International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:3, 2013

Proximal Parallel Alternating Direction Method for Monotone Structured Variational Inequalities

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Abstract—In this paper, we focus on the alternating direction method, which is one of the most effective methods for solving structured variational inequalities(VI). In fact, we propose a proximal parallel alternating direction method which only needs to solve two strongly monotone sub-VI problems at each iteration. Convergence of the new method is proved under mild assumptions. We also present some preliminary numerical results, which indicate that the new method is quite efficient.

Keywords—structured variational inequalities, proximal point method, global convergence

I. Introduction

THIS paper considers the structured monotone variational inequalities(denoted by SVI) with linear constraint:

$$\text{Find } u^* \in \Omega, \text{ such that } (u-u^*)^\top T(u^*) \geq 0, \ u \in \Omega, \quad (1)$$

with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \ T(u) = \begin{pmatrix} f(x) \\ g(y) \end{pmatrix},$$

$$\Omega = \{(x,y) | x \in \mathcal{X}, y \in \mathcal{Y}, Ax + By = b\},$$

where $\mathcal{X} \subset \mathcal{R}^n$ and $\mathcal{Y} \subset \mathcal{R}^m$ are given nonempty closed convex sets; $f : \mathcal{X} \to \mathcal{R}^n$ and $g : \mathcal{Y} \to \mathcal{R}^m$ are given continuous monotone operators; $A \in \mathcal{R}^{r\times n}$ and $B \in \mathcal{R}^{r\times m}$ are given full-rank matrices; $b \in \mathcal{R}^r$ is a given vector. Wide applications of SVI in various fields can be found in Glowinski[1], Glowinski and Le Tallec[2], Eckstein and Fukushima[3], He et al.[4], and Pardalos et al.[13].

By attaching a Lagrange multiplier vector $\lambda \in \mathcal{R}^r$ to the linear constraint $Ax + By = b$, SVI can be transformed into the following compact form[7,8,11]:

$$\text{Find } w^* \in W, \text{such that } (w - w^*)^\top Q(w^*) \geq 0, \quad \forall w \in \mathcal{W} \quad (2)$$

where

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, Q(w) = \begin{pmatrix} f(x) - A^\top\lambda \\ g(y) - B^\top\lambda \\ Ax + By - b \end{pmatrix}, \mathcal{W} = \mathcal{X}\times\mathcal{Y}\times\mathcal{R}^r.$$

Problem (2) is referred as SVI_2.

For the purpose of parallel computing, He[5] proposed the following parallel splitting augmented Lagrangian method. In his method, the new iterate $\tilde{w} = (\tilde{x}, \tilde{y}, \tilde{\lambda}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r$ is generated from a given triplet $w = (x, y, \lambda) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r$ via the following procedure:

Given $(x, y, \lambda) \in \mathcal{W}$*, find* $\tilde{x} \in \mathcal{X}$ *such that*

$$(x' - \tilde{x})^\top\{f(\tilde{x}) - A^\top[\lambda - H(A\tilde{x} + By - b)]\} \geq 0, \quad \forall x' \in \mathcal{X}, \quad (3)$$

and then find $\tilde{y} \in \mathcal{Y}$ *such that*

$$(y' - \tilde{y})^\top\{g(\tilde{y}) - B^\top[\lambda - H(Ax + B\tilde{y} - b)]\} \geq 0, \quad \forall y' \in \mathcal{Y}, \quad (4)$$

finally, update λ *via*

$$\tilde{\lambda} = \lambda - H(A\tilde{x} + B\tilde{y} - b), \quad (5)$$

where $H \in R^{r\times r}$ is a given penalty parameter of the linear constraint. To ensure convergence of the iterative sequence, the new iterate is generated by an additional descent step.

Note that He's method (3)-(5) has to solve two monotone sub-VI problems in each iteration. In many cases, solving these problems are quite difficult. Motivated by the alternating direction method in [4,12,16] and the proximal method in [9,10], in this paper, we propose a proximal parallel descent method, which needs to solve two strongly monotone sub-VI problems in each iteration.

