International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:4, 2013

Analysis of $GI/M(n)/1/N$ queue with single working vacation and vacation interruption

P. Vijaya Laxmi, V. Goswami and V. Suchitra

Abstract—This paper presents a finite buffer renewal input single working vacation and vacation interruption queue with state dependent services and state dependent vacations, which has a wide range of applications in several areas including manufacturing, wireless communication systems. Service times during busy period, vacation period and vacation times are exponentially distributed and are state dependent. As a result of the finite waiting space, state dependent services and state dependent vacation policies, the analysis of these queueing models needs special attention. We provide a recursive method using the supplementary variable technique to compute the stationary queue length distributions at pre-arrival and arbitrary epochs. An efficient computational algorithm of the model is presented which is fast and accurate and easy to implement. Various performance measures have been discussed. Finally, some special cases and numerical results have been depicted in the form of tables and graphs.

Keywords—State Dependent Service; Vacation Interruption; Supplementary Variable; Single Working Vacation; Blocking Probability.

I. Introduction

QUEUEING system with server vacations have been studied extensively for the last two decades because these systems are appropriate for modeling computer and telecommunication systems. Past work may be divided into three categories: (i) server vacation (ii) working vacation and (iii) vacation interruption. In classical server vacation, server completely stops service during the vacation. A comprehensive review of vacation models can be found in [1]-[2].

Working vacation policy has practical application background in optimal design of the system. When the number of customers in the system is relatively few, we set a lower speed operating period in order to minimize the operating cost. An analysis of $M/M/1$ queue with multiple working vacations is first examined in [3]. They modeled a wavelength division multiplexing (WDM) optical access network using multiple wavelengths which can be reconfigured. The stochastic decomposition structures of the system indices in the $M/M/1$ queue with multiple and single working vacations is obtained in [4] and [5]. Later, their work was extended to a $GI/M/1/MWV$ queue in [6], to an $M/G/1/MWV$ queue in [7] and $M/G/1/MWV$ with vacation interruption in [8]. Analysis of finite buffer $GI/M/1$ queue with single and multiple working vacations have been discussed in [9] and [10], respectively and the corresponding infinite buffer queue with SWV has been discussed in [11]. The renewal input infinite buffer continuous- and discrete- queues with single working vacation is obtained in [12]. Using quasi birth and death process and matrix-geometric solution method an $M/M/1$ queue with multiple working vacations and N-policy have been studied in [13].

P. Vijaya Laxmi is with the Department of Applied mathematics, Andhra University, Visakhapatnam, 530003, India. e-mail: (vijaya_iit2003@yahoo.co.in, Tel: +91-9989578075).
V. Goswami is with School of Computer Application, KIIT University, Bhubaneswar 751024, India.
V. Suchitra is with Department of Applied mathematics, Andhra University, Visakhaptnam, 530003, India.

During the working vacation policy, the server can still provide service for the main jobs during the vacation period with the lower rate rather than stopping completely. Meanwhile, the server can come back from the vacation to the normal working level once some indices of the system, such as the number of the main jobs, achieves the certain value. The server may come back from the vacation without completing the vacation. Such policy is called vacation interruption. An $M/M/1$ queue with working vacations and vacation interruption using matrix-analysis method have been investigated in [14]. Under such a vacation interruption policy, the server can come back to the normal working level no matter whether the vacation ends. Performance analysis on $GI/M/1$ queue with working vacation and vacation interruption is obtained in [15]. Further, $GI/M/1$ queue with setup period and working vacation and vacation interruption proposed in [16]. An $M/G/1$ queue with single working vacation and vacation interruption under Bernoulli schedule has been studied in [17]. They obtained the distribution for the stationary queue length at departure epochs using the matrix analytic method.

State dependent services are defined if a customer is routed to one of the other service stations. State dependent routing probabilities cannot prevent the routing of jobs to already congested disk stations but they preferentially route jobs to the least loaded disks. Such types of queueing systems are common in applications to communications and computer networks. In a packet switched networks the service provider can charge users a state dependent price, which depends on the extent to which the network is congested. Because of the randomness of packet arrivals, congestion is bound to happen unless the carrier deploys excessive large amount of capacity. To accommodate the needs of emerging applications, guaranteed services are now being introduced to packet switched networks. We present state dependent service processes that would arise if the work processes are allowed to depend on the current queue levels. Now a days, the service provider can charge users a state dependent price, which depends on the extent to which the network is congested in a packet-switched integrated-services network. The economic implications of both state dependent and long-term average pricing has been discussed in [18].

The stationary queue length distributions of $M(n)/G/1/K$ and $G/M(n)/1/K$ queues using quasi-birth-death process has been studied in [19]. Recursion formulae have been derived when the state dependent rates are all different to compute the stationary queue length distribution. An unified algorithm for computing the stationary queue length distributions in $M(k)/G/1/N$ and $G/M(k)/1/N$ queues when some of the state dependent rates are same discussed in [20]. Analysis and computational algorithm for finite buffer renewal input queues with state dependent vacations and services has been discussed in [21]. Using the supplementary variable technique they developed a recursive algorithm. Analysis and computational algorithm for $M(n)/G/1/K$ queues with state dependent vacations has also been studied in [22].

The existing literature in the study of working vacations queueing systems mainly focus on multiple working vacations and single working vacation. There have been no results that presented the analytic and computational aspects of the finite buffer $GI/M/1/N$ queue with single working vacation and vacation interruption under state dependent services. This would motivate us to investigate such queueing model. We connect the single working vacation and vacation interruption policies and assume that during the vacation period if there are jobs in the system after a service completion, the server will come back to the normal working level. Thus, the vacation time and vacation interruption controls when the system ends the vacation period or the lower speed operation period.

