounded relative to each finitely enumerable subset of I.

Extending Corollary 2.2.12 of [4] to a locally convex space we have the following result.

Lemma 1: If X is a totally bounded subset of a locally convex space, then for each $\varepsilon > 0$ there exist totally bounded

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[1][6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [3] or [9].

[2]In [8] we have presented a constructive proof of an approximate version of Tychonoff's fixed point theorem.

[3]Formulations of Tychonoff's fixed point theorem in this paper follow those in [5].

[4]A set S is finitely enumerable if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S.

sets $K_1, \dots, K_n$, each of diameter less than or equal to ε, such that $X = \cup_{i=1}^n K_i$.
The diameter of K_i is defined as follows.

$$\sup_{x,y\in K_i} \sum_{i\in F} p_i(x-y).$$

In the appendix we present a proof of this lemma.

Our Tychonoff's fixed point theorem is stated as follows;

Theorem 1: (Tychonoff's fixed point theorem for uniformly continuous and sequentially locally non-constant functions) Let X be a compact (totally bounded and complete) and convex subset of a locally convex space E, and g be a uniformly continuous and sequentially locally non-constant unction from X to itself. Then, g has a fixed point.

If X is an n-dimensional simplex Δ this lemma is expressed as follows.

Lemma 2: If Δ is an n-dimensional simplex, for each $\varepsilon > 0$ there exist totally bounded sets $H_1, \dots, H_n$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^n H_i$.

Uniform continuity of a function in a locally convex space is expressed as follows;

Definition 2: (Uniform continuity of a function in a locally convex space) Let X, Y be subsets of a locally convex space. A function $g : X \longrightarrow Y$ is uniformly continuous in X if for each $\varepsilon > 0$ and each finitely enumerable subset G of J, which is also an index set, there exist $\delta > 0$ and a finitely enumerable subset F of I such that if $x, y \in X$ and $\sum_{i\in F} p_i(x-y) < \delta$, then $\sum_{j\in G} q_j(g(x)-g(y)) < \varepsilon$, where $(q_j)_{j\in J}$ is a family of seminorms on Y.
In a metric space a seminorm should be replaced by a metric or a norm in this definition.

Let us consider an n-dimensional simplex Δ, x be a point in Δ, and consider a uniformly continuous function f from Δ into itself. Uniform continuity of functions in Δ is expressed as follows;

> For each $\varepsilon > 0$ there exist $\delta > 0$ such that if $x, y \in \Delta$ and $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$.

According to [9] and [10] f has an approximate fixed point. It means

For each $\varepsilon > 0$ there exists $x \in \Delta$ such that $|x - f(x)| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{x\in\Delta} |x - f(x)| = 0.$$

Then,

$$\inf_{x\in H_i} |x - f(x)| = 0,$$

for some H_i such that $\cup_{i=1}^n H_i = \Delta$.

If X is a compact and convex subset of a locally convex space, there exists a finitely enumerable ε-approximation $\{x^0, x^1, \dots, x^n\}$ to X. Every point in X is within ε for at least one x^j. Consider the following set

$$X_\varepsilon = \left\{ \sum_{j=0}^n \alpha_j x^j \mid \sum_{j=1}^n \alpha_j = 1,\ \alpha_j \geq 0 \right\}.$$

Since X is convex, $X_\varepsilon \subset X$ and they are homeomorphic. X_ε lies in the finite dimensional linear subspace of X spanned by $x^0, x^1, \dots, x^n$. There is a natural identification of this space with an n-dimensional simplex Δ in the Euclidean space with vertices $v^0 = (1, 0, 0, \dots, 0)$, $v^1 = (0, 1, 0, \dots, 0)$, $\dots$, $v^n = (0, 0, \dots, 1)$. Thus, there is a natural identification of X with Δ, and so a uniformly continuous function g from X into itself has an approximate fixed point. Therefore,

$$\inf_{x\in X} \sum_{j\in F} p_j(x - g(x)) = 0.$$

Then, by Lemma 1

$$\inf_{x\in K_i} \sum_{j\in F} p_j(x - g(x)) = 0,$$

for some K_i such that $\cup_{i=1}^n K_i = X$.

The notion that f has at most one fixed point in [2] is defined as follows;

Definition 3: (At most one fixed point) For all $x, y \in \Delta$, if $x \neq y$, then $f(x) \neq x$ or $f(y) \neq y$.

By reference to the notion of *sequentially at most one maximum* in [1], we define the property of *sequential local non-constancy* for $f : \Delta \longrightarrow \Delta$ as follows;

Definition 4: (Sequential local non-constancy of functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $H_1, H_2, \dots, H_m$, each of diameter less than or equal to ε, such that $\Delta = \cup_{i=1}^m H_i$, and if for all sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ in each H_i, $|f(x_n) - x_n| \longrightarrow 0$ and $|f(y_n) - y_n| \longrightarrow 0$, then $|x_n - y_n| \longrightarrow 0$.