The remainder of the paper is organized as follows. In Section 2, the new method and its contractive properties are presented. In Section 3, we prove the convergence of the proposed method. Some numerical results are given in Section 4. Finally, we give some concluding remarks in Section 5.

II. New Algorithm and its Contractive Properties

For analysis convenience, we denote

$$N = \begin{pmatrix} A^\top HA + R & \\ & B^\top HB + S \end{pmatrix}. \quad (6)$$

and

$$G = \begin{pmatrix} N & \\ & (\tau H)^{-1} \end{pmatrix}. \quad (7)$$

Note that N and G are symmetric positive definite if $R \in R^{n\times n}, S \in R^{m\times m}$ and H are symmetric positive definite. The G-norm of a vector z is denoted by $\|z\|_G$, i.e., $\|z\|_G^2 = z^\top Gz$. Set $u^k = (x^k, y^k)$ and $\tilde{u}^k = (\tilde{x}^k, \tilde{y}^k)$.

We are now in the position to state our method.

Proximal Parallel Descent Method

Step 0. Given $\varepsilon > 0, \gamma \in (0, 2), \tau \in (\frac{\sqrt{2}}{2}, \sqrt{2})$, $w^0 = (x^0, y^0, \lambda^0) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{R}^r$. Set $k := 0$.

Step 1. Find $\tilde{x}^k \in \mathcal{X}$ such that:

$$\begin{aligned}&(x - \tilde{x}^k)^\top\{f(\tilde{x}^k) - A^\top[\lambda^k - H(A\tilde{x}^k \\ &+By^k - b)] + R(\tilde{x}^k - x^k)\} \geq 0, \quad \forall x \in \mathcal{X},\end{aligned} \quad (8)$$

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then find $\tilde{y}^k \in \mathcal{Y}$ such that:

$$\begin{aligned}&(y-\tilde{y}^k)^\top\{g(\tilde{y}^k) - B^\top[\lambda^k - H(Ax^k\\&+B\tilde{y}^k - b)] + S(\tilde{y}^k - y^k)\} \geq 0, \quad \forall y \in \mathcal{Y},\end{aligned} \tag{9}$$

$$\tilde{\lambda}^k = \lambda^k - \tau H(A\tilde{x}^k + B\tilde{y}^k - b). \tag{10}$$

Step 2. Convergence Verification. If $\max\{\|A(x^k - \tilde{x}^k)\|, \|B(y^k - \tilde{y}^k)\|, \|\lambda^k - \tilde{\lambda}^k\|\} < \varepsilon$, then stop.

Step 3. Descent Step. Generate the new iterate by

$$w^{k+1} = w^k - \gamma\alpha_k G^{-1}(w^k - \tilde{w}^k), \tag{11}$$

where

$$\alpha_k = \frac{\varphi(w^k, \tilde{w}^k)}{\|w^k - \tilde{w}^k\|_G^2}, \tag{12}$$

with

$$\begin{aligned}&\varphi(w^k, \tilde{w}^k) = \tfrac{1}{2}\|u^k - \tilde{u}^k\|_N^2\\&+\tfrac{1}{2}\|A\tilde{x}^k + By^k - b\|_H^2 + \tfrac{1}{2}\|Ax^k + B\tilde{y}^k - b\|_H^2.\end{aligned} \tag{13}$$

Set $k := k + 1$ and go to Step 1.

Throughout this paper, we make the following standard assumptions.

Assumptions.

• The solution set of SVI_2, denoted by $\mathcal{W}^*$, is nonempty. Equivalently, the solution set of SVI, denoted by Ω^*, is nonempty.

• The involved subvariational inequalities (8) and (9) are solvable.

Remark 1. The proposed proximal parallel descent method reduces to the parallel splitting augmented Lagrangian method in [15] when $R = 0$, $S = 0$ and $\tau = 1$.

Remark 2. It is easy to check that $\tilde{w}^k$ is a solution of SVI_2 if and only if $Ax^k = A\tilde{x}^k$, $By^k = B\tilde{y}^k$ and $\lambda^k = \tilde{\lambda}^k$. Thus, this is the base of the stopping criterion in Step 2 of the proposed method.

