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Algebraic Riccati Matrix Equation for Eigen-Decomposition of Special Structured Matrices; Applications in Structural Mechanics

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Abstract**—**In this paper Algebraic Riccati matrix equation is used for Eigen-decomposition of special structured matrices. This is achieved by similarity transformation and then using algebraic riccati matrix equation to triangulation of matrices. The process is decomposition of matrices into small and specially structured sub-matrices with low dimensions for fast and easy finding of Eigen-pairs. Numerical and structural examples included showing the efficiency of present method.

Keywords**—**Riccati; matrix equation; eigenvalue problem; symmetric; bisymmetric; persymmetric; decomposition; canonical forms, Graphs theory; adjacency and Laplacian matrices.

I. Introduction

CALCULATION of eigenvalues and eigenvectors of a matrix is important in any engineering problems [1]. Basic and fundamental calculations for stability, vibration and buckling analysis of structural systems require to solving generalized eigenvalue problem [2, 3]. For calculation of eigenvalues and eigenvectors of a matrix the characteristic equation of the matrix should be formed and the corresponding equation of order *n* should be solved [4].

Recently canonical forms are developed and used for Eigensolution of symmetric structured matrices arising in data analyzing of symmetric and regular structures [5, 6]. There are also classical methods for Eigensolution of structured matrices based on LU decomposition, preconditioning, divide and counter algorithms and other approximate methods [7- 9].

The algebraic Riccati equation has been widely used in control system syntheses [10, 11], especially in optimal control [12, 13]. As the solution to this equation may not be unique [14], the existence conditions of solutions have been considerably investigated [15]. A review of application and solution of algebraic riccati matrix equation can be found in [16].

In this paper, a simple and efficient method is presented for computing of the eigenvalues and eigenvectors of spatial structured matrices by the use of algebraic Riccati equation. Here bisymmetric matrices and per symmetric are decomposed into sub-matrices with low dimensions for simple and fast computing of eigenvalues and eigenvectors.

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II. Basic Definitions of Graph Theory

A. Definitions from Graph Theory

A *graph* $G(N,E)$ consists of a set of elements, (G), called *nodes* and a set of elements, $E(G)$, called *edges,* together with a relation of incidence which associates two distinct nodes with each edge, known as its *ends*. Two nodes of a graph are called *adjacent* if these nodes are the end nodes of an edge. An edge is called *incident* with a node if it is an end node of the edge. The *degree* of a node is the number of edges incident with the node. A *subgraph* G_i of a graph G is a graph for which $N(G_i) \subseteq N(G)$ and $E(G_i) \subseteq E(G)$, and each edge of G_i has the same ends as in G. A *path graph* P is a simple connected graph with $N(P) = E(P)+1$ that can be drawn in a way that all of its nodes and edges lie on a single straight line. A *cycle graph* C is a simply connected graph with $N(C) = E(C)$ that can be drawn so that all of its nodes and edges lie on a circle. A *path graph* and a *cycle graph* with n nodes are denoted by P_n and C_n, respectively.

B. Matrices Associated with a Graph

Let ***G*** be a graph with *n* nodes. The adjacency matrix ***A*** is an $n \times n$ matrix in which the entry in row *i* and column *j* is 1 if node n_i is adjacent to n_j, and is zero otherwise. This matrix is symmetric and the row sums of **A** are the degrees of nodes of G. The Laplacian matrix of graph *G* is defined as:

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}. \tag{1}$$

where ***D*** is a diagonal matrix in which the i-th diagonal entry is equal to the degree of node *i* [17].

III. Similarity Transformation of Matrices

A complex scalar λ_i is called an eigenvalue of the square matrix $\mathbf{A}_{n\times n}$ if a nonzero vector $\mathbf{v}_i$ exists such that $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$. The vector $\mathbf{v}_i$ is called an eigenvector of A associated with λ_i. The set of eigenvalues of **A** is called the spectrum of **A**. A scalar λ_i is an eigenvalue of **A** if and only if

