International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:2, 2013

Explicit Solutions and Stability of Linear Differential Equations with multiple Delays

Felix Che Shu

Abstract—We give an explicit formula for the general solution of a one dimensional linear delay differential equation with multiple delays, which are integer multiples of the smallest delay. For an equation of this class with two delays, we derive two equations with single delays, whose stability is sufficient for the stability of the equation with two delays. This presents a new approach to the study of the stability of such systems. This approach avoids requirement of the knowledge of the location of the characteristic roots of the equation with multiple delays which are generally more difficult to determine, compared to the location of the characteristic roots of equations with a single delay.

Keywords—Delay Differential Equation, Explicit Solution, Exponential Stability, Lyapunov Exponents, Multiple Delays.

I. Introduction

DELAY Equations play an important role in mathematical modelling. This is a consequence of the fact that effects of delays are inherent in the dynamics of many systems which are of interest to humanity. However, solutions of Delay Differential Equations are difficult to get in explicit form in general. This causes inconveniences in many situations when dealing with Delay Differential Equations.

The classical method of solving linear Delay Differential Equations is the step method. This is an iterative procedure, by which the Delay Equation is solved on successive intervals of a suitable length, if an appropriate initial function is specified.

If the equation has more than one delay or even if it has one delay but is multidimensional, then the terms of the solution obtained using the step method do not always follow easily recognizable patterns. This makes it difficult to find explicit representations for solutions of these equations.

In [3], we gave an explicit formula for the solution of a two dimensional irreducible linear system of Delay Differential Equations. The formula was used in [4] to prove a stability result for these systems.

The use of the explicit formula in the study is advantageous compared to traditional approaches to the study of the stability of these systems in that it does not require knowledge of the roots of the characteristic function of the multidimensional system, which in general are difficult to find.

In what follows, we shall prove similar results for one dimensional equations with multiple delays. We will give a formula which is applicable to obtain an explicit representation of the solution of any linear Delay Differential Equation of the form

$$\begin{aligned} \acute{z}(t) &= az(t) + \sum_{j=1}^{d} b_j z(t - \frac{jr}{d}),\ t \geq 0, \qquad (1) \\ z(t) &= \varphi(t),\ t \in [-r\ 0], \qquad (2) \end{aligned}$$

where $(b_1, \dots, b_d) \in \mathbb{R}^d,\ r > 0,\ a \in \mathbb{R},\ d \in \{1,\ 2,\ 3\ ,\dots\}$ and φ is integrable and use it in proving a stability result for solutions of these equations when $d = 2$.

Although the method is applicable to the case $d \geq 3$, our interest in the case $d = 2$ is mainly due to the fact that the computations and notation involved can be kept simple.

II. Prerequisites

In this section, we fix notation and the conventions by which we abide in the sequel. Throughout, $\mathbb{N} := \{1,\ 2, \cdots\},\ \mathbb{N}_* := \mathbb{N} \cup \{0\},\ \mathbb{Z} := \mathbb{N}_* \cup \{-n : n \in \mathbb{N}\},\ d \in \mathbb{N},\ r > 0,\ a \in \mathbb{R}$, $(b_1, \dots, b_d) \in \mathbb{R}^d$ and we consider the equation (1), (2).

We shall deal with vectors of real numbers as well as sets of real numbers. By a vector, we shall understand an ordered collection of objects, which are not necessarily distinct. A set will be a collection of objects whose members are distinct, but not necessarily ordered. For a vector X of numbers, we will use the notation $\{x \in X : C\}$ to refer to the entries of X for which the condition C holds and $\sharp\{x \in X : C\}$ will denote the number of entries of X, for which C holds. We shall use the same notation for sets but it will be clear from the context whether we are dealing with sets or vectors. We shall use the usual notation $(x_1, x_2, \dots, x_n)$ for n-dimensional row vectors with entries $x_1, \dots, x_n$.

$\emptyset$ shall denote either the empty set or the empty vector i.e., the vector with no entries.

θ will be a numerical symbol with value 1 and hence $\theta x = x\theta = x$ for all $x \in \mathbb{R}$.

