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Dense chaos in coupled map lattices

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Abstract—This paper is mainly concerned with a kind of coupled map lattices (CMLs). New definitions of dense δ-chaos and dense chaos (which is a special case of dense δ-chaos with $\delta = 0$) in discrete spatiotemporal systems are given and sufficient conditions for these systems to be densely chaotic or densely δ-chaotic are derived.

Keywords—Discrete spatiotemporal systems; coupled map lattices; dense δ-chaos; Li-Yorke pairs.

I. Introduction

CHAOTIC properties of a dynamical system is ardently discussed since the introduction of the term chaos in 1975 by Li and Yorke[1]. Let I be a closed interval on real line. If a dynamical system (I, f) has a uncountable set $S \subset I$ in which (x, y) is a Li-Yorke pair for $\forall x, y \in S : x \neq y$ (the definition of Li-Yorke pairs will be seen follow), then (I, f) is said to be chaotic in the sense of Li-Yorke. While, the definition of chaos in the sense of Li-Yorke is inconveniently in engineering applications. In 1989, R. L. Devaney[2] stated a definition of chaos, known as Devaney chaos today. A map f is said to be chaotic in the sense of Devaney on I if f is transitive on I, the set of periodic points of f is dense in I and f has sensitive dependence on initial conditions. Then, in 1992, Banks[3] proved that if $f : (X, d) \to (X, d)$ is transitive and has dense periodic points then f has sensitive dependence on initial conditions (where X is a compact metric space with no isolated point). This causes that Devaney's chaoticity is preserved under topological conjugation on generally infinite metric space. And then, in 1992, Lubomir Snoha[4] say that f is densely chaotic if the set of Li-Yorke pairs is dense in $I \times I$. In 2005, the definition of densely δ-chaotic is given by Schweizer and Smital[5]. That is, f is densely δ-chaotic if the set of Li-Yorke pairs modulus δ is dense in $I \times I$.

The coupled map lattices(CMLs) as spatiotemporal chaotic systems were proposed in 1983 by Kaneko[6]. Since it is a simple model with most essential features of spatiotemporal chaos, the CMLs have been extensively studied in the fields of bifurcation and chaos, pattern formation, physical biology and engineering, cryptography, thermodynamics and chaotic dynamics. (see, for instance, [7-12] and some references therein.)

In this paper, we consider the CML of the form

$$x_{m+1,n} = (1-\varepsilon)f(x_{m,n}) + \tfrac{1}{2}\varepsilon[f(x_{m,n-1}) + f(x_{m,n+1})], \quad (1)$$

where $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ denotes the real numbers) is a function, $m \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, $n \in \mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$, and $\varepsilon \in [0, 1]$ is a constant.

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Let $\mathbb{N}_t = \{t, t+1, \cdots\}$ with integer $t \in \mathbb{Z}$ and denote $\Omega = \{(0, n) | n \in \mathbb{Z}\} = \{\cdots, (0, -1), (0, 0), (0, 1), \cdots\}$.

For any sequence $\phi = \{\phi_{m,n}\}$ defined on Ω, it is easy to construct by induction a double-indexed sequence $x = \{x_{m,n} | m = 0, 1, 2, \cdots; n = \cdots, -1, 0, 1, \cdots\}$ that equals the initial condition ϕ on Ω and satisfies system (1) on $\mathbb{N}_1 \times \mathbb{Z}$.

In fact, from (1), for any $n \in \mathbb{Z}$, one can calculate a sequence $x_1 = \{x_{1,n}\}_{n=-\infty}^{\infty} = (\cdots, x_{1,-1}, x_{1,0}, x_{1,1}, \cdots)$ by using the initial condition ϕ. Then, by induction, for any $m \in \mathbb{N}_1$, one can calculate a sequence $x_m = \{x_{m,n}\}_{n=-\infty}^{\infty}$, so as to obtain $x = \{x_{m,n} | m = 0, 1, 2, \cdots; n = \cdots, -1, 0, 1, \cdots\}$ satisfying system (1), which is said to be a solution of system (1) with initial condition ϕ.