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$\det(\mathbf{A}-\lambda_i\mathbf{I})=0$. That is true if and only if λ_i is a root of the characteristic polynomial. Two matrices **A** and **B** are said to be similar if there is a nonsingular matrix U such that:

$$\mathbf{B}=\mathbf{U}^{-1}\mathbf{A}\mathbf{U} \tag{2}$$

The mapping A → B is called a similarity transformation. It can be shown that similarity transformations preserve the eigenvalues of matrices:

$$\mathbf{A}\mathbf{v}_i=\lambda\mathbf{v}_i, \tag{3}$$

$$\mathbf{U}^{-1}\mathbf{A}\mathbf{U}\mathbf{U}^{-1}\mathbf{v}_i=\mathbf{U}^{-1}\lambda\mathbf{v}_i, \tag{4}$$

By substituting $\mathbf{B}=\mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ and $\mathbf{y}_i=\mathbf{U}^{-1}\mathbf{v}_i$, we will have:

$$\mathbf{B}\mathbf{y}_i=\lambda\mathbf{y}_i, \tag{5}$$

Equation (5) which is a standard representation of Eigen-problems means that λ_i are also the eigenvalues of the matrix **B** [18].

IV. Bisymmetric and Per Symmetric Matrixes

A. Bisymmetric Matrix

In mathematics, a bisymmetric matrix is a square matrix that is symmetric about both of its main diagonals. More precisely, an n×n matrix **M** is bisymmetric if and only if it satisfies $\mathbf{M}=\mathbf{M}^t$ and **M×S = S×M**, where S is the n×n exchange matrix.

$$\mathbf{S}=\begin{bmatrix} & & & 1\\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix}, \tag{6}$$

B. Persymmetric Matrix

In mathematics, persymmetric matrix may refer to a square matrix which is symmetric in the northeast-to-southwest diagonal or a square matrix such that the values on each line perpendicular to the main diagonal are the same for a given line. If B is persymmetric matrix

$$\mathbf{B}^t=\mathbf{S}\mathbf{B}\mathbf{S} \tag{7}$$

where, **S** is the exchange matrix.

V. Algebraic Riccati Matrix Equation

The matrix equation

$$\mathbf{X}\mathbf{B}\mathbf{X}+\mathbf{X}\mathbf{A}-\mathbf{D}\mathbf{X}-\mathbf{C}=\mathbf{0}, \tag{8}$$

is called algebraic riccati matrix equation. Where $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$, with appropriate dimensions, are known matrices and **X** should be determined. Solutions of the algebraic riccati matrix equation (8) are important in many applications. Potter (1966) who has solved a special case of the equation, but the closed form of the problem has not been solved [19]. Additional particular solutions are obtained by the decomposition of **C** into a sum of three matrices. Unfortunately, there is no procedure for determining every permissible decomposition of **C**. This solution of riccati equation by decomposition of **C** is as the following:

$$\begin{cases} \mathbf{X}\mathbf{B}\mathbf{X}=\mathbf{M},\\ \mathbf{X}\mathbf{A}=\mathbf{N},\\ -\mathbf{D}\mathbf{X}=\mathbf{P},\\ \mathbf{M}+\mathbf{N}+\mathbf{P}=\mathbf{C}. \end{cases} \tag{9}$$

VI. Decomposition Of Specially Structured Matrix

Consider the following blocked matrix:

$$\mathbf{L}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}, \tag{10}$$

where $\mathbf{A},\mathbf{D}\in R^{n\times n}$. It is desired to find a similarity transformation form of **L**. we use matrix **U** as

$$\mathbf{U}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{X} & \mathbf{I}\end{bmatrix}, \tag{11}$$

It is obvious that

$$\mathbf{U}^{-1}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{X} & \mathbf{I}\end{bmatrix}, \tag{12}$$

Eigenvalues of **L** can be determined as

$$\mathbf{L}v_i=\lambda\mathbf{v}_i, \tag{13}$$

Similarity transformation of **L** can be written as

$$K=\mathbf{U}^{-1}L\mathbf{U}, \tag{14}$$

$$\mathbf{K}=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{X} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{X} & \mathbf{I}\end{bmatrix} \tag{15}$$

expanding and then simplification of (15) yields

$$\mathbf{K} = \begin{bmatrix} \mathbf{A}+\mathbf{BX} & \mathbf{B} \\ -\mathbf{XBX}-\mathbf{XA}+\mathbf{DX}+\mathbf{C} & \mathbf{D}-\mathbf{XB} \end{bmatrix} \tag{16}$$