Given the equation (1), we associate with it, the sequence $\{E_m\}_{m \in \mathbb{Z}}$ of sets, defined by

$$E_m := \begin{cases} \emptyset & : \ m < 0 \\ \{\theta\} & : \ m = 0 \\ \cup_{j=1}^{d} b_j E_{m-j} & : \ m \geq 1, \end{cases}$$

and define $E := \cup\{E_m : m \in \mathbb{Z}\}$. In this definition, $b_j E_{m-j} := \{b_j x : x \in E_{m-j}\}$.

Remark II.1. (i) To enable elements of E_m to be distinct and hence, that E_m is actually a set, products $b_j x$ in the definition of E_m shall not be commutative. Note that elements of E are algebraic expressions involving the coefficients $b_1, \dots, b_d$ in (1) and θ. If $x \in \mathbb{R}$, then we

Felix Che Shu, Department of Mathematics, University of Buea, Cameroon e-mail: shu.felix@ubuea.cm.

define $x\emptyset := \emptyset$. The symbol 0 will be used to represent the real number $zero$. Expressions having numerical value $zero$ shall also be represented by this symbol, where it is convenient to do so.

We shall write x^m for the product $\overbrace{x \cdots x}^{m\times}$ and replace expressions having value 1 by θ, where it is convenient.

(ii) Let $m \geq 1$ and $n \geq 1$, then

$$E_m = \cup_{j_1=1}^d b_{j_1} E_{m-j_1} = \cup_{j_1=1}^d \cup_{j_2=1}^d b_{j_1} b_{j_2} E_{m-j_1-j_2}.$$

If we proceed this way, then after n recursions we arrive at $E_m =$

$$\cup_{j_1=1}^d \cdots \cup_{j_n=1}^d b_{j_1} \cdots b_{j_n} E_{m-j_1-\cdots-j_n}. \quad (3)$$

For any $n \geq 1$ and $m \geq 1$, $E_{m-j_1-\cdots-j_n}$=

$$\begin{cases} \emptyset & : \quad m - j_1 - \cdots - j_n < 0 \\ \{\theta\} & : \quad m - j_1 - \cdots - j_n = 0. \end{cases} \quad (4)$$

From (3), it follows that for any $n \geq 1$ and $(j_1, \ldots, j_n)$ such that $j_l \in \{1, \ldots, d\}$, $l = 1, \ldots, n$, we have $b_{j_1} \cdots b_{j_n} E_{m-j_1-\cdots-j_n} \subseteq E_m$ and hence by (4), if $m - j_1 - \cdots - j_n = 0$, then $b_{j_1} \cdots b_{j_n} \in E_m$.

(iii) From (ii), it is obvious that for $m \geq 1$, elements of E_m are precisely the elements $b_{j_1} \cdots b_{j_n}$ of E for which $m = j_1 + \cdots + j_n$, $j_l \in \{1, \ldots, d\}$, $l \in \{1, \ldots, n\}$, for some $n \geq 1$.

The following notation will be used in many of our arguments:

Definition II.1. *If $x \in E$, then we define*

(i) $\varepsilon(x) := \begin{cases} (j_1, \ldots, j_n) & : \quad x \neq \theta, \ x = b_{j_1} \cdots b_{j_n} \\ \emptyset & : \quad x = \theta. \end{cases}$

(ii) $\beta(x) := \emptyset$, *if $x = \theta$ and* $\beta(x) := \{i_1, \ldots, i_k\}$, *if $x \neq \theta$, $1 \leq i_1 < \ldots < i_k \leq d$,* where $i_1, \ldots, i_k$ are the distinct entries of $\varepsilon(x)$. As an example, if $\varepsilon(x) = (2, 1, 2, 3, 1, 2, 3)$ then $\beta(x) := \{1, 2, 3\}$.

(iii) $m_x(j) := \sharp\{l \in \varepsilon(x) : l = j\}, \quad j \in \{1, \ldots, d\}$.

(v) $\chi(x) := ((i_1, m_x(i_1)), \ldots, (i_k, m_x(i_k)))$, if $x \neq \theta$, $\beta(x) = \{i_1, \ldots, i_k\}$ and $\chi(x) := (0, 0)$ if $x = \theta$.

