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Complexity Reduction Approach with Jacobi Iterative Method for Solving Composite Trapezoidal Algebraic Equations

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Abstract—In this paper, application of the complexity reduction approach based on half- and quarter-sweep iteration concepts with Jacobi iterative method for solving composite trapezoidal (CT) algebraic equations is discussed. The performances of the methods for CT algebraic equations are comparatively studied by their application in solving linear Fredholm integral equations of the second kind. Furthermore, computational complexity analysis and numerical results for three test problems are also included in order to verify performance of the methods.

Keywords—Complexity reduction approach, Composite trapezoidal scheme, Jacobi method, Linear Fredholm integral equations.

I. Introduction

INTEGRAL equations provide an important tool for solving various scientific problems, such as boundary value problems for both ordinary and partial differential equations. Their historical progress is strongly connected to the solution of boundary value problems in potential theory and had a significance impact on the development of functional analysis [7]. Consequently, in this paper, numerical solutions for inhomogeneous second kind linear integral equations of Fredholm type given in the form

$$\varphi(x) - \int_{\alpha}^{\beta} K(x,t)\varphi(t)dt = f(x), x \in [\alpha, \beta] \qquad (1)$$

are considered. The function $f(x) \in L^2[\alpha, \beta]$ is given, $K(x,t) \in L^2([\alpha, \beta] \times [\alpha, \beta])$ is the kernel of the integral equation and $\varphi(x)$ is the unknown function to be determined.

There is a large literature on numerical methods for solving problem (1), for instance refer [2-6, 8-11, 13, 14, 16]. The applications of numerical methods for problem (1) mostly lead to dense linear systems and sometimes the condition number of the corresponding matrices is large. The computational complexity of setting-up the matrix and then solving the corresponding linear system is huge when the order of the matrix is large.

Consequently, in this paper, implementation and performance of the complexity reduction approach based on half- [1] and quarter-sweep [12] iteration concepts with Jacobi iterative method for solving composite trapezoidal (CT) algebraic equations arise from the discretization of the problem (1) are shown. The basic idea of the complexity reduction approach is to speed-up the computational time by reducing the computational complexity of the solution method. The combinations of half- and quarter-sweep iteration concepts with Jacobi iterative method are known as the Half-Sweep Jacobi (HSJ) and Quarter-Sweep Jacobi (QSJ) methods respectively. Meanwhile, the standard Jacobi method also can be called as Full-Sweep Jacobi (FSJ) method.

The rest of this paper is organized as follows. The derivations of the CT algebraic equations are explained in Section II. In Section III, the formulations of the FSJ, HSJ and QSJ methods for solving generated CT algebraic equations are elaborated. The latter section of this paper will discuss the computational complexity of the Jacobi methods with corresponding CT algebraic equations for solving problem (1). Some numerical results are presented in Section V to assert the performance of the methods and concluding remarks are given in final section.

II. Trapezoidal Algebraic Equations

In this section, application of the CT scheme for discretizing problem (1) will be discussed. An implementation of the CT scheme for problem (1) leads to CT algebraic equations which will be solved by using FSJ, HSJ and QSJ methods. However, for HSJ and QSJ methods, the standard CT algebraic equation needs to modify by combining the half- and quarter-sweep iteration approaches respectively.

Let interval $[\alpha, \beta]$ divided uniformly into even N subintervals and the discrete set of points of x and t given by $x_i = \alpha + ih(i = 0, 1, 2, \cdots, N-2, N-1, N)$ and $t_j = \alpha + jh(j = 0, 1, 2, \cdots, N-2, N-1, N)$ respectively, where the constant step size, h is defined as follows

$$h = \frac{\beta - \alpha}{N}. \qquad (2)$$

Before further explanation, the following notations will be applied for simplicity

$$\left.\begin{array}{l} K_{i,j} = K(x_i, t_j) \\ \widehat{\varphi}_i = \widehat{\varphi}(x_i) \\ \widehat{\varphi}_j = \widehat{\varphi}(t_j) \\ f_i = f(x_i) \end{array}\right\}. \qquad (3)$$