We define sequential local non-constancy for functions $g : X \longrightarrow X$ in a locally convex space as follows;

Definition 5: (Sequential local non-constancy of functions in a locally convex space) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets $K_1, K_2, \dots, K_m$, each of diameter less than or equal to ε, such that $X = \cup_{i=1}^m K_i$, and if for all sequences $(x_n)_{n\geq 1}$, $(y_n)_{n\geq 1}$ in each K_i, $\sum_{i\in F} p_i(g(x_n) - x_n) \longrightarrow 0$ and $\sum_{i\in F} p_i(g(y_n) - y_n) \longrightarrow 0$, then $\sum_{i\in F} p_i(x_n - y_n) \longrightarrow 0$.

Now we show the following lemma.

Lemma 3: Let g be a uniformly continuous function from X into itself, and assume that $\inf_{x\in K_i} \sum_{j\in F} p_j(g(x) - x) = 0$. If the following property holds:

> For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in K_i$, $\sum_{j\in F} p_j(g(x) - x) < \delta$ and $\sum_{j\in F} p_j(g(y) - y) < \delta$, then $\sum_{j\in F} p_j(x - y) \leq \varepsilon$.

Then, there exists a point $z \in \Delta$ such that $g(z) = z$, that is, a fixed point of g.

Proof: Choose a sequence $(x_n)_{\geq 1}$ in K_i such that $\sum_{i\in F} p_i(g(x_n) - x_n) \longrightarrow 0$. Compute N such that $\sum_{i\in F} p_i(g(x_n) - x_n) < \delta$ for all $n \geq N$. Then, for $m, n \geq N$ we have $\sum_{i\in F} p_i(x_m - x_n) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $(x_n)_{n\geq 1}$ is a Cauchy sequence in K_i, and converges to a limit $z \in K_i$. The continuity of g yields $\sum_{i\in F} p_i(g(z) - z) = 0$ for each $F \subset I$, that is, $g(z) = z$. ■

Let us prove Tychonoff's fixed point theorem (Theorem 1).

Proof: Assume $\inf_{x\in K_i} \sum_{j\in F} p_j(f(x) - x) = 0$. Choose a sequence $(z_n)_{n\geq 1}$ in $K_i \subset \Delta$ such that $\sum_{i\in F} p_i(f(z_n) - z_n) \longrightarrow 0$. We prove that the following condition holds.

For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in K_i$, $\sum_{j\in F} p_j(f(x) - x) < \delta$ and $\sum_{j\in F} p_j(f(y) - y) < \delta$, then $\sum_{j\in F} p_j(x - y) \le \varepsilon$.

Assume that the set

$$T = \{(x, y) \in K_i \times K_i : \sum_{j\in F} p_j(x - y) \ge \varepsilon\}$$

is nonempty and compact[5]. Since the mapping $(x, y) \longrightarrow \max(\sum_{i\in F} p_i(f(x) - x), \sum_{i\in F} p_i(f(y) - y))$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n\ge1}$ such that

$$\lambda_n = 0 \Rightarrow$$
$$\inf_{(x,y)\in T} \max\left(\sum_{i\in F} p_i(f(x) - x), \sum_{i\in F} p_i(f(y) - y)\right) < 2^{-n},$$

$$\lambda_n = 1 \Rightarrow$$
$$\inf_{(xyu)\in T} \max\left(\sum_{i\in F} p_i(f(x) - x), \sum_{i\in F} p_i(f(y) - y)\right) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $\sum_{i\in F} p_i(f(x) - x) < 2^{-n-1}$, $\sum_{i\in F} p_i(f(y) - y) < 2^{-n-1}$, we have $(x, y) \notin T$ and $\sum_{i\in F} p_i(x - y) \le \varepsilon$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(x_n, y_n) \in T$ such that $\max(\sum_{i\in F} p_i(f(x_n) - x_n), \sum_{i\in F} p_i(f(y_n) - y_n)) < 2^{-n}$, and if $\lambda_n = 1$, set $x_n = y_n = z_n$. Then, $\sum_{i\in F} p_i(f(x_n) - x_n) \longrightarrow 0$ and $\sum_{i\in F} p_i(f(y_n) - y_n) \longrightarrow 0$, so $\sum_{i\in F} p_i(x_n - y_n) \longrightarrow 0$. Computing N such that $\sum_{i\in F} p_i(x_N - y_N) < \varepsilon$, we must have $\lambda_N = 1$. By Lemma 3 f has a fixed point.

We have completed the proof. ■

Appendix

First we show the following lemma which is an extension of Proposition 2.2.11 of [4] to a locally convex space.