Remark II.2. (i) If $x, \ y \in E$, then $\chi(x) = \chi(y) \iff \beta(x) = \beta(y)$ and $m_x(j) = m_y(j)$ for all $j \in \beta(x) \iff \varepsilon(x)$ is a permutation of $\varepsilon(y)$.

Also note that if $x \in E$, $x \neq \theta$ and (v, w) is an entry of the vector $\chi(x)$, then $v \geq 1$ and $w \geq 1$.

(ii) If $x, \ y \in E$, $\varepsilon(x) := (j_1, \ldots, j_n)$ and $\varepsilon(y) := (k_1, \ldots, k_m)$ then we define the vector $(\varepsilon(x), \varepsilon(y))$ by

$$(\varepsilon(x), \varepsilon(y)) := (j_1, \ldots, j_n, k_1, \ldots, k_m).$$

For the product xy we have

$$\varepsilon(xy) = \begin{cases} (\varepsilon(x), \varepsilon(y)) & : \quad x \neq \theta \text{ and } y \neq \theta \\ \varepsilon(x) & : \quad y = \theta \\ \varepsilon(y) & : \quad x = \theta \\ \emptyset & : \quad x = \theta \text{ and } y = \theta. \end{cases}$$

(iii) From (ii), if $x, \ y \in E$, then $m_{xy}(j) = m_x(j) + m_y(j), \ j \in \{1, \ldots, d\}$.

(iv) We will make it a convention that the order of the indices in the definitions of $\chi(x)$ and $\beta(x)$ is the same, and is such that $1 \leq i_1 < \cdots < i_k \leq d$.

Definition II.2. *For $x \in E$, we define* $p(x) := \sum_{\{i \in \beta(x)\}} m_x(i)$, $q(x) := \sum_{\{i \in \beta(x)\}} i m_x(i)$ and $v(x) := \prod_{\{i \in \beta(x)\}} (m_x(i))!$.

Note that $p(\theta) = q(\theta) = 0$ and $v(\theta) = 1$ since $\beta(\theta) = \emptyset$. The following lemmas are a consequence of the preceding definitions and remarks:

Lemma II.1. (i) *If $x, y \in E$, then $p(xy) = p(x) + p(y)$ and $q(xy) = q(x) + q(y)$. Further,*

(ii) *if $\beta(x) \cap \beta(y) = \emptyset$, then $v(xy) = v(x)v(y)$.*

Proof: (i) If $x = y = \theta$, then $xy = \theta$, hence since $p(\theta) = 0$, it follows that $p(xy) = p(x) + p(y)$.

If $x = \theta$ and $y \neq \theta$, then $xy = y$. Therefore $p(xy) = p(y)$ and hence $p(xy) = p(x) + p(y)$, since $p(x) = 0$. If $y = \theta$ and $x \neq \theta$, then the argument is similar.

Assume now that $x \neq \theta$ and $y \neq \theta$. Let $\chi(x) = ((i_1, m_x(i_1)), \ldots, (i_k, m_x(i_k)))$ and $\chi(y) = ((j_1, m_y(j_1)), \ldots, (j_l, m_y(j_l)))$.

Now $\beta(xy) = (\beta(x) \backslash \beta(y)) \cup (\beta(y) \backslash \beta(x)) \cup (\beta(x) \cap \beta(y))$, which is a disjoint union. Using Remark II.2(iii),

$$\begin{aligned} p(xy) &= \sum_{\{i \in \beta(xy)\}} m_{xy}(i) \\ &= \sum_{\{i \in \beta(x) \backslash \beta(y)\}} m_{xy}(i) + \sum_{\{i \in \beta(y) \backslash \beta(x)\}} m_{xy}(i) + \sum_{\{i \in \beta(x) \cap \beta(y)\}} m_{xy}(i) \\ &= \sum_{\{i \in \beta(x) \backslash \beta(y)\}} m_x(i) + \sum_{\{i \in \beta(y) \backslash \beta(x)\}} m_y(i) + \sum_{\{i \in \beta(x) \cap \beta(y)\}} (m_x(i) + m_y(i)) \\ &= \left(\sum_{\{i \in \beta(x) \backslash \beta(y)\}} m_x(i) + \sum_{\{i \in \beta(x) \cap \beta(y)\}} m_x(i) \right) \\ &\quad + \left(\sum_{\{i \in \beta(y) \backslash \beta(x)\}} m_y(i) + \sum_{\{i \in \beta(x) \cap \beta(y)\}} m_y(i) \right) \\ &= \sum_{\{i \in \beta(x)\}} m_x(i) + \sum_{\{i \in \beta(y)\}} m_y(i) = p(x) + p(y). \end{aligned}$$