As discussed in [10], a CT algebraic equation for approximation values of φ is

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$$\widehat{\varphi}_i - \sum_{j=0}^{N} w_j K_{i,j} \widehat{\varphi}_j = f_i, i = 0, 1, 2, \cdots, N-2, N-1, N \quad (4)$$

where solution $\widehat{\varphi}$ is an approximation of the exact solution φ to (1) and w_j is the weights of CT scheme that satisfy the following conditions

$$w_j = \begin{cases} \frac{h}{2}, & j = 0, N \\ h, & otherwise \end{cases} . \quad (5)$$

The standard CT algebraic equation as derived in (4) is also can be referred as full-sweep CT (FSCT) algebraic equation.

Now, let consider the following finite grid networks that show the even distribution of node points for formulating half-sweep CT (HSCT) and quarter-sweep CT (QSCT) algebraic equations.

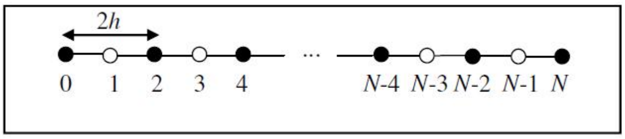

Fig. 1. Distribution of uniform node points for the half-sweep case.

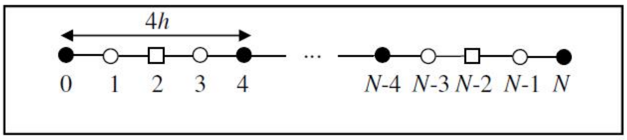

Fig. 2. Distribution of uniform node points for the quarter-sweep case.

By referring Figs. 1 and 2, only node points of type • will be involved during iteration process. Meanwhile, solution for the remaining points will be computed directly after the convergence criterion is satisfied. Thus, by employing the half- and quarter-sweep iteration concepts, the HSCT and QSCT algebraic equations are as follow

$$\widehat{\varphi}_i - \sum_{j=0,2,4}^{N} w_j K_{i,j} \widehat{\varphi}_j = f_i, i = 0, 2, 4, \cdots, N-4, N-2, N \quad (6)$$

and

$$\widehat{\varphi}_i - \sum_{j=0,4,8}^{N} w_j K_{i,j} \widehat{\varphi}_j = f_i, i = 0, 4, 8, \cdots, N-8, N-4, N \quad (7)$$

respectively. The weights, w_j for HSCT and QSCT algebraic equations will satisfy the following relation

$$w_j = \begin{cases} h, & j = 0, N \\ 2h, & otherwise \end{cases} \quad (8)$$

and

$$w_j = \begin{cases} 2h, & j = 0, N \\ 4h, & otherwise \end{cases} \quad (9)$$

respectively.

Following the conventional process, FSCT, HSCT and QSCT algebraic equations can be written as the following matrix form

$$A\widehat{\varphi} = f \quad (10)$$

where $A = (a_{i,j}) \in \mathbb{R}^{(\frac{N}{p}+1)\times(\frac{N}{p}+1)}$ is a real matrix with

$$a_{i,j} = \begin{cases} 1 - w_j K_{i,j}, & i = j \\ -w_j K_{i,j}, & i \neq j \end{cases} \quad (11)$$

and $\widehat{\varphi}, f \in \mathbb{R}^{\frac{N}{p}+1}$. The value of p, which corresponds to one, two and four denote the FSCT, HSCT and QSCT algebraic equations. It is obvious that, applications of the half- and quarter-sweep iteration concepts reduce order of the matrix from $(N+1)$ to $(\frac{N}{2}+1)$ and $(\frac{N}{4}+1)$ respectively.

III. Jacobi Iterative Methods

The FSJ, HSJ and QSJ iterative methods to solve the corresponding FSCT, HSCT and QSCT algebraic equations are formulated in terms of a splitting of the matrix A

$$A = D - L - U \quad (12)$$

where the components of $D = (d_{i,j})$, $L = (l_{i,j})$ and $U = (u_{i,j})$ are defined by

$$d_{i,j} = \begin{cases} a_{i,j}, & i = j \\ 0, & i \neq j \end{cases} \quad (13)$$

$$l_{i,j} = \begin{cases} -a_{i,j}, & i > j \\ 0, & i \leq j \end{cases} \quad (14)$$

and

$$u_{i,j} = \begin{cases} -a_{i,j}, & i < j \\ 0, & i \geq j \end{cases} \quad (15)$$

respectively.