Lemma 4: Let X be a totally bounded subset of a locally convex space, x_0 a point of X, and r a positive number. Then, there exists a closed, totally bounded subset K of X such that $U(x_0, F, r) \subset K \subset V(x_0, F, 8r)$, where

$$U(x_0, F, r) = \{x \in X : \sum_{i\in F} p_i(x - x_0) < r\},$$

and

$$V(x_0, F, 8r) = \{x \in X : \sum_{i\in F} p_i(x - x_0) \le 8r\}.$$

F is a finitely enumerable subset of I.

Proof: With $G_1 = \{x_0\}$, construct inductively a sequence $(G_n)_{n\ge1}$ of finitely enumerable subset of X such that

1) $\sum_{i\in F} p_i(x - G_n) < 2^{-n+1}r$ for each x in $U(x_0, F, r)$,
2) $\sum_{i\in F} p_i(x - G_n) < 2^{-n+3}r$ for each x in G_{n+1},

where

$$\sum_{i\in F} p_i(x - G_n) = \inf y \in G_n \sum_{i\in F} p_i(x - y).$$

[5]See Theorem 2.2.13 of [4].

Assume that $G_1, \dots, G_n$ have been constructed and let $\{x_1, \dots, x_N\}$ be a $2^{-n}r$-approximation to X. Write $\{1, \dots, N\}$ as a union of subsets A and B such that

$$\sum_{i\in F} p_i(x_i - G_n) < 2^{-n+3}r \text{ if } i \in A,$$

$$\sum_{i\in F} p_i(x_i - G_n) > 2^{-n+2}r \text{ if } i \in B.$$

Then,

$$G_{n+1} = \{x_i : i \in A\}$$

satisfies the condition (2). Let x be a point of $U(x_0, F, r)$. By the induction hypothesis, there exists y in G_n with $\sum_{i\in F} p_i(x - y) < 2^{-n+1}r$. Choosing i in $\{1, \dots, N\}$ such that $\sum_{i\in F} p_i(x - x_i) < 2^{-n}r$ (Note that $\{x_1, \dots, x_N\}$ is a $2^{-n}r$-approximation to X), we have

$$\sum_{i\in F} p_i(x_i - G_n) \le \sum_{i\in F} p_i(x_i - y) \le \sum_{i\in F} p_i(x - x_i) + \sum_{i\in F} p_i(x - y) < 2^{-n+2}r.$$

Thus, $i \notin B$, so $i \in A$ and $x_i \in G_{n+1}$. Since $\sum_{i\in F} p_i(x - x_i) < 2^{-(n+1)+1}r$, the set G_{n+1} satisfies the condition (1).

Let K be the closure of $\cup_{n\ge1} G_n$ in X. From (1) $U(x_0, F, r) \subset K$. On the other hand, given $x \in K$ and a natural number n, we can find $m \ge n$ and $y \in G_m$ such that $\sum_{i\in F} p_i(x - y) < 2^{-n+4}r$. By (2), there exist points $y_m = y$, $y_{m-1} \in G_{m-1}, \dots, y_n \in G_n$ such that $\sum_{i\in F} p_i(y_{i+1} - y_i) < 2^{-i+3}r$ for $n \le i \le m - 1$. Thus,

$$\sum_{i\in F} p_i(y - G_n) \le \sum_{i\in F} p_i(y - y_n) \le \sum_{i=n}^{m-1} \sum_{i\in F} p_i(y_{i+1} - y_i) < \sum_{i=n}^{\infty} 2^{-i+3}r = 2^{-n+4}r, \quad (1)$$

and

$$\sum_{i\in F} p_i(x - G_n) \le \sum_{i\in F} p_i(x - y) + \sum_{i\in F} p_i(y - G_n) < 2^{-n+4}r + 2^{-n+4}r = 2^{-n+5}r.$$

It follows that $\cup_{i=1}^n G_i$ is a finitely enumerable $2^{-n+5}r$-approximation to K. Since n is arbitrary, we conclude that K is totally bounded.

Taking $n = 1$ in (1), we see that $\sum_{i\in F} p_i(y - x_0) < 8r$ for each y in $\cup_{n\ge1} G_n$, hence $K \subset V(x_0, F, 8r)$. ■

Now the proof of Lemma 1 is as follows.

Proof of Lemma 1: Given $\varepsilon > 0$, construct an $\frac{\varepsilon}{16}$-approximation to X, By Lemma 4, for each i in $\{1, \dots, n\}$ there exists a closed, totally bounded set K_i such that $U(x_i, F, \frac{\varepsilon}{16}) \subset K_i \subset V(x_i, F, \frac{\varepsilon}{2})$. Clearly $X = \cup_{i=1}^n K_i$, and also $\sum_{i\in F} p_i(x - y) \le \varepsilon$ for all x, y in K_i, so the diameter of K_i is smaller than or equal to ε. ■

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