The proof of the second assertion of (i) is similar.

$$\begin{aligned} (ii) \ v(xy) &= \prod_{\{i \in \beta(xy)\}} m_x(i)! \\ &= \prod_{\{i \in \beta(x)\}} m_x(i)! \prod_{\{i \in \beta(y)\}} m_x(i)! = v(x)v(y). \end{aligned}$$

■

Lemma II.2. *For any $j \in \{1, \ldots, d\}$ and $x \in E$,*

(i) $p(b_j x) = p(x) + 1$

(ii) $q(b_j x) = q(x) + j$.

Proof: By Lemma II.1(i), $p(b_j x) = p(b_j) + p(x)$ and $q(b_j x) = q(b_j) + q(x)$. Since $\varepsilon(b_j) = (j)$, it follows that $\beta(b_j) = \{j\}$ and $m_{b_j}(l) = \delta_{jl},\ l = 1, \dots, d$, where δ is the kronecker symbol. Therefore, $p(b_j) = m_{b_j}(j) = 1$ and $q(b_j) = j m_{b_j}(j) = j$. ■

Lemma II.3. (i) *For $m \geq 0$ and $x \in E$, $q(x) = m$ if and only if $x \in E_m$.*

(ii) *If $d = 1$, then for all $m \geq 1$, $\{x \in E_m : 1 \notin \beta(x)\} = \emptyset$.*

(iii) *If $d = 2$, then for all $m \geq 2$, $\{x \in E_m : 1 \notin \beta(x)\} = \begin{cases} \{b_2^k\} & : \ m = 2k \\ \emptyset & : \ m \text{ odd} . \end{cases}$*

(iv) *For $d \geq 1$, $\{x \in E_m : 1 \notin \beta(x)\} \neq \emptyset$ for all $m \geq 2 \iff d \geq 3$.*

Proof: (i) For $m = 0$, $x \in E_0 \iff x = \theta \iff q(x) = 0$.

Let now $x \in E$, $m \geq 1$ and $\chi(x) = ((i_1, m_x(i_1)), \dots, (i_k, m_x(i_k)))$ for some $k \geq 1$. For $n \geq 1$, let

$$\Omega_{mn} := \{b_{j_1} b_{j_2} \cdots b_{j_n} \in E : \ j_l \in \{1, \dots, d\}, \quad l = 1, \dots, n \text{ and } \sum_{l=1}^{n} j_l = m\},$$

then by Remark II.1(iii),

$$\begin{aligned} x \in E_m &\iff x \in \cup_{n=1}^{\infty} \Omega_{mn} \\ &\iff i_1 m_x(i_1) + \cdots + i_k m_x(i_k) = m \qquad (5) \\ &\iff q(x) = m, \qquad (6) \end{aligned}$$

where (5) and (6) follow from the definitions of the quantities $\chi(x)$ and $q(x)$.

(ii) Here, we simply note that if $d = 1$, then $E_m = \{b_1^m\}$ for all $m \geq 1$ and so $\{1\} = \beta(x)$ for all $x \in E_m$.

(iii) Let $d = 2$, $k \geq 1$ and $m = 2k$, then

$$\begin{aligned} & x \in E_{2k} \text{ and } 1 \notin \beta(x) \\ &\iff x \in b_1 E_{2k-1} \cup b_2 E_{2(k-1)} \text{ and } 1 \notin \beta(x) \\ &\iff x \in b_2 E_{2(k-1)} \text{ and } 1 \notin \beta(x), \qquad (7) \end{aligned}$$

where (7) follows from the fact that $1 \in \beta(b_1 x)$ for all $x \in E$. Therefore,

$$x \in E_{2k} \text{ and } 1 \notin \beta(x) \iff x \in b_2 E_{2(k-1)} \text{ and } 1 \notin \beta(x). \quad (8)$$

We will now prove the assertion by induction on k. When $k = 1$, then $E_2 = \{b_1^2, b_2\}$ and thus the assertion is true for $k = 1$. Assume that the assertion is true for $k = n$, for some $n \geq 1$, i.e., $x \in E_{2n}$ and $1 \notin \beta(x) \iff x = b_2^n$. Using (8) and the assumption of the induction in this order, for $k = n+1$, it follows that $x \in E_{2(n+1)}$ and $1 \notin \beta(x) \iff x \in b_2 E_{2n}$ and $1 \notin \beta(x) \iff x = b_2^{n+1}$.