The FSJ, HSJ and QSJ methods begin with an arbitrary initial datum, $\widehat{\varphi}^{(0)} \in \mathbb{R}^{N+1}$, $\widehat{\varphi}^{(0)} \in \mathbb{R}^{\frac{N}{2}+1}$ and $\widehat{\varphi}^{(0)} \in \mathbb{R}^{\frac{N}{4}+1}$ respectively, and then produce a sequence of vectors, $\widehat{\varphi}^{(k)}$ for $k = 1, 2, 3, \cdots$ by solving

$$D\widehat{\varphi}^{(k+1)} = (L+U)\widehat{\varphi}^{(k)} + f, k = 0, 1, 2, \cdots. \quad (16)$$

It is noted that, D must be invertible for the FSJ, HSJ and QSJ methods to be applicable. The iteration matrices of the FSJ, HSJ and QSJ methods i.e. T_{FSJ}, T_{HSJ} and T_{QSJ} respectively are given by

$$T_{FSJ} = T_{HSJ} = T_{QSJ} = D^{-1}(L+U). \quad (17)$$

The iterative steps of the methods are then defined, respectively, by

FSJ:

$$\widehat{\varphi}^{(k+1)} = T_{FSJ}\widehat{\varphi}^{(k)} + D^{-1}f \tag{18}$$

HSJ:

$$\widehat{\varphi}^{(k+1)} = T_{HSJ}\widehat{\varphi}^{(k)} + D^{-1}f \tag{19}$$

QSJ:

$$\widehat{\varphi}^{(k+1)} = T_{QSJ}\widehat{\varphi}^{(k)} + D^{-1}f \tag{20}$$

for $k = 0, 1, 2, \cdots$. The FSJ, HSJ and QSJ methods are convergent if and only if the spectral radius of the iteration matrices is less than unity i.e. $\rho(T_{FSJ}) < 1$, $\rho(T_{HSJ}) < 1$ and $\rho(T_{QSJ}) < 1$ respectively.

Based on (18)-(20), the generalized algorithm of FSJ, HSJ and QSJ methods associated with FSCT, HSCT and QSCT algebraic equations respectively to solve problem (1) would be described in Algorithm 1. The value of p which corresponds to one, two and four represent the FSJ, HSJ and QSJ methods.

Algorithm 1. FSJ, HSJ and QSJ algorithms
i. Set $\widehat{\varphi}^{(0)}$ and ϵ
ii. Iteration cycle
  for $k = 0, 1, 2, \cdots$ until convergence **do**
    for $i = 0, p, 2p, \cdots, N-2p, N-p, N$
    Compute
    $\widehat{\varphi}_i^{(k+1)} = \frac{1}{a_{i,i}}(f_i - \sum_{j=0,p,2p}^{i-p} a_{i,j}\widehat{\varphi}_j^{(k)} - \sum_{j=i+p,i+2p,i+3p}^{N} a_{i,j}\widehat{\varphi}_j^{(k)})$
iii. Convergence test. If the converge criterion is satisfied i.e. the maximum norm $\| \widehat{\varphi}^{(k+1)} - \widehat{\varphi}^{(k)} \|_\infty \leq \epsilon$ (where ϵ is the convergence criterion), go to Step (iv), otherwise, repeat the iteration cycle (i.e., go to Step (ii))
iv. Stop