In 2003, Chen and Liu[13] initiated the study of chaos in the sense of Li-Yorke for a certain type of discrete spatiotemporal systems by using a method similar to the discussion in 1D discrete systems. Then, in 2005, Chen et al. [14] showed one kind of close relationship between a 2D discrete system and an infinite-dimensional discrete system, thus introducing a new definition of chaos for 2D systems in the sense of Devaney. And in 2007, a definition about chaos in the sense of Li-Yorke in discrete spatiotemporal systems is given by Tian and Chen [15]. Along the same line, this paper introduced definitions about dense δ-chaos and dense chaos in discrete spatiotemporal systems. Some sufficient conditions for system (1) to be densely δ-chaotic or densely chaotic is derived.

II. Dense δ-chaos in metric spaces

A metric (or distance) on a set X is a function

$$d : X \times X \to \mathbb{R}^+ = [0, \infty)$$

with the following properties:

(1) $d(x, y) \geq 0$ for all $x, y \in X$ with $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$,

where $d(x, y)$ is called the distance between x and y. The pair (X, d) is called a metric space.

Definition 2.1 Let (X, d) and $(Y, \widehat{d})$ be two metric spaces, and $h : X \to Y$ be an one-to-one and onto map. If there exist two positive constants $\alpha, \beta > 0$ such that:

$$\alpha d(x, y) \leq \widehat{d}(\widetilde{x}, \widetilde{y}) \leq \beta d(x, y), \text{ for all } x, y \in X,$$

where $\widetilde{x} = h(x)$ and $\widetilde{y} = h(y)$, then the metric space (X, d) is said to be equivalent to the metric space $(Y, \widehat{d})$ (with respect to the map h). If $X = Y$ and $h(x) = x$ for all $x \in X$, then d is said to be equivalent to $\widehat{d}$.

Referring to the definitions of dense chaos and dense δ-chaos for a real compact interval I in [4,5], definitions of dense chaos and dense δ-chaos defined on a metric space are naturally generalized as follows.

Definition 2.2 Let (X,d) is a metric space. A pair of points $\{x,y\} \in X$ is said to be a Li-Yorke pair if one has simultaneously

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$
$$\text{and } \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

Denote the set of Li-Yorke pairs of f by

$$LY_d(f) = \{(x,y) \in X \times X : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0, \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}.$$

And denote the set of Li-Yorke pairs with modulus δ ($\delta \geq 0$ is a constant) by

$$LY_d(f,\delta) = \{(x,y) \in X \times X : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \delta, \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}.$$

Definition 2.3 Let $f : X \to X$ be a map on a metric space (X,d). The map f is said to be densely chaotic if

$$\overline{LY_d(f)} = X \times X$$

where $\overline{LY_d(f)}$ denotes the closure of the set $LY_d(f)$.

Definition 2.4 Let $f : X \to X$ be a map on a metric space (X,d). The map f is said to be densely δ-chaotic if

$$\overline{LY_d(f,\delta)} = X \times X.$$

According the definitions of dense chaos and dense δ-chaos, one can obtain that dense chaos is a special case of dense δ-chaos with $\delta = 0$. As one knows, similar to dense chaos, Li-Yorke chaos is also concerned with Li-Yorke pairs. However, different from the dense chaoticity which examines whether Li-Yorke pairs distribute in every conner of the whole space, Li-Yorke's chaoticity tests that there exists an uncountable subset S of X or not, this subset contains no periodic point and $\{(x,y) : \forall x, y \in S, x \neq y\}$ is a Li-Yorke pair.

Theorem 2.1 Assume that $I \in \mathbb{R}$, $f : I \to I$ is a function, d_1, d_2 are two metrics in I, and d_1 is equivalent to d_2. Then f is densely chaotic on (I,d_1) if and only if f is densely chaotic on (I,d_2).

Proof: Necessity. Since d_1 is equivalent to d_2, there exist two constants $\alpha, \beta > 0$ such that

$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y), \text{ for any } x, y \in I.$$

Let $(a,b) \in I \times I$ is a Li-Yorke pair of f in (I,d_1), i.e.

$$\limsup_{k\to\infty} d_1(f^k(a), f^k(b)) > 0$$
$$\text{and } \liminf_{k\to\infty} d_1(f^k(a), f^k(b)) = 0.$$

Denote $r = \limsup_{k\to\infty} d_1(f^k(a), f^k(b)) > 0$, then

$$\begin{aligned} \alpha r &= \alpha \limsup_{k\to\infty} d_1(f^k(a), f^k(b)) \\ &\leq \limsup_{k\to\infty} d_2(f^k(a), f^k(b)) \\ &\leq \beta \limsup_{k\to\infty} d_1(f^k(a), f^k(b)) \\ &= \beta r. \end{aligned}$$

Thus $\limsup_{k\to\infty} d_2(f^k(a), f^k(b)) > 0$.