The iterative w^k is a proximal solution of SVI_2 if the new method stops at Step 2. Thus, it is assumed, without loss of generality, that the new method generates an infinite sequence $\{w^k\}$.

Lemma 1. Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (8)-(10) from the given triplet w^k, and $\varphi(w^k, \tilde{w}^k)$ be defined in (13). Then we have

$$\begin{aligned}&\varphi(w^k, \tilde{w}^k) \geq \tfrac{2\tau-\sqrt{2}}{2\tau}[\|A(x^k - \tilde{x}^k)\|_H^2\\&+\|B(y^k - \tilde{y}^k)\|_H^2] + \tfrac{2-\sqrt{2}\tau}{2\tau^2}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2.\end{aligned} \tag{14}$$

Proof First, by using the Cauchy-Schwartz Inequality, we have

$$\begin{aligned}&(Ax^k - A\tilde{x}^k)^\top(\lambda^k - \tilde{\lambda}^k)\\\geq\ &-\tfrac{1}{2}(\sqrt{2}\|A(x^k - \tilde{x}^k)\|_H^2 + \tfrac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2).\end{aligned} \tag{15}$$

$$\begin{aligned}&(By^k - B\tilde{y}^k)^\top(\lambda^k - \tilde{\lambda}^k)\\\geq\ &-\tfrac{1}{2}(\sqrt{2}\|B(y^k - \tilde{y}^k)\|_H^2 + \tfrac{1}{\sqrt{2}}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2).\end{aligned} \tag{16}$$

From (10) and (13), we have

$$\begin{aligned}&\varphi(w^k, \tilde{w}^k)\\=\ &\frac{1}{2}\|u^k - \tilde{u}^k\|_N^2 + \frac{1}{2}\|A\tilde{x}^k + B\tilde{y} - b + B(y^k - \tilde{y}^k)\|_H^2\\&+\frac{1}{2}\|A\tilde{x}^k + B\tilde{y} - b + A(x^k - \tilde{x}^k)\|_H^2\\=\ &\frac{1}{2}\|u^k - \tilde{u}^k\|_N^2 + \|A\tilde{x}^k + B\tilde{y} - b\|_H^2 + \frac{1}{2}\|y^k - \tilde{y}^k\|_{B^\top HB}^2\\&+\frac{1}{2}\|x^k - \tilde{x}^k\|_{A^\top HA}^2 + (A\tilde{x}^k + B\tilde{y}^k - b)^\top H[A(x^k - \tilde{x}^k)\\&+B(y^k - \tilde{y}^k)]\\\geq\ &\|x^k - \tilde{x}^k\|_{A^\top HA}^2 + \|y^k - \tilde{y}^k\|_{B^\top HB}^2 + \frac{1}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2\\&+\frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top[A(x^k - \tilde{x}^k) + B(y^k - \tilde{y}^k)].\end{aligned}$$

Substituting (15) and (16) in the above inequality, we have the assertion (14). The proof is completed.

Next, we prove some contractive properties of the sequence generated by the proposed method.

Lemma 2. Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (8)-(10) from the given triplet $w^k = (x^k, y^k, \lambda^k)$. Then, for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, we have

$$\begin{aligned}&(\lambda^k - \tilde{\lambda}^k)^\top(\tau H)^{-1}(\tilde{\lambda}^k - \lambda^*) + (x^k - \tilde{x}^k)^\top R(\tilde{x}^k - x^*)\\&+(y^k - \tilde{y}^k)^\top S(\tilde{y}^k - y^*)\\\geq\ &(A\tilde{x}^k - Ax^*)^\top H(By^k - B\tilde{y}^k) + (B\tilde{y}^k - By^*)^\top\\&H(Ax^k - A\tilde{x}^k) + \tfrac{1-\tau}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}}^2.\end{aligned} \tag{17}$$