This paper presents a finite buffer renewal input single server queueing system with single state dependent exponential working vacation and vacation interruption. The general uncorrelated arrival process seems to be more appropriate and reasonable than exponential distribution, as the memoryless property of the exponential distribution does not always meet the need of applications. Also, the general uncorrelated arrival process can include as the special cases of exponential, deterministic, Erlang, hyperexponential distributions, etc. The service times during a service period, service time during a single vacation period and vacation times are exponentially distributed. Using the supplementary variable technique and treating the remaining inter-arrival time as the supplementary variable, we render a recursive method to develop the steady-state system length distributions at pre-arrival and arbitrary epochs. The recursive method is easy and powerful to implement. An iterative algorithm has been presented for computing the stationary system length distributions. Some performance measures such as blocking probability, the expected queue length, the expected waiting time, etc., have been evaluated. Numerical results have been represented in the form of tables and graphs. This model has potential applications in the design and analysis of queueing system where one of the main objective is to reduce congestion by controlling service and vacation rates.

The rest of the paper is organized as follows. Section II presents the description and analysis of the model. Computational algorithm to compute the stationary system length distribution at pre-arrival and arbitrary epochs is presented in Section III. Various performance measures are evaluated in Section IV. Section V demonstrates numerical results to show the effectiveness of the model parameters. Section VI concludes the paper.

II. Description of the Model

Let us consider a renewal input $GI/M(n)/1/N$ queue where N is the finite capacity of the system. We assume that the inter-arrival times of successive arrivals are independent and identically distributed (i.i.d.) random variables with cumulative distribution function $A(x)$, probability density function $a(x), x \geq 0$, Laplace-Stiletjes (L.-S.) transform $A^*(\theta)$ and mean inter-arrival time $1/\lambda = -A^{*(1)}(0)$, where $h^{(1)}(0)$ denotes the first derivative of $h(\theta)$ evaluated at $\theta = 0$. Whenever the system becomes empty the server takes exactly one vacation. The vacation times follow an exponential distribution and are state dependent with rate $\gamma_i, 0 \leq i \leq N$. The customers are served by a single server in the order of first come first served (FCFS). The service times during a regular busy period (vacation) are assumed to be exponentially distributed random variables with rate μ_i $(\eta_i), 1 \leq i \leq N$, when there are i customers present in the system before beginning a service.

The server takes only one working vacation when the system becomes empty. After returning from SWV, he remains idle in the system if no customer is present in the queue otherwise, a regular service period resumes. Note that the SWV can be thought of as a post-processing time during which the server works at a lower service rate rather than completely stop working. At a service completion epoch during vacation if the server finds customers in the queue, he returns to the regular busy period without completing the remaining vacation which is termed as vacation interruption (VI). The model is denoted by $GI/M(n)/1/N/SWV - VI$. Let $\mu,\ \eta,\ \gamma$ be the mean service rates during regular busy period, working vacation period and mean vacation rate, respectively and are given by

$$\mu = \frac{\sum_{i=1}^{N} \mu_i}{N}, \quad \eta = \frac{\sum_{i=1}^{N} \eta_i}{N}, \quad \gamma = \frac{\sum_{i=0}^{N} \gamma_i}{N}.$$

The traffic intensity is given by $\rho = \lambda/\mu$. Let us define the state of the system and the joint probabilities as

- $N_s(t)$ = Number of customers present in the system, including the one in service,
- $U(t)$ = Remaining inter-arrival time for the next arrival,
- $\zeta(t) = \begin{cases} 0, & \text{if the server is in working vacation period, or idle.} \\ 1, & \text{if the server is in regular busy period.} \end{cases}$

$$\begin{aligned} \pi_{i,0}(u,t)du &= P(N_s(t) \leq i,\ u \leq U(t) \leq u+du,\ \zeta(t)=0), \\ &\quad u \geq 0,\ 0 \leq i \leq N, \\ \pi_{i,1}(u,t)du &= P(N_s(t) \leq i,\ u \leq U(t) \leq u+du,\ \zeta(t)=1), \\ &\quad u \geq 0,\ 0 \leq i \leq N. \end{aligned}$$

A. Analysis of the Model

In this section, we obtain queue length distributions at arbitrary epochs. We first develop the differential-difference equations that relate the distribution of number of customers in the system at the end of vacation period and regular service period. The supplementary variable technique is used to relate the state of the system at two consecutive time epochs t and $t+dt$. Using probability arguments and taking limit as $t \to \infty$, the steady state differential-difference equations are given by

$$\begin{aligned}
-\pi_{0,0}^{(1)}(x) &= \mu_1\pi_{1,1}(x) + \eta_1\pi_{1,0}(x) - \gamma_0\pi_{0,0}(x), && (1)\\
-\pi_{i,0}^{(1)}(x) &= -(\gamma_i+\eta_i)\pi_{i,0}(x) + a(x)\pi_{i-1,0}(0),\\
&\quad 1 \le i \le N-1, && (2)\\
-\pi_{N,0}^{(1)}(x) &= -(\gamma_N+\eta_N)\pi_{N,0}(x)\\
&\quad + a(x)\Big(\pi_{N-1,0}(0) + \pi_{N,0}(0)\Big), && (3)\\
-\pi_{0,1}^{(1)}(x) &= \gamma_0\pi_{0,0}(x), && (4)\\
-\pi_{i,1}^{(1)}(x) &= -\mu_i\pi_{i,1}(x) + \mu_{i+1}\pi_{i+1,1}(x) + \gamma_i\pi_{i,0}(x)\\
&\quad + \eta_{i+1}\pi_{i+1,0}(x) + a(x)\pi_{i-1,1}(0),\\
&\quad 1 \le i \le N-1, && (5)\\
-\pi_{N,1}^{(1)}(x) &= -\mu_N\pi_{N,1}(x) + \gamma_N\pi_{N,0}(x)\\
&\quad + a(x)\Big(\pi_{N-1,1}(0) + \pi_{N,1}(0)\Big), && (6)
\end{aligned}$$