Similarly, we have

$$\liminf_{k\to\infty} d_2(f^k(a), f^k(b)) = 0.$$

So, (a,b) is a Li-Yorke pair of f in (I,d_2).

And because f is densely chaotic on (I,d_1), i.e.

$$\overline{LY_{d_1}(f)} = I \times I,$$

then

$$\overline{LY_{d_2}(f)} = I \times I.$$

That is to say, f is densely chaotic on (I,d_2).

By the similar argument, the sufficiency is follows immediately. ■

Remark For some constants $\delta > 0$, if f is densely δ-chaotic on (I,d_1), f may be not densely δ-chaotic on (I,d_2) even though d_1 is equivalent to d_2. In fact f is densely $\alpha\delta$-chaotic on (I,d_2) (where α is a positive constant).

III. A new definition of dense chaos in spatiotemporal systems

In this section, new definitions of dense chaos and dense δ-chaos in discrete spatiotemporal systems are introduced.

Let $\mathbb{R}_\infty^\infty$ be a set of (bi-directional) 1D real sequences, i.e.,

$$\begin{aligned} \mathbb{R}_\infty^\infty &= \{\{a_n\}_{n=-\infty}^\infty\} \\ &= \{(\cdots, a_{-1}, a_0, a_1, \cdots) : a_n \in \mathbb{R}, n \in \mathbb{Z}\}. \end{aligned}$$

Obviously, several different metrics can be defined on $\mathbb{R}_\infty^\infty$. For example, for two sequences

$$x_1 = \{x_{1,n}\}_{n=-\infty}^\infty \in \mathbb{R}_\infty^\infty,$$
$$x_2 = \{x_{2,n}\}_{n=-\infty}^\infty \in \mathbb{R}_\infty^\infty,$$

one may defines

$$d_1(x_1,x_2) = \sum_{n=-\infty}^\infty \frac{|x_{1,n}-x_{2,n}|}{2^{|n|}}; \quad (2)$$
$$d_2(x_1,x_2) = sup\{|x_{1,n}-x_{2,n}| : n \in \mathbb{Z}\}; \quad (3)$$
$$d_3(x_1,x_2) = \sum_{n=-\infty}^\infty \frac{1}{2^{|n|}} \frac{|x_{1,n}-x_{2,n}|}{1+|x_{1,n}-x_{2,n}|}. \quad (4)$$
$$d_4(x_1,x_2) = 0 \text{ if } x_1 = x_2 \text{ and } d_4(x_1,x_2) = 1 \text{ if } x_1 \neq x_2; \quad (5)$$

Then, it is easy to prove that $d_i (i = 1,2,3,4)$ are metrics on $\mathbb{R}_\infty^\infty$. Moreover, d_i is not equivalent to d_j ($i \neq j, i,j = 1,2,3,4$).

let I be a subset of $\mathbb{R}$ and denote

$$\begin{aligned} I_\infty^\infty &= \{\{a_n\}_{n=-\infty}^\infty\} \\ &= \{(\cdots, a_{-1}, a_0, a_1, \cdots) : a_n \in I, n \in \mathbb{Z}\}. \end{aligned}$$

It is obvious that (I_∞^∞, d) is a metric subspace of $(\mathbb{R}_\infty^\infty, d)$.

Let $f : I \to I$ be a function and

$$x = \{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$$

be a solution of system (1) with initial condition

$$\phi = \{\phi_n = \phi_{0,n}\}_{n=-\infty}^\infty,$$

where $\phi_n \in I$ for all $n \in \mathbb{Z}$.

And denote

$$x_m = \{x_{m,n}\}_{n=-\infty}^\infty = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots)$$

for all $m \in \{0,1,2,\cdots\}$.

Let

$$\begin{aligned} x_{m+1} &= \{x_{m+1,n}\}_{n=-\infty}^\infty \\ &= (\cdots, x_{m+1,-1}, x_{m+1,0}, x_{m+1,1}, \cdots) \\ &= F(x_m), \end{aligned}$$

where $x_0 = \phi = \{x_{0,n} = \phi_{0,n}\}_{n=-\infty}^\infty$ and

$x_{m+1,n} = (1-\varepsilon) f(x_{m,n}) + \frac{1}{2}\varepsilon[f(x_{m,n-1}) + f(x_{m,n+1})]$, $m \in \mathbb{N}_0$, $n \in \mathbb{Z}$.