Proof Since $w^* \in \mathcal{W}^*$ and $\tilde{x}^k \in \mathcal{X}, \tilde{y}^k \in \mathcal{Y}$, we have

$$(\tilde{x}^k - x^*)^\top(f(x^*) - A^\top\lambda^*) \geq 0, \tag{18}$$

$$(\tilde{y}^k - y^*)^\top(g(y^*) - B^\top\lambda^*) \geq 0, \tag{19}$$

and

$$Ax^* + By^* - b = 0.$$

On the other hand, from (8)(9), it follows that

$$(x^* - \tilde{x}^k)^\top(f(\tilde{x}^k) - A^\top[\lambda^k - H(A\tilde{x}^k + By^k - b)] + R(\tilde{x}^k - x^k)) \geq 0, \tag{20}$$

and

$$(y^* - \tilde{y}^k)^\top(g(\tilde{y}^k) - B^\top[\lambda^k - H(Ax^k + B\tilde{y}^k - b)] + S(\tilde{y}^k - y^k)) \geq 0. \tag{21}$$

Adding (18) and (20), and using (10) and the monotonicity of operator f, we get

$$\begin{aligned}&(A\tilde{x}^k - Ax^*)^\top[\tilde{\lambda}^k - \lambda^* + \tfrac{\tau-1}{\tau}(\lambda^k - \tilde{\lambda}^k)]\\&+(x^k - \tilde{x}^k)^\top R(\tilde{x}^k - x^*) \geq (A\tilde{x}^k - Ax^*)^\top H(By^k - B\tilde{y}^k).\end{aligned} \tag{22}$$

Similarly, combining (19) and (21), and using (10) and the monotonicity of operator g, it follows that

$$\begin{aligned}&(B\tilde{y}^k - By^*)^\top[\tilde{\lambda}^k - \lambda^* + \tfrac{\tau-1}{\tau}(\lambda^k - \tilde{\lambda}^k)]\\&+(y^k - \tilde{y}^k)^\top S(\tilde{y}^k - y^*) \geq (B\tilde{y}^k - By^*)^\top H(Ax^k - A\tilde{x}^k).\end{aligned} \tag{23}$$

Hence, combining (22) and (23), and $Ax^* + By^* = b$, we get

$$\begin{array}{ll} & (A\tilde{x}^k + B\tilde{y}^k - b)^\top(\tilde{\lambda}^k - \lambda^*) + (x^k - \tilde{x}^k)^\top R(\tilde{x}^k - x^*) \\ & +(y^k - \tilde{y}^k)^\top S(\tilde{y}^k - y^*) \\ \geq & (A\tilde{x}^k - Ax^*)^\top H(By^k - B\tilde{y}^k) + (B\tilde{y}^k - By^*)^\top \\ & H(Ax^k - A\tilde{x}^k) \\ & -\frac{\tau - 1}{\tau}(A\tilde{x}^k + B\tilde{y}^k - b)^\top(\lambda^k - \tilde{\lambda}^k), \end{array}$$

the assertion (17) follows from (10) and the above inequality. The lemma is proved.

Lemma 3. Let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (8)-(10) from the given triplet w^k; G be defined by (7) and $\varphi(w^k, \tilde{w}^k)$ be defined by (13). Then, for any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, we have

$$(w^k - w^*)^\top G(w^k - \tilde{w}^k) \geq \varphi(w^k, \tilde{w}^k). \quad (24)$$

Proof In order to offset the unknown vector x^*, y^* in the right side of (17), adding $(A\tilde{x}^k - Ax^*)^\top H(Ax^k - A\tilde{x}^k) + (B\tilde{y}^k - By^*)^\top H(By^k - B\tilde{y}^k)$ to both side of it and using the notation of G and $Ax^* + By^* = b$, we obtain

$$\begin{array}{l} (\tilde{w}^k - w^*)^\top G(w^k - \tilde{w}^k) \geq (A\tilde{x}^k + B\tilde{y}^k - b)^\top \\ H(By^k - B\tilde{y}^k + Ax^k - A\tilde{x}^k) + \frac{1-\tau}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}}. \end{array} \quad (25)$$