If the algebraic riccati equation, $-\mathbf{XBX}-\mathbf{XA}+\mathbf{DX}+\mathbf{C}=\mathbf{0}$, can be solved, then we can decompose (10) as

$$\begin{bmatrix} \mathbf{A}+\mathbf{BX} & \mathbf{B} \\ \mathbf{0} & \mathbf{D}-\mathbf{XB} \end{bmatrix} \tag{17}$$

So the eigenvalues of **L** can be found;

$$eig(\mathbf{L}) = eig(\mathbf{A}+\mathbf{BX}) \cup eig(\mathbf{D}-\mathbf{XB}). \tag{18}$$

VII. Decomposition of Bisymmetric Matrices

Consider bisymmetric matrix **L**

$$\mathbf{L} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{bmatrix}, \tag{19}$$

L is bisymmetric matrix so it is required to have

$$\mathbf{SBS} = \mathbf{B}^T, \quad \mathbf{AS} = \mathbf{SA}. \tag{20}$$

Similarity transformation of **L** can be written

$$\mathbf{L} = \begin{bmatrix} \mathbf{A}+\mathbf{BX} & \mathbf{B} \\ -\mathbf{XBX}-\mathbf{XA}+\mathbf{AX}+\mathbf{B}^T & \mathbf{A}-\mathbf{XB} \end{bmatrix} \tag{21}$$

If the matrix equation $-\mathbf{XBX}-\mathbf{XA}+\mathbf{AX}+\mathbf{B}^T=\mathbf{0}$ can be solved, we can write decomposed form of **L**. For this case $\mathbf{X}=\mathbf{S}$, satisfies in the algebraic riccati matrix equation Eq. (21) so:

$$eig(\mathbf{L}) = eig(\mathbf{A}+\mathbf{BS}) \cup eig(\mathbf{A}-\mathbf{SB}). \tag{22}$$

VIII. Examples

Example 1 *(Numerical)*: Consider the following sub-matrices:

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 6 & -3 \\ 2 & -3 & 15 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 9 & 6 & 3 \\ 8 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^t & \mathbf{SAS} \end{bmatrix}$$

In this example **A** is symmetric and **B** is persymmetric so we can calculate the eigenvalues of M using present method by eigenvalues of the following sub-matrices:

$$eig(\mathbf{M}) = eig(\mathbf{A}+\mathbf{BS}) \cup eig(\mathbf{A}-\mathbf{BS}).$$

eig(**A**+**BS**)=[0.6833, 9.1077, 30.2089],
eig(**A-BS**)=[-13.6225, 7.3721, 16.2504].

So the eigenvalues of matrix **M**:

eig(**M**)=[-13.6225, 0.6833, 7.3721, 9.1077, 16.2504, 30.2089].

Example 2 *(graph theory):*
Consider the graph (**G**) as;

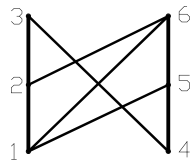

Fig. 1 Graph (**G**).

Adjacency matrix of graph (**G**) M and its sub-matrices **A**, **B** can be formed as:

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & & & 1 & 1 \\ 1 & 0 & 1 & & & 1 \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ 1 & & & 1 & 0 & 1 \\ 1 & 1 & & & 1 & 0 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} & 1 & 1 \\ & & 1 \\ 1 & & \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^t & \mathbf{SAS} \end{bmatrix}.$$

Directly calculation of the eigenvalues of **M** yields:

eig(**M**)= (-1.7912, -1.6180, -1.0000, 0.6180, 1.0000, 2.7912)

Now we can decompose **M** to (**A**+**BS**) and (**A-BS**) so eigenvalues of **M**:

$$eig(\mathbf{M}) = eig(\mathbf{A}+\mathbf{BS}) \cup eig(\mathbf{A}-\mathbf{BS}),$$

$$\mathbf{A}+\mathbf{BS} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

eig(**A+BS**)=(-1.7912, 1.0000, 2.7912),

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$$\mathrm{A - BS} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

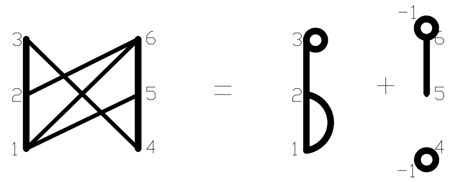
eig(**A-BS**)=(-1.61803, -1.0000, 0.61803).