Then, one can see that system (1) is equivalent to a system in the form of

$$x_{m+1} = F(x_m),\ x_m \in I_\infty^\infty \subseteq \mathbb{R}_\infty^\infty,\ m = 0,1,2,\cdots. \quad (6)$$

The map F of system (6) is said to be induced by system (1). And (f,F) is a pair of maps associated with the two systems (1) and (6).

Obviously, a double-indexed sequence $\{x_{m,n} : m \in \mathbb{N}_0, n \in \mathbb{Z}\}$ is a solution of system (1) if and only if the sequence $\{x_m\}_{m=0}^\infty$ is a solution of system (6), where $x_m = \{x_{m,n}\}_{n=-\infty}^\infty, m \in \mathbb{N}_0$.

Definition 3.1 Let (I_∞^∞, d) is a metric space. For any (x, y) in $I_\infty^\infty \times I_\infty^\infty$, $\forall \varepsilon > 0$, a point (x_1, y_1) in $I_\infty^\infty \times I_\infty^\infty$ is said to be in ε-neighborhood of (x, y), if

$$d(x, x_1) < \varepsilon \text{ and } d(y, y_1) < \varepsilon.$$

The ε-neighborhood of (x, y) is denoted by $B((x, y), \varepsilon)$.

Definition 3.2 Let I be a subset of $\mathbb{R}$, $f : I \to I$ is a function and $F : I_\infty^\infty \to I_\infty^\infty$ is a map on (I_∞^∞, d) induced by system (1). If the map F is chaotic on I_∞^∞, i.e., system (6) is chaotic on (I_∞^∞, d), then system (1) is said to be chaotic on (I_∞^∞, d).

In particular, if F is densely chaotic on (I_∞^∞, d), then system (1) is said to be densely chaotic on (I_∞^∞, d). If F is densely δ-chaotic on (I_∞^∞, d), then system (1) is said to be densely δ-chaotic on (I_∞^∞, d).

The following conclusion is easy to check.

Theorem 3.1 Assume that $I \in \mathbb{R}$, $f : I \to I$ is a function. $I_\infty^\infty = \{x = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots) : x_{m,n} \in I, n \in \mathbb{Z}, m \in \mathbb{N}_0\}$, $F : I_\infty^\infty \to I_\infty^\infty$ is a map induced by system (1) with the function f. d_1, d_2 are two metrics in I_∞^∞, and d_1 is equivalent to d_2. Then F is densely chaotic on (I_∞^∞, d_1) if and only if F is densely chaotic on (I_∞^∞, d_2).

IV. Main Results

In this section, we will consider that system (1) is densely δ-chaotic or not (i.e. F is densely δ-chaotic or not) when f is densely δ-chaotic.

Theorem 4.1 Assume that $I \subset \mathbb{R}$, and define the distance in I with $d(a, b) = |a - b|$ for any two points $a, b \in I$ (where $|\cdot|$ denotes modulus). $f : I \to I$ is a function. Let

$$\Delta_\infty^\infty = \{x = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots) : x_{m,n} = a, \quad a \in I, n \in \mathbb{Z}, m \in \mathbb{N}_0\},$$

$F : \Delta_\infty^\infty \to \Delta_\infty^\infty$ is a map induced by system (1) with the function f. If the function f is densely δ-chaotic ($\delta \geq 0$ is a constant), then system (1) is densely δ-chaotic on $(\Delta_\infty^\infty, d_1)$. Where d_1 is defined by (2).

Proof: If $(a, b) \in I \times I (a \neq b)$ is a Li-Yorke pair with modulus δ of f, i.e.,

$$\limsup_{n\to\infty} |f^k(a) - f^k(b)| > \delta$$
$$\text{and } \liminf_{n\to\infty} |f^k(a) - f^k(b)| = 0.$$

In the following, we show that $(x^*, y^*) \in I_\infty^\infty \times I_\infty^\infty$ is a Li-Yorke pair with modulus δ of F, where

$$x^* = \{x_n = a\}_{n=-\infty}^\infty,$$
$$y^* = \{y_n = b\}_{n=-\infty}^\infty.$$

In fact,

$$F(x^*) = \{(1-\varepsilon)f(a) + \tfrac{1}{2}\varepsilon[f(a) + f(a)]\}_{n=-\infty}^\infty = \{f(a)\}_{n=-\infty}^\infty,$$

then $F^k(x^*) = \{f^k(a)\}_{n=-\infty}^\infty$. Similarly, $F^k(y^*) = \{f^k(b)\}_{n=-\infty}^\infty$.