Therefore, it follows from (10) and (25) that

$$\begin{array}{ll} & (\tilde{w}^k - w^*)^\top G(w^k - \tilde{w}^k) \\ \geq & \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top(By^k - B\tilde{y}^k + Ax^k - A\tilde{x}^k) \\ & +\frac{1-\tau}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}}, \end{array}$$

which implies that

$$\begin{array}{ll} & (w^k - w^*)^\top G(w^k - \tilde{w}^k) \\ \geq & \|w^k - \tilde{w}^k\|^2_G + \frac{1}{\tau}(\lambda^k - \tilde{\lambda}^k)^\top(By^k - B\tilde{y}^k + Ax^k \\ & -A\tilde{x}^k) + \frac{1-\tau}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}} \\ \geq & \frac{1}{2}[\|Ax^k - A\tilde{x}^k\|^2_H + \frac{1}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}} \\ & +2(\lambda^k - \tilde{\lambda}^k)^\top(Ax^k - A\tilde{x}^k)] \\ & +\frac{1}{2}[\|By^k - B\tilde{y}^k\|^2_H + \frac{1}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}} \\ & +2(\lambda^k - \tilde{\lambda}^k)^\top(By^k - B\tilde{y}^k)] \\ & +\frac{\tau^2 - \tau - 1}{\tau^2}\|\lambda^k - \tilde{\lambda}^k\|^2_{H^{-1}} + \frac{1}{2}\|u^k - \tilde{u}^k\|^2_M \\ \geq & \frac{1}{2}\|u^k - \tilde{u}^k\|^2_M + \frac{1}{2}\|(Ax^k - A\tilde{x}^k) + \frac{H^{-1}}{\tau}(\lambda^k - \tilde{\lambda}^k)\|^2_H \\ & +\frac{1}{2}\|(By^k - B\tilde{y}^k) + \frac{H^{-1}}{\tau}(\lambda^k - \tilde{\lambda}^k)\|^2_H \\ = & \frac{1}{2}\|u^k - \tilde{u}^k\|^2_M + \frac{1}{2}\|A\tilde{x}^k + By^k - b\|^2_H \\ & +\frac{1}{2}\|Ax^k + B\tilde{x}^k - b\|^2_H \\ = & \varphi(w^k, \tilde{w}^k), \end{array}$$

where the last inequality follows from $\tau \in (\frac{\sqrt{2}}{2}, \sqrt{2})$ and the first equality follows from (10). Hence, this lemma is proved.

As $\varphi(w^k, \tilde{w}^k) \geq 0$, (24) shows that $-G^{-1}(w^k - \tilde{w}^k)$ is a descent direction of the distant function $\|w - w^*\|^2_G$ at $w = w^k$. Therefore, it is natural to design the Descent Step.

Next, the optimal step size along the descent direction is investigated from computational point of view. For this purpose, we investigate the iterative scheme with an undeterminate step size denoted by α, that is

$$w^{k+1}(\alpha) = w^k - \alpha(w^k - \tilde{w}^k), \quad (26)$$

and let

$$\Theta_k(\alpha) := \|w^k - w^*\|^2_G - \|w^{k+1}(\alpha) - w^*\|^2_G \quad (27)$$

be the profit function of the k-th iteration. Obviously, we cannot maximize $\Theta_k(\alpha)$ directly. In this paper, we follow the identical strategy in [5,6] to obtain the optimal step size α_k. The following lemma explains the reason of choosing α_k in the form of (12).