Finally eigenvalues of **M** can be formed as:

$$eig(\mathbf{M}) = eig(\mathbf{A} + \mathbf{BS}) \cup eig(\mathbf{A} - \mathbf{BS}),$$

eig(**M**)= (-1.7912, -1.6180, -1.0000, 0.6180, 1.0000, 2.7912).

According the above calculation, we can decompose the graph **G** to sub-graph $\mathbf{G_1}$ and $\mathbf{G_2}$ in the following form:



$$(G) = (G_1 \cup G_2)$$

Fig. 2 Graph (**G**) and its decomposition and healed form.

Example 3 (*structural mechanics*):

Consider the truss models $\mathbf{G_1}$, $\mathbf{G_2}$, $\mathbf{G_3}$, $\mathbf{G_4}$ and their adjacency and Laplacian matrices of the truss model as:

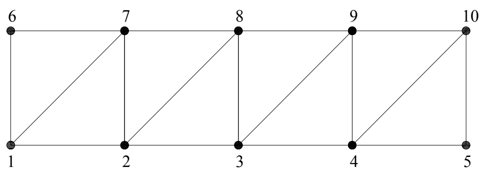


Fig. 3 Graph model of truss $\mathbf{G_1}$.

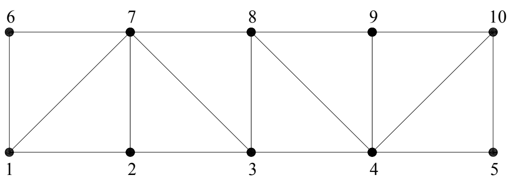


Fig. 4 Graph model of truss $\mathbf{G_2}$.

$$Lap(G_2) = \begin{bmatrix} 3 & -1 & & & & -1 & -1 & & & \\ -1 & 3 & -1 & & & & -1 & & & \\ & -1 & 4 & -1 & & & -1 & -1 & & \\ & & -1 & 5 & -1 & & & -1 & -1 & -1 \\ & & & -1 & 2 & & & & & -1 \\ -1 & & & & & 2 & -1 & & & \\ -1 & -1 & -1 & & & -1 & 5 & -1 & & \\ & & -1 & -1 & & & -1 & 4 & -1 & \\ & & & -1 & & & & -1 & 3 & -1 \\ & & & -1 & -1 & & & & -1 & 3 \end{bmatrix},$$

$$Adj(G_2) = \begin{bmatrix} & 1 & & & & 1 & 1 & & & \\ 1 & & 1 & & & & 1 & & & \\ & 1 & & 1 & & & 1 & 1 & & \\ & & 1 & & 1 & & & 1 & 1 & 1 \\ & & & 1 & & & & & & 1 \\ 1 & & & & & & 1 & & & \\ 1 & 1 & 1 & & & 1 & & 1 & & \\ & & 1 & 1 & & & 1 & & 1 & \\ & & & 1 & & & & 1 & & 1 \\ & & & 1 & 1 & & & & 1 & \end{bmatrix},$$

$$Adj(G_1) = \begin{bmatrix} & 1 & & & & 1 & 1 & & & \\ 1 & & 1 & & & & 1 & 1 & & \\ & 1 & & 1 & & & & 1 & 1 & \\ & & 1 & & 1 & & & & 1 & 1 \\ & & & 1 & & & & & & 1 \\ 1 & & & & & & 1 & & & \\ 1 & 1 & & & & 1 & & 1 & & \\ & 1 & 1 & & & & 1 & & 1 & \\ & & 1 & 1 & & & & 1 & & 1 \\ & & & 1 & 1 & & & & 1 & \end{bmatrix},$$

$$Lap(G_1) = \begin{bmatrix} 3 & -1 & & & & -1 & -1 & & & \\ -1 & 4 & -1 & & & & -1 & -1 & & \\ & -1 & 4 & -1 & & & & -1 & -1 & \\ & & -1 & 4 & -1 & & & & -1 & -1 \\ & & & -1 & 2 & & & & & -1 \\ -1 & & & & & 2 & -1 & & & \\ -1 & -1 & & & & -1 & 4 & -1 & & \\ & -1 & -1 & & & & -1 & 4 & -1 & \\ & & -1 & -1 & & & & -1 & 4 & -1 \\ & & & -1 & -1 & & & & -1 & 3 \end{bmatrix}$$