where $\pi_{i,0}(0)$ and $\pi_{i,1}(0)$ are the respective rates of arrivals i.e., an arrival is about to occur. Let us define the Laplace transforms of $\pi_{i,0}(x)$ and $\pi_{i,1}(x)$ as $\pi_{i,0}^*(\theta) = \int_0^\infty e^{-\theta x}\pi_{i,0}(x)dx$, $\pi_{i,1}^*(\theta) = \int_0^\infty e^{-\theta x}\pi_{i,1}(x)dx$, $Re\,\theta \ge 0$, so that, $\pi_{i,0} \equiv \pi_{i,0}^*(0)$ and $\pi_{i,1} \equiv \pi_{i,1}^*(0)$, where $\pi_{i,0}(\pi_{i,1})$ is the joint probability that there are i customers in the system and the server is on working vacation (busy period) at an arbitrary epoch. Multiplying (1) to (6) by $e^{-\theta x}$ and integrating with respect to x from 0 to ∞ yields

$$\begin{aligned}
(\gamma_0-\theta)\pi_{0,0}^*(\theta) &= \mu_1\pi_{1,1}^*(\theta) +\\
&\quad \eta_1\pi_{1,0}^*(\theta) - \pi_{0,0}(0), && (7)\\
(\gamma_i+\eta_i-\theta)\pi_{i,0}^*(\theta) &= A^*(\theta)\pi_{i-1,0}(0) - \pi_{i,0}(0),\\
&\quad 1 \le i \le N-1, && (8)\\
(\gamma_N+\eta_N-\theta)\pi_{N,0}^*(\theta) &= A^*(\theta)(\pi_{N-1,0}(0) + \pi_{N,0}(0))\\
&\quad -\pi_{N,0}(0), && (9)\\
-\theta\pi_{0,1}^*(\theta) &= \gamma_0\pi_{0,0}^*(\theta) - \pi_{0,1}(0) && (10)\\
(\mu_i-\theta)\pi_{i,1}^*(\theta) &= \mu_{i+1}\pi_{i+1,1}^*(\theta) + \gamma_i\pi_{i,0}^*(\theta)\\
&\quad +\eta_{i+1}\pi_{i+1,0}^*(\theta)\\
&\quad +A^*(\theta)\pi_{i-1,1}(0) - \pi_{i,1}(0),\\
&\quad 1 \le i \le N-1, && (11)\\
(\mu_N-\theta)\pi_{N,1}^*(\theta) &= \gamma_N\pi_{N,0}^*(\theta) - \pi_{N,1}(0)\\
&\quad +A^*(\theta)(\pi_{N,1}(0)\\
&\quad +\pi_{N-1,1}(0)). && (12)
\end{aligned}$$

Now using the equations (7) to (12) we obtain the following important lemma:

Lemma:

$$\sum_{i=0}^{N}\pi_{i,0}(0) + \sum_{i=0}^{N}\pi_{i,1}(0) = \lambda. \qquad (13)$$

The left hand side is the mean number of entrances into the system per unit time and is equal to the mean arrival rate λ.

Proof: Adding (7) to (12) and taking limit as $\theta \to 0$, we obtain the desired result using the normalization condition $\sum_{i=0}^{N}\pi_{i,0} + \sum_{i=0}^{N}\pi_{i,1} = 1$. Substituting $\theta = (\gamma_N + \eta_N)$ in (9), we get

$$\pi_{N-1,0}(0) = \left(\frac{1 - A^*(\gamma_N+\eta_N)}{A^*(\gamma_N+\eta_N)}\right)\pi_{N,0}(0).$$

From (9), we have

$$\pi_{N,0}^*(\theta) = \frac{(A^*(\theta) - A^*(\gamma_N+\eta_N))}{(\gamma_N+\eta_N-\theta)A^*(\gamma_N+\eta_N)}\pi_{N,0}(0).$$

Substituting $\theta = (\gamma_i + \eta_i)$ in (8), we get

$$\pi_{i-1,0}(0) = \frac{\pi_{i,0}(0)}{A^*(\gamma_i+\eta_i)} = \left(\frac{1 - A^*(\gamma_N+\eta_N)}{\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}\right)\pi_{N,0}(0), \quad i = N-1,\ldots,1.$$

From (8), we obtain

$$\begin{aligned}
\pi_{i,0}^*(\theta) &= \frac{A^*(\theta)\pi_{i-1,0}(0) - \pi_{i,0}(0)}{(\gamma_i+\eta_i-\theta)}\\
&= \frac{(1 - A^*(\gamma_N+\eta_N))}{(\gamma_i+\eta_i-\theta)\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}\\
&\quad \times \frac{(A^*(\theta) - A^*(\gamma_i+\eta_i))}{(\gamma_i+\eta_i-\theta)\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)},\\
&\quad i = N-1,\ldots,1.
\end{aligned}$$

Substituting $\theta = \mu_N$ in (12) and $\theta = \mu_i$ in (11), we get

$$\begin{aligned}
\pi_{N-1,1}(0) &= \frac{1 - A^*(\mu_N)}{A^*(\mu_N)}\pi_{N,1}(0) - \frac{\gamma_N}{A^*(\mu_N)}\pi_{N,0}^*(\mu_N),\\
\pi_{i-1,1}(0) &= \frac{\pi_{i,1}(0)}{A^*(\mu_i)} - \frac{\mu_{i+1}\pi_{i+1,1}^*(\mu_i)}{A^*(\mu_i)}\\
&\quad -\frac{\eta_{i+1}\pi_{i+1,0}^*(\mu_i)}{A^*(\mu_i)} - \frac{\gamma_i\pi_{i,0}^*(\mu_i)}{A^*(\mu_i)},\\
&\quad i = N-1,\ldots,2,
\end{aligned}$$

where $\pi_{i,1}^*(\theta)$ are given by the following:

$$\begin{aligned}
\pi_{N,1}^*(\theta) &= \frac{\gamma_N\pi_{N,0}^*(\theta) + A^*(\theta)(\pi_{N-1,1}(0) + \pi_{N,1}(0))}{(\mu_N-\theta)}\\
&\quad -\frac{\pi_{N,1}(0)}{(\mu_N-\theta)},\\
\pi_{i,1}^*(\theta) &= \frac{\gamma_i\pi_{i,0}^*(\theta) + \mu_{i+1}\pi_{i+1,1}^*(\theta) + \eta_{i+1}\pi_{i+1,0}^*(\theta)}{(\mu_i-\theta)}\\
&\quad +\frac{A^*(\theta)\pi_{i-1,1}(0) - \pi_{i,1}(0)}{(\mu_i-\theta)},\\
&\quad i = N-1,\ldots,1.
\end{aligned}$$