Lemma 4. Let w^* be an arbitrary point in $\mathcal{W}^*$. For the given w^k, let $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k)$ be generated by (8-10), $\varphi(w^k, \tilde{w}^k)$ be defined by (13) and $\Theta_k(\alpha)$ be defined by (27). Then, we have

$$\Theta_k(\alpha) \geq \Phi_k(\alpha) := 2\alpha\varphi(w^k, \tilde{w}^k) - \alpha^2\|w^k - \tilde{w}^k\|^2_G. \quad (28)$$

Proof From (24) and (26), we have

$$\begin{array}{ll} & \|w^{k+1}(\alpha) - w^*\|^2_G \\ = & \|w^k - \alpha(w^k - \tilde{w}^k) - w^*\|^2_G \\ = & \|w^k - w^*\|^2_G - 2\alpha(w^k - w^*)^\top G(w^k - \tilde{w}^k) \\ & +\alpha^2\|w^k - \tilde{w}^k\|^2_G \\ \leq & \|w^k - w^*\|^2_G - 2\alpha\varphi(w^k, \tilde{w}^k) + \alpha^2\|w^k - \tilde{w}^k\|^2_G. \end{array}$$

Therefore, the assertion (28) follows from (27) and the above inequality directly. The proof is completed.

Lemma 4 shows that $\Phi_k(\alpha)$ is a lower bound of $\Theta_k(\alpha)$ for any $\alpha > 0$, and this motivates us to maximize $\Phi_k(\alpha)$ to accelerate the convergence of the new method. Note that $\Phi_k(\alpha)$ is a quadratic function of α and it reaches its maximum at α_k defined by (12). From the numerical point of view, it is necessary to attach a relax factor γ to the optimal step size α_k obtained theoretically to achieve faster convergence.

The following Theorem shows that the sequence $\{w^k\}$ is Fejer monotone with respect to $\mathcal{W}^*$.

Theorem 1. For any $w^* = (x^*, y^*, \lambda^*) \in \mathcal{W}^*$, there exists a positive scalar η such that the sequence $\{w^k\}$ generated by the new method satisfies

$$\|w^{k+1} - w^*\|^2_G \leq \|w^k - w^*\|^2_G - \eta\varphi(w^k, \tilde{w}^k). \quad (29)$$

Proof It follows from Lemma 4 that

$$\begin{array}{ll} & \|w^k - w^*\|^2_G - \|w^{k+1} - w^*\|^2_G \\ \geq & \Phi_k(\gamma\alpha_k) \\ = & 2\gamma\alpha_k\varphi(w^k, \tilde{w}^k) - (\gamma\alpha_k)^2\|w^k - \tilde{w}^k\|^2_G \\ = & \gamma(2-\gamma)\alpha_k\varphi(w^k, \tilde{w}^k) \\ \geq & c\gamma(2-\gamma)\varphi(w^k, \tilde{w}^k). \end{array}$$

where the last inequality follows from the fact $\alpha_k \geq c > 0$. The assertion is proved with $\eta = c\gamma(2-\gamma)$.

From the definition of η, it is easy to conclude that γ should be in (0,2) in order to ensure that the new iterate w^{k+1} is closer to $\mathcal{W}^*$ than w^k.

The following corollary is concluded immediately from Theorem 1 and the definition of $\varphi(w^k, \tilde{w}^k)$.

Corollary 1. Let w^* be an arbitrary point in $\mathcal{W}^*$, and the sequence $\{w^k\}$ be generated by the new method. Then, we have

(1) The sequence $\{w^k\}$ is bounded.

(2) The sequence $\{\|w^k - w^*\|_G\}$ is non-increasing.

(3) $\lim_{k\to\infty} \|u^k - \tilde{u}^k\|_N = 0$.

(4) $\lim_{k\to\infty} \|A\tilde{x}^k + By^k - b\|_H = 0$ and $\lim_{k\to\infty} \|Ax^k + B\tilde{y}^k - b\|_H = 0$.

III. Global convergence

Now we state the convergence of the new method.

Theorem 2. The sequence $\{w^k\}$ generated by the new method converges to some $w^\infty \in \mathcal{W}^*$.