$$Adj(G_3) = \begin{bmatrix} & 1 & & 1 & 1 & 1 & & \\ 1 & & 1 & & & 1 & & \\ & 1 & & 1 & & 1 & 1 & 1 \\ 1 & & 1 & & & & & 1 \\ 1 & & & & & 1 & & 1 \\ 1 & 1 & 1 & & 1 & & 1 & \\ & & 1 & & & 1 & & 1 \\ & & 1 & 1 & 1 & & 1 & \end{bmatrix},$$

$$Lap(G_3) = \begin{bmatrix} 4 & -1 & & -1 & -1 & -1 & & \\ -1 & 3 & -1 & & & -1 & & \\ & -1 & 5 & -1 & & -1 & -1 & -1 \\ -1 & & -1 & 3 & & & & -1 \\ -1 & & & & 3 & -1 & & -1 \\ -1 & -1 & -1 & & -1 & 5 & -1 & \\ & & -1 & & & -1 & 3 & -1 \\ & & -1 & -1 & -1 & & -1 & 4 \end{bmatrix},$$

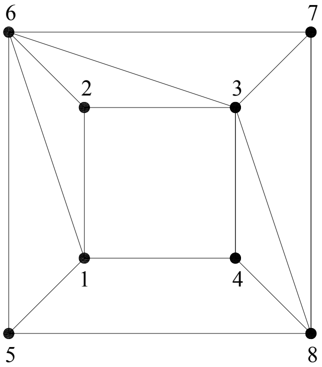


Fig. 5 Graph model of truss $\mathbf{G_3}$.

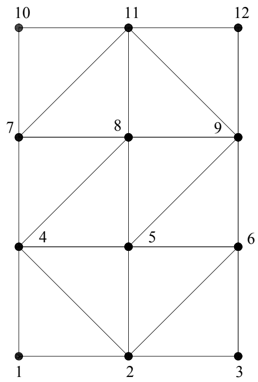


Fig. 6 Graph model of truss $\mathbf{G_4}$.

$$Adj(\mathbf{G}_4) = \begin{bmatrix} & 1 & & 1 & & & & & & & & \\ 1 & & 1 & 1 & 1 & 1 & & & & & & \\ & 1 & & & & 1 & & & & & & \\ 1 & 1 & & & 1 & & 1 & 1 & & & & \\ & 1 & & 1 & & 1 & & 1 & 1 & & & \\ & 1 & 1 & & 1 & & & & 1 & & & \\ & & & 1 & & & & 1 & & 1 & 1 & \\ & & & 1 & 1 & & 1 & & 1 & & 1 & \\ & & & & 1 & 1 & & 1 & & & 1 & 1 \\ & & & & & & 1 & & & & 1 & \\ & & & & & & 1 & 1 & 1 & 1 & & 1 \\ & & & & & & & & 1 & & 1 & \end{bmatrix},$$

$$Lap(\mathbf{G}_4) = \begin{bmatrix} 2 & -1 & & -1 & & & & & & & & \\ -1 & 5 & -1 & -1 & -1 & -1 & & & & & & \\ & -1 & 2 & & & -1 & & & & & & \\ -1 & -1 & & 5 & -1 & & -1 & -1 & & & & \\ & -1 & & -1 & 5 & -1 & & -1 & -1 & & & \\ & -1 & -1 & & -1 & 4 & & & -1 & & & \\ & & & -1 & & & 4 & -1 & & -1 & -1 & \\ & & & -1 & -1 & & -1 & 5 & -1 & & -1 & \\ & & & & -1 & -1 & & -1 & 5 & & -1 & -1 \\ & & & & & & -1 & & & 2 & -1 & \\ & & & & & & -1 & -1 & -1 & -1 & 5 & -1 \\ & & & & & & & & -1 & & -1 & 2 \end{bmatrix},$$

In all of these examples adjacency and Laplacian matrices are persymmetric and sub-matrices hold in the defined conditions so we can decompose to smaller sub-matrices for easy and fast computing of their eigenvalues.

IX. Concluding Remarks

In this paper, a simple method by using the algebraic riccati matrix equation is presented for calculating the eigenvalues of adjacency and Laplacian matrices of bisymmetric and persymmetric matrices of structural and graph theory models. Examples studied here show that the results obtained by the present method are exact solution method for the problem. The calculated eigenvalues are exact values, and can efficiently be used for solution of the models whose structural matrices are or can be transformed into the presented form. The present method can be used in combinatorial optimization problems such as the ordering and partitioning of structural models.

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