For $\theta = \gamma_i + \eta_i$, $\pi_{i,0}^*(\theta)$ are given by

$$\begin{aligned}
\pi_{N,0}^*(\theta) &= -A^{*(1)}(\theta)\left(\pi_{N-1,0}(0) + \pi_{N,0}(0)\right),\\
\pi_{i,0}^*(\theta) &= -\Big(A^{*(1)}(\theta)\pi_{i-1,0}(0)\Big), \quad 1 \le i \le N-1.
\end{aligned}$$

For $\theta = \mu_i$, $\pi^*_{i,1}(\theta)$ are given by

$$\pi^*_{N,1}(\theta) = -(\gamma_N \pi^{*(1)}_{N,0}(\theta) + A^{*(1)}(\theta)(\pi_{N-1,1}(0) + \pi_{N,1}(0))),$$
$$\pi^*_{i,1}(\theta) = -(\gamma_i \pi^{*(1)}_{i,0}(\theta) + \mu_{i+1}\pi^{*(1)}_{i+1,0}(\theta) + \eta_{i+1}\pi^{*(1)}_{i+1,0}(\theta) + A^{*(1)}(\theta)\pi_{i-1,1}(0)), \quad 1 \le i \le N-1.$$

B. Relation Between Steady State Distribution at Pre-Arrival and Arbitrary Epochs

Let $\pi^-_{i,j}$, $0 \le i \le N$, $j = 0, 1$ denote the pre-arrival epoch probability, that is, an arrival sees i customers in the system and the server is in state j at arrival epoch. Applying Bayes' theorem, we have

$$\pi^-_{i,j} = \lim_{t\to\infty} \frac{P[N_s(t) = i, \zeta(t) = j, U(t) = 0]}{P[U(t) = 0]}.$$

Further, using (13) in the above expression, we obtain

$$\pi^-_{i,j} = \frac{1}{\lambda}\pi_{i,j}(0), \; 0 \le i \le N, \; j = 0, 1. \quad (14)$$

From the above set of expressions one can evaluate pre-arrival epoch probabilities.

To obtain the steady state probabilities at arbitrary epochs, we develop a relation between pre-arrival and arbitrary epoch probabilities. Setting $\theta = 0$ in (9), (8), (12) to (7) and using (14), we obtain

$$\pi_{0,0} = \frac{\pi^-_{0,1}}{\gamma_0}, \quad (15)$$
$$\pi_{N,0} = \frac{\lambda}{\gamma_N + \eta_N}\pi^-_{N-1,0}, \quad (16)$$
$$\pi_{i,0} = \frac{\lambda}{\gamma_i + \eta_i}(\pi^-_{i-1,0} - \pi^-_{i,0}) \quad 1 \le i \le N-1, \quad (17)$$
$$\pi_{N,1} = \frac{\lambda}{\mu_N}\left[\left(\frac{\gamma_N}{\gamma_N + \eta_N}\right)\pi^-_{N-1,0} + \pi^-_{N-1,1}\right], \quad (18)$$
$$\pi_{i,1} = \frac{\lambda}{\mu_i}\left[\frac{\eta_i}{\gamma_i + \eta_i}\pi^-_{i,0} + \frac{\gamma_i}{\gamma_i + \eta_i}\pi^-_{i-1,0} + \pi^-_{i-1,1}\right], \quad 1 \le i \le N-1, \quad (19)$$
$$\pi_{0,1} = 1 - \sum_{i=0}^{N}\pi_{i,1} - \sum_{i=1}^{N}\pi_{i,1}.. \quad (20)$$

As the state probabilities at pre-arrival epochs are known from (14), we can evaluate the arbitrary epoch probabilities using (15) to (20).

Remark 1: $\mu_i = \mu$, $\gamma_i = \gamma$, $\eta_i = \eta$, for all $i = 1, \ldots, N$, that is, there is state independent services and state independent single working vacation with vacation interruption. Our model reduces to $GI/M/1/N$ queue with single working vacation with vacation interruption.

Remark 2: $\eta_i \to 0$ for $1 \le i \le N$, that is, there is state dependent services and state dependent single vacation. In this case, the model reduces to $GI/M/1/N$ queue with state dependent single vacation.

Remark 3: $\mu_i = \mu$, $\gamma_i = \gamma$, $\eta_i \to 0$, for all $i = 1, \ldots, N$, that is, there is state independent services and state independent single vacation without vacation interruption. The model reduces to $GI/M/1/N$ queue with single vacations and our results match with the results of Tian et al. [2].

Remark 4: $\mu_i = \mu$, $\gamma_i \to \infty$, $\eta_i \to 0$ for all $i = 1, \ldots, N$, that is, there is state independent services and no vacations. The model reduces to $GI/M/1/N$ queue without vacations and our results match with the results available in literature.

III. Computational Algorithm

In this section, we present a computational algorithm for the steady state probabilities that summarizes the results obtained in the previous section.