Proof It follows from (3) of Corollary 1 that

$$\lim_{k\to\infty} \|x^k - \tilde{x}^k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|y^k - \tilde{y}^k\| = 0. \tag{30}$$

Moreover, (8), (9), (30) and (4) of Corollary 1 implies that

$$\left\{ \begin{array}{l} \lim_{k\to\infty}(x - \tilde{x}^k)^\top \{f(\tilde{x}^k) - A^\top \lambda^k\} \ge 0, \quad \forall x \in \mathcal{X}, \\ \lim_{k\to\infty}(y - \tilde{y}^k)^\top \{g(\tilde{y}^k) - B^\top \lambda^k\} \ge 0, \quad \forall y \in \mathcal{Y}. \end{array} \right. \tag{31}$$

Since $\{w^k\}$ is bounded, thus $\{\tilde{u}^k\}$ and $\{\lambda^k\}$ are also bounded. Let u^∞ be a cluster point of $\{\tilde{u}^k\}$ and λ^∞ be a cluster point of $\{\lambda^k\}$, respectively. The subsequence $\{\tilde{u}^{k_j}\}$ converges to u^∞ and the subsequence $\{\tilde{\lambda}^{k_j}\}$ converges to λ^∞. It follows from (31) that

$$\left\{ \begin{array}{l} \lim_{k\to\infty}(x - \tilde{x}^{k_j})^\top \{f(\tilde{x}^{k_j}) - A^\top \lambda^{k_j}\} \ge 0, \quad \forall x \in \mathcal{X}, \\ \lim_{k\to\infty}(y - \tilde{y}^{k_j})^\top \{g(\tilde{y}^{k_j}) - B^\top \lambda^{k_j}\} \ge 0, \quad \forall y \in \mathcal{Y}. \end{array} \right. \tag{32}$$

Consequently, we have

$$\left\{ \begin{array}{l} (x - \tilde{x}^\infty)^\top \{f(\tilde{x}^\infty) - A^\top \lambda^\infty\} \ge 0, \quad \forall x \in \mathcal{X}, \\ (y - \tilde{y}^\infty)^\top \{g(\tilde{y}^\infty) - B^\top \lambda^\infty\} \ge 0, \quad \forall y \in \mathcal{Y}. \end{array} \right. \tag{33}$$

On the other hand, from (4) of Corollary 1, we get

$$\begin{array}{rl} & \|Ax^\infty + By^\infty - b\| \\ = & \lim_{j\to\infty} \|A\tilde{x}^{k_j} + B\tilde{y}^{k_j} - b\| \\ = & \lim_{j\to\infty} \|A\tilde{x}^{k_j} + By^{k_j} - b + B(\tilde{y}^{k_j} - y^{k_j})\| \\ \le & \lim_{j\to\infty} \|A\tilde{x}^{k_j} + By^{k_j} - b\| + \lim_{j\to\infty} \|B\| \cdot \|\tilde{y}^{k_j} - y^{k_j}\| \\ = & 0, \end{array}$$

From (33) and the above inequality, we get that $w^\infty \in \mathcal{W}^*$. From (10), we have

$$\begin{array}{rl} & \|\lambda^k - \tilde{\lambda}^k\|_{H^{-1}} \\ = & \tau\|H(A\tilde{x}^k + B\tilde{y} - b)\|_{H^{-1}} \\ = & \tau\|A\tilde{x}^k + B\tilde{y} - b\|_H \\ = & \tau\|A\tilde{x}^k + By^k - b + B(\tilde{y} - y^k)\|_H \\ \le & \tau\|A\tilde{x}^k + By^k - b\|_H + \tau\|y^k - \tilde{y}^k\|_{B^\top HB} \\ \le & \tau\|A\tilde{x}^k + By^k - b\|_H + \tau\|B^\top HB\| \cdot \|y^k - \tilde{y}^k\| \end{array}$$

Therefore, from (30), (4) of Corollary 1 and the above inequality, it follows that

$$\lim_{k\to\infty} \|\lambda^k - \tilde{\lambda}^k\| = 0.$$

and then we get

$$\lim_{j\to\infty} \tilde{w}^{k_j} = w^\infty.$$

Since $\lim_{k\to\infty} \|w^k - \tilde{w}^k\|_G = 0$ and $\{w^{k_j}\} \to u^\infty$, for any given $\varepsilon > 0$, there is an integer l, such that

$$\|w^{k_j} - \tilde{w}^{k_j}\|_G < \frac{\varepsilon}{2}, \quad \text{and} \quad \|\tilde{w}^{k_j} - w^\infty\|_G < \frac{\varepsilon}{2}.$$