After the iteration process, additional calculation is required for HSJ and QSJ methods to calculate the remaining points. In this paper, second order Lagrange interpolation (SOLI) technique [11] will be utilized to compute the remaining points. The formulations to calculate remaining points using SOLI technique for HSJ and QSJ methods are defined as

$$\widehat{\varphi}_i = \begin{cases} \frac{3}{8}\widehat{\varphi}_{i-1} + \frac{3}{4}\widehat{\varphi}_{i+1} - \frac{1}{8}\widehat{\varphi}_{i+3}, & i = 1, 3, 5, \cdots, N-3 \\ \frac{3}{4}\widehat{\varphi}_{i-1} + \frac{3}{8}\widehat{\varphi}_{i+1} - \frac{1}{8}\widehat{\varphi}_{i-3}, & i = N-1 \end{cases} \tag{21}$$

and

$$\widehat{\varphi}_i = \begin{cases} \frac{3}{8}\widehat{\varphi}_{i-2} + \frac{3}{4}\widehat{\varphi}_{i+2} - \frac{1}{8}\widehat{\varphi}_{i+6}, & i = 2, 6, 10, \cdots, N-6 \\ \frac{3}{4}\widehat{\varphi}_{i-2} + \frac{3}{8}\widehat{\varphi}_{i+2} - \frac{1}{8}\widehat{\varphi}_{i-6}, & i = N-2 \\ \frac{3}{8}\widehat{\varphi}_{i-1} + \frac{3}{4}\widehat{\varphi}_{i+1} - \frac{1}{8}\widehat{\varphi}_{i+3}, & i = 1, 3, 5, \cdots, N-3 \\ \frac{3}{4}\widehat{\varphi}_{i-1} + \frac{3}{8}\widehat{\varphi}_{i+1} - \frac{1}{8}\widehat{\varphi}_{i-3}, & i = N-1 \end{cases} \tag{22}$$

respectively.

IV. Computational Complexity Analysis

An estimation amount of the computational work has been conducted in order to evaluate the computational complexity of FSJ, HSJ and QSJ methods with corresponding CT approximation equations for solving problem (1). The computational work is estimated by considering the arithmetic operations performed per iteration. In estimating the computational work for FSJ, HSJ and QSJ methods, the value of each element in A is store in advance.

From Algorithm 1, it can be observed that N, $\frac{N}{2}$ and $\frac{N}{4}$ additions/subtractions (ADD/SUB) operations are involved for FSJ, HSJ and QSJ methods respectively in computing a value for each node point in the solution domain. Whereas, $N+2$, $\frac{N}{2}+2$ and $\frac{N}{4}+2$ multiplications/divisions (MUL/DIV) operations are required for FSJ, HSJ and QSJ methods respectively.

However, for HSJ and QSJ methods, the iteration process are carried out only on $\frac{N}{2}+1$ and $\frac{N}{4}+1$ node points respectively. Thus, additional two ADD/SUB and six MUL/DIV arithmetic operations are required to calculate a remaining node point after convergence by using SOLI technique. Hence, the total arithmetic operations involved for FSJ, HSJ and QSJ methods for solving problem (1) are summarized in Table I.

TABLE I
Total Computing Operations Involved for the FSJ, HSJ and QSJ Methods

Methods	Per Iteration			After Convergence
	Arithmetic Operations		Total Computing Operations	
	ADD/SUB [a]	MUL/DIV [b]	[a+b]	
FSJ	N^2+N	N^2+3N+2	$2N^2+4N+2$	-
HSJ	$\frac{N^2}{4}+\frac{N}{2}$	$\frac{N^2}{4}+\frac{3N}{2}+2$	$\frac{N^2}{2}+2N+2$	$4N$
QSJ	$\frac{N^2}{16}+\frac{N}{4}$	$\frac{N^2}{16}+\frac{3N}{4}+2$	$\frac{N^2}{8}+N+2$	$6N$

V. Numerical Simulations and Results

To investigate the performance of the FSJ, HSJ and QSJ methods, the following second kind linear Fredholm integral equations which will generate nonsingular matrix A by using CT scheme were used as the test problems.