Step 1: For $0 \le i \le N$, calculate $\pi_{i,0}(0)$ and $\pi^*_{i,0}(\theta)$ in terms of $\pi_{N,0}(0)$ as follows

$$\pi^*_{i,0}(\theta) = \zeta_{i,\theta}\pi_{N,0}(0), \quad 0 \le i \le N.$$

Algorithm for calculating $\pi_{i,0}(0)$, and $\pi_{i,1}(0)$ is as follows:

- Calculate $\pi_{i,0}(0)$, $0 \le i \le N$, as follows

$$\pi_{N,0}(0) = 1,$$
$$\pi_{i,0}(0) = \frac{1 - A^*(\gamma_N + \eta_N)}{\prod_{j=i+1}^{N} A^*(\gamma_j + \eta_j)}, \; i = N-1, \ldots, 1, 0.$$

- Calculate $\zeta_{i,\theta}$ as follows

if $i = N$ **then**
  if $\theta = \gamma_N + \eta_N$ **then**
    $\zeta_{N,\theta} = -\left(\frac{A^{*(1)}(\theta)}{A^*(\gamma_N+\eta_N)}\right)$
  else
    $\zeta_{N,\theta} = \frac{A^*(\theta) - A^*(\gamma_N+\eta_N)}{A^*(\gamma_N+\eta_N)(\gamma_N+\eta_N-\theta)}$
  end if
else if $1 \le i \le N-1$ **then**
  if $\theta = \gamma_i + \eta_i$ **then**
    $\zeta_{i,\theta} = -\frac{A^{*(1)}(\theta)(1-A^*(\gamma_N+\eta_N))}{\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}$
  else
    $\zeta_{i,\theta} = \frac{(1-A^*(\gamma_N+\eta_N))(A^*(\theta)-A^*(\gamma_i+\eta_i))}{(\gamma_i+\eta_i-\theta)\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}$
  end if
end if

- Calculate $\zeta^{(l)}_{i,\theta}$ as follows

if $i = N$ **then**
  if $\theta = \gamma_N + \eta_N$ **then**
    $\zeta^{(l)}_{N,\theta} = -\left(\frac{A^{*(l+1)}(\theta)}{(l+1)A^*(\gamma_N+\eta_N)}\right)$
  else
    $\zeta^{(l)}_{N,\theta} = \frac{A^{*(l)}(\theta) + A^*(\gamma_N+\eta_N)l\zeta^{(l-1)}_{N,\theta}}{(\gamma_N+\eta_N-\theta)A^*(\gamma_N+\eta_N)}$
  end if
else if $1 \le i \le N-1$ **then**
  if $\theta = \gamma_i + \eta_i$ **then**
    $\zeta^{(l)}_{i,\theta} = -\frac{A^{*(l+1)}(\theta)(1-A^*(\gamma_N+\eta_N))}{(l+1)\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}$
  else

$$\zeta^{(l)}_{i,\theta} = \frac{A^{*(l)}(\theta)(1-A^*(\gamma_N+\eta_N)) + l\zeta^{(l-1)}_{i,\theta}(\theta)\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}{(\gamma_i+\eta_i-\theta)\prod_{j=i}^{N} A^*(\gamma_j+\eta_j)}$$

end if
end if

Step 2: For $i = N, N-1, \ldots, 1$, calculate $\pi_{i,1}(0)$ and $\pi_{i,1}(\theta)$ in terms of $\pi_{N,1}(0)$ and $\pi_{N,0}(0)$ as given below.

$$\pi_{i,1}(0) = t_i \pi_{N,0}(0) + d_i \pi_{N,1}(0),\ 0 \le i \le N,$$
$$\pi^*_{i,1}(\theta) = e_{i,\theta}\pi_{N,0}(0) + f_{i,\theta}\pi_{N,1}(0),\ 1 \le i \le N,$$

where $t_i, d_i, e_{i,\theta}, f_{i,\theta}$ are computed as follows.

- Calculate t_i and d_i as follows.

$$t_N = 0,\ \ d_N = 1,\ \ t_{N-1} = -\frac{\gamma_N \zeta_{N,\mu_N}}{A^*(\mu_N)},$$
$$d_{N-1} = \frac{1 - A^*(\mu_N)}{A^*(\mu_N)},$$
$$t_{i-1} = \frac{t_i - \mu_{i+1}e_{i+1,\mu_i} - \eta_{i+1}\zeta_{i+1,\mu_i}}{A^*(\mu_i)} - \frac{\gamma_i \zeta_{i,\mu_i}}{A^*(\mu_i)},$$
$$i = N-1, N-2, \ldots, 1,$$
$$d_{i-1} = \frac{d_i - \mu_{i+1}f_{i+1,\mu_i}}{A^*(\mu_i)},$$
$$i = N-1, N-2, \ldots, 1.$$

- Calculate $e_{i,\theta}$ and $f_{i,\theta}$ as follows

if $i = N$ **then**
  if $\theta = \mu_N$ **then**
    $e_{N,\theta} = -\gamma_N \zeta^{(1)}_{N,\theta} - A^{*(1)}(\theta)t_{N-1}$
    $f_{N,\theta} = \frac{-A^{*(1)}(\theta)}{A^*(\mu_N)}$
  else
    $e_{N,\theta} = \frac{\gamma_N\zeta_{N,\theta} + A^*(\theta)t_{N-1}}{\mu_N - \theta}$
    $f_{N,\theta} = \frac{A^*(\theta) - A^*(\mu_N)}{A^*(\mu_N)(\mu_N - \theta)}$
  end if
else if $1 \le i \le N-1$ **then**
  if $\theta = \mu_i$ **then**
    $e_{i,\theta} = -(\gamma_i \zeta^{(1)}_{i,\theta} + \mu_{i+1}e^{(1)}_{i+1,\theta} + \eta_{i+1}\zeta^{(1)}_{i+1,\theta} + A^{*(1)}(\theta)t_{i-1})$
    $f_{i,\theta} = -\left(\mu_{i+1}f^{(1)}_{i+1,\theta} + A^{*(1)}(\theta)d_{i-1}\right)$
  else
    $e_{i,\theta} = \frac{\gamma_i\zeta_{i,\theta} + \mu_{i+1}e_{i+1,\theta} + \eta_{i+1}\zeta_{i+1,\theta} + A^*(\theta)t_{i-1} - t_i}{\mu_i - \theta}$
    $f_{i,\theta} = \frac{\mu_{i+1}f_{i+1,\theta} + A^*(\theta)d_{i-1} - d_i}{\mu_i - \theta}$
  end if
end if