Therefore, for any $k \ge k_l$, it follows from (2) of Corollary 1 that

$$\begin{array}{rl} & \|w^k - w^\infty\|_G \\ \le & \|w^{k_j} - w^\infty\|_G \\ \le & \|w^{k_j} - \tilde{w}^{k_j}\|_G + \|\tilde{w}^{k_j} - w^\infty\|_G < \varepsilon. \end{array}$$

Thus, the sequence $\{w^k\}$ converges to w^∞, which is a solution of SVI_2. This completes the proof.

IV. Numerical results

In this section, we present some numerical results for the proposed proximal parallel descent method, which is denoted by New Algorithm. All codes are written in Matlab 7.0 and run on a AMD-3200+ personal computer. We also code the algorithm proposed by Ye et al.[14], which is denoted by Ye and Han's method.

We test the problem studied in [14], i.e.

$$\min\{c^\top x | x \in \Omega_1 \cap \Omega_2\},$$

where

$$\Omega_1 = \{x | \|x\| \le r_1, x \in R^n\} \quad \text{and} \quad \Omega_2 = \{x | \|x - b\| \le r_2, x \in R^n\}.$$

By introducing an auxiliary variable y, the above problem can be rewritten into the following:

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & x + y - b = 0 \\ & x \in \odot_{r_1}, \quad y \in \odot_{r_2}, \end{array} \tag{34}$$

where $\odot_r$ denotes a ball centered on zero point with radius r. Therefore, (34) coincides with the SVI in the following sense:

$$u = \left(\begin{array}{c} x \\ y \end{array} \right), \; T(u) = \left(\begin{array}{c} c \\ 0 \end{array} \right), \; \mathcal{X} = \odot_{r_1}, \quad \mathcal{Y} = \odot_{r_2},$$

$$A = I_n, \quad B = I_n.$$

The data of the problem are illustrate as follows: $b = 10(1, 1, \cdots, 1)^\top, c = (1, 1, \cdots, 1)^\top$; $r_1 = 0.5\|b\|$ and $r_2 = 0.6\|b\|$; $\tau = 1.414$; $\gamma = 1.98$ for New Algorithm and $\gamma = 1.59$ for Ye and Han's method. For convenience, we set $H = R = S = I_n$. The stopping criterion is

$$\|A(x^k - \tilde{x}^k)\| + \|B(y^k - \tilde{y}^k)\| + \|\lambda^k - \tilde{\lambda}^k\| < 10^{-6}.$$

The respective number of iteration (no. iter.), and seconds of CPU computing (CPU(s)) of New Algorithm and Ye and Han's

TABLE I
NUMERICAL RESULTS FOR TESTED PROBLEM.

Dimension	Method	no. iter.	CPU(s)
	Ye and Han's method	49	0.0401
$n = 10$	New Algorithm	36	0.0300
	Ye and Han's method	51	0.3305
$n = 50$	New Algorithm	37	0.2904
	Ye and Han's method	53	2.2132
$n = 100$	New Algorithm	39	2.0429

method are reported in Table 1, where n denotes the dimension of the tested problem.

For different problem dimension n, the numerical results in Table 1 show that New Algorithm outperforms Ye and Han's method in the sense that New Algorithm need fewer iteration and less CPU time, which clearly illustrate its efficiency.

V. CONCLUSION

In this paper, a new proximal parallel descent method is introduced for solving structured variational inequality problems. Numerical experiments show that the new method is effective. Furthermore, methods that follow the same framework of the new method, but allow inexact solution of the sub-VI deserve further investigation in the future.

ACKNOWLEDGMENT

The authors would like to thank the reviewers for their valuable suggestions. This work is supported by the Foundation of Zhejiang Provincial Education Department Under Grant Y201225096, and the Foundation of Shandong Natural Science Project Under Grant ZR2012AL08.

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