Then

$$d_1(F^k(x^*), F^k(y^*)) = d_1(\{f^k(a)\}_{n=-\infty}^\infty, \{f^k(b)\}_{n=-\infty}^\infty) = \sum_{n=-\infty}^\infty \frac{|f^k(a) - f^k(b)|}{2^{|n|}} = 3|f^k(a) - f^k(b)|.$$

The following results are straightforward.

$$\limsup_{n\to\infty} d_1(F^k(x^*), F^k(y^*)) = 3\limsup_{n\to\infty} |f^k(a) - f^k(b)| > 3\delta > \delta;$$

$$\liminf_{n\to\infty} d_1(F^k(x^*), F^k(y^*)) = 3\liminf_{n\to\infty} |f^k(a) - f^k(b)| = 0.$$

Therefore, (x^*, y^*) is a Li-Yorke pair with modulus δ of F.

Now we prove that F is densely δ-chaotic.

For every $(x, y) \in \Delta_\infty^\infty \times \Delta_\infty^\infty$, denote

$$x = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots),$$
$$y = (\cdots, y_{n,-1}, y_{n,0}, y_{n,1}, \cdots),$$

where $x_{m,p} = x_{m,p+1}, y_{n,q} = y_{n,q+1}, p, q \in \mathbb{Z}$. Then $(x_{m,0}, y_{n,0}) \in I \times I$.

Since $f : I \to I$ is densely δ-chaotic, for any $\varepsilon > 0$, one has

$$B((x_{m,0}, y_{n,0}), \tfrac{\varepsilon}{3}) \cap LY_d(f, \delta) \neq \phi.$$

Put

$$(a, b) \in B((x_{m,0}, y_{n,0}), \tfrac{\varepsilon}{3}) \cap LY_d(f, \delta),$$

then

$$|x_{m,0} - a| < \tfrac{\varepsilon}{3},$$
$$|y_{n,0} - b| < \tfrac{\varepsilon}{3},$$
$$\limsup_{n\to\infty} |f^k(a) - f^k(b)| > \delta,$$
$$\text{and } \liminf_{n\to\infty} |f^k(a) - f^k(b)| = 0.$$

From the above discussion, we know that (x^*, y^*) is a Li-Yorke pair with modulus δ of F, where

$$x^* = \{\cdots, a, a, a, \cdots\},$$
$$y^* = \{\cdots, b, b, b, \cdots\}.$$

And

$$d_1(x, x^*) = \sum_{p=-\infty}^\infty \frac{|x_{m,p} - a|}{2^{|p|}} = 3|x_{m,0} - a| < \varepsilon,$$
$$d_1(y, y^*) = \sum_{q=-\infty}^\infty \frac{|y_{n,q} - b|}{2^{|q|}} = 3|y_{n,0} - b| < \varepsilon.$$

This implies that

$$(x^*, y^*) \in B((x, y), \varepsilon) \cap LY_{d_1}(F, \delta),$$

i.e.,

$$\overline{LY_{d_1}(F, \delta)} = \Delta_\infty^\infty \times \Delta_\infty^\infty.$$

We thus conclude that system (1) is densely δ-chaotic on $(\Delta_\infty^\infty, d_1)$. ∎

In particular, if $\delta = 0$, it means that Theorem 4.1 is right for dense chaos. This conclusion is described in Theorem 4.2.

Theorem 4.2 Assume that $I \subset \mathbb{R}$, and define the distance in I as $d(a, b) = |a - b|$ for any two points $a, b \in I$. $f : I \to I$ is a function. Let

$$\Delta_\infty^\infty = \{x = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots) | x_{m,n} = a, \quad a \in I, n \in \mathbb{Z}, m \in \mathbb{N}_0\},$$

$F : \Delta_\infty^\infty \to \Delta_\infty^\infty$ is a map induced by system (1) with the function f. If the function f is densely chaotic, then system (1) is densely chaotic on $(\Delta_\infty^\infty, d_1)$. Where d_1 is defined by (2).