Test Problem 1 [15]

$$\varphi(x) - \int_0^1 (4xt - x^2)\varphi(t)dt = x, x \in [0, 1] \tag{23}$$

and the analytical solution is given by

$$\varphi(x) = 24x - 9x^2.$$

Test Problem 2 [10]

$$\varphi(x) - \int_0^1 (x^2 + t^2)\varphi(t)dt = x^6 - 5x^3 + x + 10, x \in [0, 1] \tag{24}$$

with the analytical solution

$$\varphi(x) = x^6 - 5x^3 + \frac{1045}{28}x^2 + x + \frac{2141}{84}.$$

Test Problem 3 [13]

$$\varphi(x) - \int_0^{\frac{\pi}{2}} (\frac{1}{2}xt)\varphi(t)dt = sin(x) - \frac{x}{2}, x \in [0, \frac{\pi}{2}] \quad (25)$$

and the analytical solution is of the form

$$\varphi(x) = sin(x).$$

For the comparative analysis, the following criteria i.e.
k - Number of iterations
CPU - CPU time in seconds
$RMSE$ - Root mean squared error [6]
are considered. The value of initial datum, $\widehat{\varphi}^{(0)}$ is set to be zero for all the test problems. The algorithm codes for the simulations were developed using Borland C++ Version 5.02 and performed on a PC with processor Intel(R) Core(TM) 2 CPU T5500 (1.66Hz, 1.67Hz) and 1022MB RAM. Throughout the simulations, the convergence test considered $\epsilon = 10^{-12}$ and carried out on eight different values of N. The simulation results of the tested FSJ, HSJ and QSJ methods for test problems 1 to 3 are recorded in Tables II to IV. Based on Tables II to IV, percentage gains in terms of CPU time for HSJ and QSJ iterative methods compared to the FSJ method are tabulated in Table V. Meanwhile, in Table VI to VIII, the ratio between two consecutive error i.e. $R_N = \frac{|RMSE_N|}{|RMSE_{2N}|}$ is given.

TABLE II
NUMERICAL RESULTS FOR TEST PROBLEM 1

N	Methods	k	CPU	$RMSE$
60	FSJ	441	0.54	2.298936×10^{-02}
	HSJ	431	0.22	9.259511×10^{-02}
	QSJ	417	0.18	3.809464×10^{-01}
120	FSJ	447	0.98	5.710787×10^{-03}
	HSJ	441	0.62	2.288247×10^{-02}
	QSJ	431	0.26	9.216458×10^{-02}
240	FSJ	450	2.39	1.423744×10^{-03}
	HSJ	447	1.44	5.697425×10^{-03}
	QSJ	441	0.70	2.282893×10^{-02}
480	FSJ	452	7.34	3.554802×10^{-04}
	HSJ	450	2.49	1.422073×10^{-03}
	QSJ	447	0.96	5.690738×10^{-03}
960	FSJ	453	22.90	8.881543×10^{-05}
	HSJ	452	9.03	3.552712×10^{-04}
	QSJ	450	2.63	1.421237×10^{-03}
1920	FSJ	453	88.25	2.219716×10^{-05}
	HSJ	453	27.28	8.878930×10^{-05}
	QSJ	452	8.59	3.551667×10^{-04}
3840	FSJ	454	346.04	5.548459×10^{-06}
	HSJ	453	102.93	2.219390×10^{-05}
	QSJ	453	25.58	8.877624×10^{-05}
7680	FSJ	454	1375.82	1.387004×10^{-06}
	HSJ	454	380.54	5.548051×10^{-06}
	QSJ	453	94.82	2.219227×10^{-05}

TABLE III
NUMERICAL RESULTS FOR TEST PROBLEM 2

N	Methods	k	CPU	$RMSE$
60	FSJ	122	0.18	2.156110×10^{-02}
	HSJ	120	0.03	8.639489×10^{-02}
	QSJ	116	0.02	3.481134×10^{-01}
120	FSJ	123	0.26	5.354126×10^{-03}
	HSJ	122	0.10	2.142512×10^{-02}
	QSJ	120	0.07	8.585374×10^{-02}
240	FSJ	123	0.73	1.334163×10^{-03}
	HSJ	123	0.45	5.337100×10^{-03}
	QSJ	122	0.28	2.135749×10^{-02}
480	FSJ	124	1.94	3.330043×10^{-04}
	HSJ	123	0.77	1.332033×10^{-03}
	QSJ	123	0.36	5.328644×10^{-03}
960	FSJ	124	6.47	8.318457×10^{-05}
	HSJ	124	2.60	3.327378×10^{-04}
	QSJ	123	1.13	1.330976×10^{-03}
1920	FSJ	124	25.63	2.078786×10^{-05}
	HSJ	124	9.16	8.315125×10^{-05}
	QSJ	124	2.08	3.326056×10^{-04}
3840	FSJ	125	97.57	5.195932×10^{-06}
	HSJ	124	32.82	2.078369×10^{-05}
	QSJ	124	8.87	8.313472×10^{-05}
7680	FSJ	124	366.20	1.298852×10^{-06}
	HSJ	125	119.66	5.195412×10^{-06}
	QSJ	124	26.49	2.078163×10^{-05}