Let now $d = 2$, $k \geq 1$ and $m = 2k + 1$, then arguments similar to those made in the case $m = 2k$, lead to the following conclusion:

$$x \in E_{2k+1} \text{ and } 1 \notin \beta(x) \iff x = b_2^k b_1 \text{ and } 1 \notin \beta(x). \quad (9)$$

Since $1 \in \{1, 2\} = \beta(b_2^k b_1)$, the right hand side of (9) is a contradiction, implying that $\{x \in E_m : 1 \notin \beta(x)\} = \emptyset$.

(iv) Assume first that $d \geq 3$. If $m \geq 2$, then $m = 2k + j$, where $j \in \{0, 1\}$ and $k \geq 1$. If $m = 2k$, i.e. $j = 0$, then it follows from (i) that $b_2^k \in E_{2k}$, since $q(b_2^k) = 2k$. Similarly, if $m = 2k + 1$, i.e. $j = 1$, then it also follows from (i) that $b_3 b_2^{k-1} \in E_{2k+1}$. Note that the assumption that $d \geq 3$ is relevant, in order that b_3 should exist. It is easy to see that $1 \notin \beta(b_2^k)$ and $1 \notin \beta(b_3 b_2^{k-1})$.

For the other direction of the equivalence, assume on the contrary that $d \leq 2$. We have to show that there exists $m \geq 2$, such that $\{x \in E_m : 1 \notin \beta(x)\} = \emptyset$. If $d = 2$ and $m \geq 2$ is odd, then by (iii), $\{x \in E_m : 1 \notin \beta(x)\} = \emptyset$. If $d = 1$ and $m \geq 2$ is arbitrary, then by (ii), $\{x \in E_m : 1 \notin \beta(x)\} = \emptyset$. ■

Lemma II.4. *Let $d \geq 1$. If $n \geq 1$ and $\psi : E \to \mathbb{R}$, then*

$$\sum_{m=1}^{n} \sum_{\{x \in E_m\}} \psi(x) = \sum_{j=1}^{d \wedge n} \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} \psi(b_j x).$$

Proof: Since $E_m = \cup_{j=1}^{d} b_j E_{m-j}$, a disjoint union, we have

$$\begin{aligned} \sum_{m=1}^{n} \sum_{\{x \in E_m\}} \psi(x) &= \sum_{m=1}^{n} \sum_{j=1}^{d} \sum_{\{x \in E_{m-j}\}} \psi(b_j x) \\ &= \sum_{j=1}^{d} \sum_{m=1}^{n} \sum_{\{x \in E_{m-j}\}} \psi(b_j x) \\ &= \sum_{j=1}^{d} \sum_{m=1-j}^{n-j} \sum_{\{x \in E_m\}} \psi(b_j x). \end{aligned}$$

Since $E_m = \emptyset$ for $m < 0$, using the convention that $\sum_{\{x \in \emptyset\}} f(x) = 0$ for any real function f, it follows that

$$\sum_{j=1}^{d} \sum_{m=1-j}^{n-j} \sum_{\{x \in E_m\}} \psi(b_j x) = \sum_{j=1}^{d \wedge n} \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} \psi(b_j x). \quad ■$$

III. Explicit Solutions of Linear Equations with Multiple Delays

Definition III.1. (i) *We call a real valued function $z : [-r\ \infty) \to \mathbb{R}$, a solution of (1) (2), if it is continuous, satisfies (1) Lebesgue almost everywhere on $[0\ \infty)$ and (2).*

(ii) *The fundamental solution associated with (1), is the solution of (1) (2), when φ in (2) is given by $\varphi(t) = 1_{\{0\}}(t),\ t \in [-r\ 0]$.*