- Calculate $e^{(l)}_{i,\theta}$ and $f^{(l)}_{i,\theta}$ as follows

if $i = N$ **then**
  if $\theta = \mu_N$ **then**
    $e^{(l)}_{N,\theta} = -\frac{\gamma_N\zeta^{(l+1)}_{N,\theta} + A^{*(l+1)}(\theta)t_{N-1}}{l+1}$
    $f^{(l)}_{N,\theta} = -\frac{A^{*(l+1)}(\theta)}{(l+1)A^*(\mu_N)}$
  else
    $e^{(l)}_{N,\theta} = \frac{\gamma_N\zeta^{(l)}_{N,\theta} + A^{*(l)}(\theta)t_{N-1} + le^{(l-1)}_{N,\theta}}{\mu_N - \theta}$
    $f^{(l)}_{N,\theta} = \frac{A^{*(l)}(\theta) + A^*(\mu_N)lf^{(l-1)}_{N,\theta}}{A^*(\mu_N)(\mu_N - \theta)}$
  end if
else if $1 \le i \le N-1$ **then**
  if $\theta = \mu_i$ **then**
    $e^{(l)}_{i,\theta} = -\frac{\gamma_i\zeta^{(l+1)}_{i,\theta} + \mu_{i+1}e^{(l+1)}_{i+1,\theta} + \eta_{i+1}\zeta^{(l+1)}_{i+1,\theta}}{l+1} + \frac{A^{*(l+1)}(\theta)t_{i-1}}{l+1}$
    $f^{(l)}_{i,\theta} = -\frac{\mu_{i+1}f^{(l+1)}_{i+1,\theta} + A^{*(l+1)}(\theta)d_{i-1}}{l+1}$
  else
    $e^{(l)}_{i,\theta} = \frac{\gamma_i\zeta^{(l)}_{i,\theta} + \mu_{i+1}e^{(l)}_{i+1,\theta} + \eta_{i+1}\zeta^{(l)}_{i+1,\theta} + A^{*(l)}(\theta)t_{i-1}}{\mu_i - \theta} + \frac{le^{(l-1)}_{i,\theta}}{\mu_i - \theta}$
    $f^{(l)}_{i,\theta} = \frac{\mu_{i+1}f^{(l)}_{i+1,\theta} + A^{*(l)}(\theta)d_{i-1} + lf^{(l-1)}_{i,\theta}}{\mu_i - \theta}$
  end if
end if

Step 3: For $i = 0, 1, 2, \ldots, N$, calculate $\pi_{i,1}(0)$ in terms of $\pi_{N,0}(0)$ as follows

$$\pi_{i,1}(0) = (t_i + kd_i)\pi_{N,0}(0),$$

where
$$k = \frac{\psi_0 - \eta_1\zeta_{1,\gamma_0} - \mu_1 e_{1,\gamma_0}}{\mu_1 f_{1,\gamma_0}}.$$

Step 4: Determine $\pi_{N,0}(0)$ from equation (13) as

$$\pi_{N,0}(0) = \lambda\left[1 + \sum_{i=0}^{N}(t_i + kd_i) + \sum_{i=0}^{N-1}\left(\frac{1 - A^*(\gamma_N + \eta_N)}{\prod_{j=i+1}^{N} A^*(\gamma_j + \eta_j)}\right)\right]^{-1}.$$

IV. Performance Measures

In this section, we discuss some operating characteristics such as the average number of customers in the queue (L_q), average number of customers in the system (L_s), probability that the server is in idle period (P_{idle}), probability that the server is in regular busy period (P_b), probability that the server is in working vacation period (P_{wv}) and the blocking probability of the server (P_{loss}). They are given by:

$$L_q = \sum_{i=1}^{N}(i-1)\pi_{i,0} + \sum_{i=1}^{N}(i-1)\pi_{i,1},$$
$$L_s = \sum_{i=1}^{N} i\pi_{i,0} + \sum_{i=1}^{N} i\pi_{i,1},$$
$$P_{idle} = \pi_{0,1}, P_b = \sum_{i=1}^{N}\pi_{i,1}, P_{wv} = \sum_{i=0}^{N}\pi_{i,0},$$
$$P_{loss} = \pi^-_{N,0} + \pi^-_{N,1}.$$

The average waiting time of a customer in the queue (W_q) using Little's rule is given by $W_q = L_q/\hat{\lambda}$, where $\hat{\lambda} = \lambda(1 - P_{loss})$ is the effective arrival rate.

V. Numerical Results

In this section, to validate the computational algorithm some numerical results are presented in the form of tables and graphs. Here we assume that $\mu_i = i * 0.2,\ \eta_i = i * 0.15, \gamma_i = i * 0.05,$ for $1 \le i \le N$, $\rho = 0.7$ and $N = 20$ with mean $\mu = 2.1$, $\gamma = 0.525$ and $\eta = 1.575$.