TABLE IV
NUMERICAL RESULTS FOR TEST PROBLEM 3

N	Methods	k	CPU	$RMSE$
60	FSJ	62	0.04	1.366109×10^{-03}
	HSJ	60	0.02	1.591867×10^{-03}
	QSJ	58	0.01	2.504905×10^{-03}
120	FSJ	62	0.09	1.302078×10^{-03}
	HSJ	62	0.05	1.357905×10^{-03}
	QSJ	60	0.03	1.582288×10^{-03}
240	FSJ	63	0.47	1.284182×10^{-03}
	HSJ	62	0.20	1.298065×10^{-03}
	QSJ	62	0.07	1.353719×10^{-03}
480	FSJ	63	1.09	1.278724×10^{-03}
	HSJ	63	0.30	1.282186×10^{-03}
	QSJ	62	0.16	1.296047×10^{-03}
960	FSJ	63	4.80	1.276863×10^{-03}
	HSJ	63	1.13	1.277727×10^{-03}
	QSJ	63	0.71	1.281186×10^{-03}
1920	FSJ	63	19.38	1.276149×10^{-03}
	HSJ	63	5.73	1.276365×10^{-03}
	QSJ	63	1.14	1.277229×10^{-03}
3840	FSJ	63	60.31	1.275846×10^{-03}
	HSJ	63	15.44	1.275900×10^{-03}
	QSJ	63	3.79	1.276116×10^{-03}
7680	FSJ	63	225.80	1.275708×10^{-03}
	HSJ	63	59.15	1.275721×10^{-03}
	QSJ	63	19.52	1.275775×10^{-03}

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TABLE V
PERCENTAGE GAINS OF THE HSJ AND QSJ METHODS COMPARED WITH FSJ METHOD

Methods	Test Problem 1 (%)	Test Problem 2 (%)	Test Problem 3 (%)
HSJ	36.73 - 72.35	38.35 - 83.34	44.44 - 76.46
QSJ	66.66 - 93.11	61.64 - 92.77	66.66 - 94.12

TABLE VI
R_N FOR THE TEST PROBLEM 1

N	FSJ	HSJ	QSJ
60	-	-	-
120	4.025603	4.046552	4.133328
240	4.011105	4.016283	4.037184
480	4.005129	4.006422	4.011594
960	4.002460	4.002782	4.004074
1920	4.001207	4.001284	4.001605
3840	4.000599	4.000617	4.000695
7680	4.000319	4.000306	4.000323

TABLE VII
R_N FOR THE TEST PROBLEM 2

N	FSJ	HSJ	QSJ
60	-	-	-
120	4.027006	4.032411	4.054726
240	4.013097	4.014375	4.019842
480	4.006444	4.006733	4.008053
960	4.003198	4.003251	4.003561
1920	4.001594	4.001597	4.001664
3840	4.000795	4.000793	4.000803
7680	4.000403	4.000393	4.000395

TABLE VIII
R_N FOR THE TEST PROBLEM 3

N	FSJ	HSJ	QSJ
60	-	-	-
120	1.049176	1.172296	1.583090
240	1.013936	1.046099	1.168845
480	1.004268	1.012384	1.044498
960	1.001457	1.003490	1.011599
1920	1.000559	1.001067	1.003098
3840	1.000237	1.000364	1.000872
7680	1.000108	1.000140	1.000267

VI. CONCLUDING REMARKS

In the present paper, performance of the half- and quarter-sweep iteration concepts with Jacobi iterative method for the solution of CT algebraic equations associated with the numerical solutions of the second kind linear Fredholm integral equations has been investigated. From the results obtained, it can be observed that QSJ method solved the test problems 1 to 3 with minimum number of iterations and fastest CPU time. However, in some cases, number of iterations for FSJ, HSJ and QSJ methods are similar. Whereas, accuracy of numerical solutions obtained via HSJ and QSJ methods are in good agreement compared to the FSJ method for solving test problems 1 to 3. Based on Table VI to VIII, the ratio R_N tends to four for the test problems 1 and 2, and to one for the test problem 3.