Now, we change the metric in Δ_∞^∞. For example, from d_1 to d_2 (where d_2 is defined by (3)). The following we study dense chaoticity of system (1) (dense δ-chaoticity is similar).

First, if $(a, b) \in I \times I (a \neq b)$ is a Li-Yorke pair of f, then $(x^*, y^*) \in \Delta_\infty^\infty \times \Delta_\infty^\infty$ is a Li-Yorke pair of F. Where $x^* = \{x_n = a\}_{n=-\infty}^\infty, y^* = \{y_n = b\}_{n=-\infty}^\infty$.

In fact,

$$\limsup_{k\to\infty} d_2(F^k(x^*), F^k(y^*)) = \limsup_{k\to\infty} d_2(\{f^k(a)\}_{n=-\infty}^\infty, \{f^k(b)\}_{n=-\infty}^\infty) = \limsup_{k\to\infty} sup\{|f^k(a) - f^k(b)| : n = \cdots, -1, 0, 1, \cdots\} = \limsup_{k\to\infty} |f^k(a) - f^k(b)|$$

> 0.

$$\begin{aligned}&\liminf_{k\to\infty} d_2(F^k(x^*), F^k(y^*))\\ =&\liminf_{k\to\infty} sup\{|f^k(a) - f^k(b)| : n = \cdots, -1, 0, 1, \cdots\}\\ =&\liminf_{k\to\infty} |f^k(a) - f^k(b)|\\ =&\ 0.\end{aligned}$$

Then, for every points-pair $(x, y) \in \Delta_\infty^\infty \times \Delta_\infty^\infty$, denote

$$x = (\cdots, x_{m,-1}, x_{m,0}, x_{m,1}, \cdots),$$
$$y = (\cdots, y_{n,-1}, y_{n,0}, y_{n,1}, \cdots),$$

where $x_{m,p} = x_{m,p+1}, y_{n,q} = y_{n,q+1}, p, q \in \mathbb{Z}$. Then $(x_{m,0}, y_{n,0}) \in I \times I$.

Since $f : I \to I$ is densely chaotic, for any $\varepsilon > 0$, one has

$$B((x_{m,0}, y_{n,0}), \varepsilon) \cap LY_d(f) \neq \phi.$$

Put $(a, b) \in B((x_{m,0}, y_{n,0}), \varepsilon) \cap LY_d(f)$, then

$$|x_{m,0} - a| < \varepsilon,$$
$$|y_{n,0} - b| < \varepsilon,$$
$$\limsup_{k\to\infty} |f^k(a) - f^k(b)| > 0,$$
$$\text{and } \liminf_{k\to\infty} |f^k(a) - f^k(b)| = 0.$$

Clearly, $(x^* = \{\cdots, a, a, a, \cdots\}, y^* = \{\cdots, b, b, b, \cdots\})$ is a Li-Yorke pair of F. And

$$\begin{aligned}d_2(x, x^*) &= sup\{|x_{m,p} - a| : p = \cdots, -1, 0, 1, \cdots\}\\ &= |x_{m,0} - a|\\ &< \varepsilon,\\ d_2(y, y^*) &= sup\{|y_{n,q} - b| : q = \cdots, -1, 0, 1, \cdots\}\\ &= |y_{n,0} - b|\\ &< \varepsilon.\end{aligned}$$

That is to say,

$$(x^*, y^*) \in B((x, y), \varepsilon) \cap LY_{d_2}(F),$$

i.e.,

$$\overline{LY_{d_2}(F)} = \Delta_\infty^\infty \times \Delta_\infty^\infty.$$

Thus system (1) is densely chaotic on $(\Delta_\infty^\infty, d_2)$.

By the similar argument, system (1) is obvious densely chaotic on $(\Delta_\infty^\infty, d_3)$ (where d_3 is defined by (4)). Notice that, d_1, d_2, d_3 are not equivalent to each other. Then, a natural question is whether system (1) densely chaotic on $(\Delta_\infty^\infty, d)$ with arbitrary metric d? Let us consider the discrete metric d_4 which is defined by (5). Obviously, with this metric, the system is not chaotic. This means that definitions of metrics in a space are impact to the consistence between the chaotic properties of f and the ones of F. Where F is induced by system (1) with the function f.

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