Table 1 presents the results of the queue length distribution at pre-arrival and arbitrary epochs for the arrival distributions:

TABLE I
QUEUE LENGTH DISTRIBUTIONS AT PRE-ARRIVAL AND ARBITRARY EPOCHS

	$M/M/1/20$		$E_3/M/1/20$	
	pre-arrival	arbitrary	pre-arrival	arbitrary
$\pi_{0,0}$	0.000552	0.000552	0.000130	0.000067
$\pi_{1,0}$	0.000517	0.000517	0.000122	0.000124
$\pi_{2,0}$	0.000455	0.000455	0.000106	0.000111
$\pi_{5,0}$	0.000221	0.000221	0.000049	0.000054
$\pi_{10,0}$	0.000025	0.000025	0.000000	0.000000
$\pi_{15,0}$	0.000000	0.000000	0.000000	0.000000
$\pi_{20,0}$	0.000000	0.000000	0.000000	0.000000
$\pi_{0,1}$	0.000025	0.000025	0.000000	0.000000
$\pi_{1,1}$	0.004074	0.004074	0.001665	0.000941
$\pi_{2,1}$	0.016723	0.016723	0.009845	0.006533
$\pi_{5,1}$	0.114585	0.114585	0.129506	0.112198
$\pi_{10,1}$	0.081577	0.081577	0.068838	0.082384
$\pi_{15,1}$	0.004860	0.004860	0.000760	0.001175
$\pi_{20,1}$	0.000056	0.000056	0.000000	0.000000
Sum	1.000000	1.000000	1.000000	1.000000
L_q	6.35128		6.35033	
W_q	4.32084		4.31995	

exponential and Erlang-3 for $\lambda = 1.89,\ \eta = 1.575,\ \rho = 0.7$ and $\gamma = 0.525$, along with the performance measures at the bottom of the table. Note that for exponential distribution the pre-arrival and arbitrary epoch probabilities are same due to the memoryless property of exponential distribution.

Figure 1 illustrates the effect of mean service rate during working vacation (η) on the average queue length (L_q) for different mean vacation rates $\gamma = 0.525,\ 1.05,\ 1.575$ (by taking $\gamma_i = i * 0.05,\ \gamma_i = i * 0.10, \gamma_i = i * 0.15$) for exponential inter-arrival time. The other parameters are as mentioned above with the mean $\mu = 2.1$. We observe that L_q decreases as η increases. Further, as the mean vacation rate (γ) increases, the L_q decreases up to certain level (near the point 2.1 where η crosses μ) and then reverses its trend. This is due to the fact that with increase of η the number of customers getting service during vacation increases resulting in the decrease of L_q with the increase in γ and when η crosses μ, it is beneficial to render regular service. Therefore, the working vacation queue utilizes the idle time effectively when $\eta < \mu$.

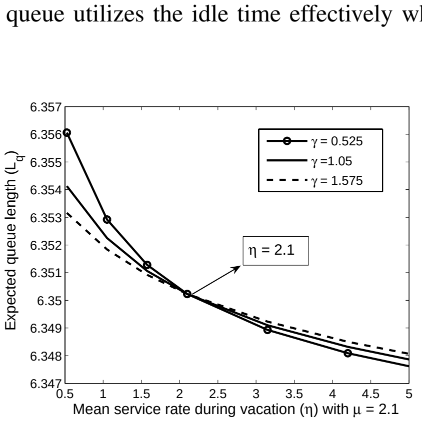

Fig. 1. Effect of mean η on L_q for different vacation rates

Figure 2 explains the effect of traffic intensity (ρ) on the expected queue length (L_q) for different inter-arrival distributions. It can be observed that as expected, L_q increases with ρ. We further observe that for a fixed ρ, HE_2 distribution yields highest and deterministic distribution the least queue lengths. Figure 3 shows the impact of traffic intensity (ρ) on the

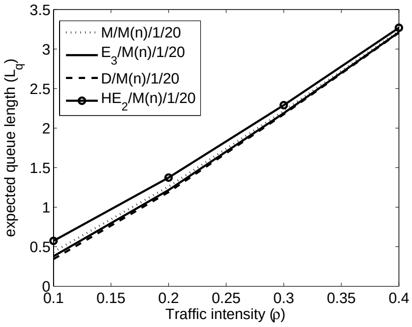

Fig. 2. Effect of ρ on L_q for different arrival distributions

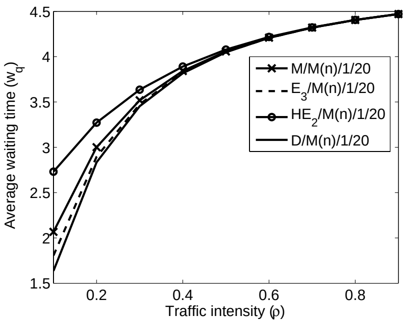

Fig. 3. Effect of ρ on W_q for different arrival distributions

average waiting time in the queue (W_q) for all the inter-arrival time distributions with parameters $\mu = 2.1$, η=1.575, and γ=0.525, (For HE_2 distribution λ_1= 0.811734, λ_2=3.2, σ_1=0.4, σ_1=0.6). It can be seen that for any inter-arrival time distribution, W_q drastically increases finally and becomes constant as ρ increases. This is because whenever the traffic intensity increases the average waiting time increases.

Figure 4 provides a comparison of expected queue lengths among the models: (i) $GI/M(n)/1/N/SWV-VI$, (ii) $GI/M/1/N/SWV$, (iii) $GI/M(n)/1/N/SV$, and (iv) $GI/M/1/N/SV$. Intuitively as expected, the queue length increases as the arrival rate increases. Moreover, the models with state dependent service (i) and (iii) perform better than the models with constant service and vacation rates. In addition to this the single working vacation model gives better results than the single vacation model.

Finally, we can conclude that:

- For the state dependent model, the expected queue length decreases with increase of mean service rate during working vacation for different vacation rates.
- The expected queue length increases with traffic intensity as well as for different buffer sizes.