REFERENCES

[1] A. R. Abdullah, "The four point Explicit Decoupled Group (EDG) method: A fast Poisson solver," *Int. J. Comput. Math.*, vol. 38, no. 1-2, pp. 61-70, 1991.
[2] E. Babolian, H. R. Marzban, and M. Salmani,"Using triangular orthogonal functions for solving Fredholm integral equations of the second kind" *Appl. Math. Comput.*, vol. 201, no. 1-2, pp. 452-464, 2008.
[3] Z. Chen, C. A. Micchelli, and Y. Xu,"Fast collocation methods for second kind integral equations" *SIAM J. Numer. Anal.*, vol. 40, no. 1, pp. 344-375, 2003.
[4] J. Dick, P. Kritzer, F. Y. Kuo, and I. H. Sloan,"Lattice-Nyström method for Fredholm integral equations of the second kind with convolution type kernels" *J. Complex.*, vol. 23, no. 4-6, pp. 752-772, 2007.
[5] A. Golbabai, and S. Seifollahi,"Numerical solution of the second kind integral equations using radial basis function networks," *Appl. Math. Comput.*, vol. 174, no. 2, pp. 877-883, 2006.
[6] A. Golbabai, and S. Seifollahi, "An iterative solution for the second kind integral equations using radial basis functions," *Appl. Math. Comput.*, vol. 181, no. 2, pp. 903-907, 2006.
[7] R. Kress, *Numerical Analysis*. New York: Springer-Verlag, 1998, ch. 12.
[8] K. Maleknejad, and M. T. Kajani, "Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions," *Appl. Math. Comput.*, vol. 145, no. 2-3, pp. 623-629, 2003.
[9] K. Maleknejad, and M. Karami, "Using the WPG method for solving integral equations of the second kind," *Appl. Math. Comput.*, vol. 166, no. 1, pp. 123-130, 2005.
[10] M. S. Muthuvalu, and J. Sulaiman, "Half-Sweep Arithmetic Mean method with composite trapezoidal scheme for solving linear Fredholm integral equations," *Appl. Math. Comput.*, vol. 217, no. 12, pp. 5442-5448, 2011.
[11] M. S. Muthuvalu, and J. Sulaiman, "Numerical solution of second kind linear Fredholm integral equations using QSGS iterative method with high-order Newton-Cotes quadrature schemes," *Malays. J. Math. Sci.*, vol. 5, no. 1, pp. 85-100, 2011.
[12] M. Othman, and A. R. Abdullah, "An efficient four points Modified Explicit Group Poisson solver," *Int. J. Comput. Math.*, vol. 76, no. 2, pp. 203-217, 2000.
[13] S. Rahbar, and E. Hashemizadeh, "A computational approach to the Fredholm integral equation of the second kind," in *Proc. World Congress on Engineering*, London, 2008, pp. 933-937.
[14] J. Saberi-Nadjafi, and M. Heidari, "Solving linear integral equations of the second kind with repeated modified trapezoid quadrature method," *Appl. Math. Comput.*, vol. 189, no. 1, pp. 980-985, 2007.
[15] W. Wang, "A new mechanical algorithm for solving the second kind of Fredholm integral equation," *Appl. Math. Comput.*, vol. 172, no. 2, pp. 946-962, 2006.
[16] J. -Y. Xiao, L. -H. Wen, and D. Zhang, "Solving second kind Fredholm integral equations by periodic wavelet Galerkin method," *Appl. Math. Comput.*, vol. 175, no. 1, pp. 508-518, 2006.

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