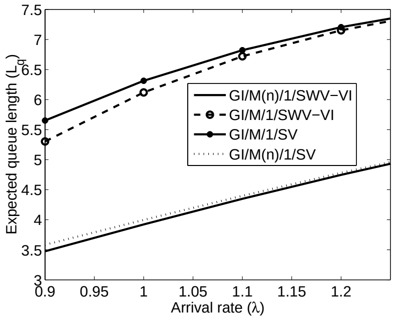

Fig. 4. Effect on λ vs L_q

- It is better to have state dependent services in addition to a working server during the vacation to give better services to the waiting lines.

VI. Conclusion

In this paper, we have carried out an analysis of a finite buffer renewal input single working vacation and vacation interruption queue with state dependent services and state dependent vacations that have potential applications in telecommunication systems, manufacturing systems, wireless and communication systems. We have developed a recursive method using the supplementary variable technique to compute the stationary queue length distributions at pre-arrival and arbitrary epochs. The inter-arrival times of customers is arbitrarily distributed while the service times during busy period, vacation period and vacation times are exponentially distributed and are state dependent. An efficient computational algorithm of the model is presented which is fast and accurate and easy to implement. We have deduced some special cases of the model by taking specific values of the parameters. Various performance measures and numerical results are illustrated in the form of tables and graphs. The incorporation of state dependent mechanism brings more closely towards the real-life application systems, in setting traffic management strategies based on performance indices.

References

[1] B. T. Doshi, *Queueing Systems with Vacations - A Survey*, Queueing Systems, Vol. 1 , pp. 29–66, 1986.

[2] N. Tian, N and Z. G. Zhang, *Vacation Queueing Models: Theory and Applications*, Springer-Verlag, New York, 2006.

[3] L. D. Servi and S. G. Finn, *The $M/M/1$ Queues with Working Vacations ($M/M/1/WV$)*, Perf. Eval., Vol. 50, pp. 41–52, 2002.

[4] W. Y. Liu, X. L. Xu, N. Tian, *Stochastic decompositions in the $M/M/1$ queue with working vacations*, Oper. Res. Lett., Vol. 35, pp. 596-600, 2007.

[5] N.Tian, X. Xhao and K. H. Wang, *The $M/M/1$ Queue with Single Working Vacation*, Int. J. Inform. Manage. Sci., Vol. 19, pp. 621–634, 2008.

[6] Y. Baba, *Analysis of a $GI/M/1$ Queue with Multiple Working Vacations*, Oper. Res. Lett., Vol. 33, 201–209, 2005.

[7] D. Wu and H. Takagi, *$M/G/1$ queue with multiple working vacations*, Perform. Eval., Vol. 63, pp. 654–681, 2006.

[8] M. Zhang and Z. Hou, *Performance analysis of $M/G/1$ queue with working vacations and vacation interruption*, Journal of Computational and Applied Mathematics, Vol. 234, pp. 2977–2985, 2010.

[9] A. D. *Banik, Analysis of Single Working Vacation in $GI/M/1/N$ and $GI/M/1$ Queueing Systems*, Int. J. Oper. Res., Vol.7, 314–333, 2010.

[10] A. D. Banik, U. C. Gupta and S. S. Pathak, *On the $GI/M/1/N$ Queue with Multiple Working Vacations - Analytic Analysis and Computation*, Appl. Math. Modell., Vol. 31, 1701–1710, 2007.

[11] J. Li and N. Tian, *Performance Analysis of a $GI/M/1$ Queue with Single Working Vacation*, Appl. Math. Comput. , Vol. 217, pp. 4960–4971, 2011.

[12] K. C. Chae, D. E. Lim and W. S. Yang, *The $GI/M/1$ queue and the $GI/Geo/1$ queue both with single working vacation*, Perform. Eval., Vol. 66, pp. 356–367, 2009.

[13] Z. Zhang and X. Xu, *Analysis for the $M/M/1$ Queue with Multiple Working Vacations and N-Policy*, Int. J. Inform. Manage. Sci., Vol. 19, pp. 495–506, 2008.

[14] J. Li and N. Tian, *The $M/M/1$ Queue with Working Vacations and Vacation Interuptions*, J. Syst. Sci. Syst. Eng., Vol. 16, pp. 121–127, 2007.

[15] J. Li, N. Tian and Z. Ma, *Performence Analysis of $GI/M/1$ Queue with Working Vacation and Vacation Interruption*, Appl. Math. Modell., Vol. 32, pp. 2715–2730, 2008.

[16] G. Zhao, X. Du and N. Tian, *$GI/M/1$ Queue with Setup Period and Working Vacation and Vacation Interruption*, Int. J. Inform. Manage. Sci., Vol. 20, pp. 351–363, 2009.

[17] S. Guo and Z. Liu, *An $M/G/1$ Queue with Single Working Vacation and Vacation Interruption under Bernoulli Schedule*, Appl. Math. Modell., in press, 2012.

[18] Q. Wang and J. M. Peha, *State Dependent Pricing and its Economic Implications*, Telecommu. Syst., Vol. 18, pp. 315–329, 2001.

[19] M. Kijima and N. Makimoto, *A Unified Approach to $GI/M(n)/1/K$ and $M(n)/G/1/K$ Queues via Finite Quasi-Birth-Death Processes*, Commun. Stat. Stoch. Models, Vol. 8, pp. 269–288, 1992.

[20] P. Yang, *Unified Algorithm for Computing the Stationary Queue Length Distribution in $M(k)/G(n)/1/N$ and $GI/M(k)/1/N$ Queues*, Queueing Systems, Vol. 17, pp. 383–401, 1994.

[21] X. Chao and R. Rahman, *Anlysis and Computational Algorithem for Queues with State Dependent Vacations I: $G/M(n)/1/K$*, J. Syst. Sci. Complex., Vol. 19, pp. 36–53, 2006.

[22] X. Chao and R. Rahman, *Anlysis and Computational Algorithem for Queues with State Dependent Vacations II: $M(n)/G/1/K$*, J. Syst. Sci. Complex., Vol. 19, pp. 191–210, 2006.