If φ in (2) is arbitrary but integrable, then it can be shown that the solution to the equation (1) (2) which we denote by z^{φ}, is given by $z^{\varphi}(t) := \varphi(t),\ t \in [-r\ 0]$ and for $t \geq 0$

$$z^{\varphi}(t) := z(t)\varphi(0) + \sum_{j=1}^{d} b_j \int_{-\frac{jr}{d}}^{0} z(t - s - \frac{jr}{d})\varphi(s)ds, \quad (10)$$

where z denotes the fundamental solution. In view of this, we shall first of all be interested in determining the fundamental solution of (1).

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:2, 2013

Proposition III.1. *Let $d \geq 1$ and consider the equation (1). Let $z(t), t \geq -r$ denote its fundamental solution and*

$$y(t) := \begin{cases} 1_{\{0\}}(t) & : \quad t \in [-r \ 0] \\ \sum_{m=0}^{[\frac{dt}{r}]} \sum_{\{x \in E_m\}} h(t, x, p(x), q(x), d) & : \quad t \geq 0, \end{cases} \tag{11}$$

where

$$h(t, x, k, l, d) := x \frac{(t - \frac{lr}{d})^k}{k!} \exp\{a(t - \frac{lr}{d})\},$$

then $y(t) = z(t)$ for all $t \geq 0$.

Proof: Since $z(t) = y(t)$ for all $t \in [-r \ 0]$, we will show by induction that $z(t) = y(t)$ on the intervals $[\frac{nr}{d} \ \frac{(n+1)r}{d}]$, $n = 0, 1, 2, \cdots$.

Let $t \in [0 \ \frac{r}{d})$, then $[\frac{dt}{r}] = 0$, hence by (11), $y(t) = h(t, \theta, p(\theta), q(\theta), d) = e^{at}$.

Solving (1) on the interval $[0 \ \frac{r}{d})$, with the initial condition $z(t) = 1_{\{0\}}(t),\ t \in [-r \ 0]$, we have $z(t) = e^{at} + \int_0^t e^{a(t-s)} \sum_{j=1}^{d} b_j z(s - \frac{jr}{d}) ds = e^{at}$, since for $s \in [0 \ \frac{r}{d})$, $z(s - \frac{jr}{d}) = 0,\ j = 1, \ldots, d$. This shows that $z(t) = y(t)$ for all $t \in [0 \ \frac{r}{d})$.

If $t = \frac{r}{d}$, then $[\frac{dt}{r}] = 1$. Since $E_0 = \{\theta\}$, $E_1 = \{b_1\}$, $p(\theta) = q(\theta) = 0$ and $p(b_1) = q(b_1) = 1$, $y(\frac{r}{d}) = \sum_{m=0}^{1} \sum_{\{x \in E_m\}} h(\frac{r}{d}, x, p(x), q(x), d)$
$= e^{a\frac{r}{d}} + \frac{b_1}{p(b_1)!}(\frac{r}{d} - \frac{r}{d}) e^{a(\frac{r}{d} - \frac{r}{d})} = e^{a\frac{r}{d}}$.

Also, since z is continuous, $z(\frac{r}{d}) = e^{a\frac{r}{d}}$. Therefore, $z(t) = y(t)$ for all $t \in [0 \ \frac{r}{d}]$.

The assertion is therefore true for $n = 0$.

Let $n \geq 1$ and assume that $z(t) = y(t)$ for all $t \in [\frac{(k-1)r}{d} \ \frac{kr}{d}]$ and $k = 1, \ldots, n$. We will now show that $z(t) = y(t)$, for all $t \in [\frac{nr}{d} \ \frac{(n+1)r}{d}]$.

By assumption, $z(t) = 1_{\{0\}}(t),\ t \in [-r \ 0]$ and for $t \in (0, \frac{nr}{d}]$,

$$z(t) = \sum_{m=0}^{[\frac{dt}{r}]} \sum_{\{x \in E_m\}} h(t, x, p(x), q(x), d). \tag{12}$$

If $1 \leq n < d$ and $s \in [\frac{nr}{d} \ \frac{(n+1)r}{d})$, then $s - \frac{jr}{d} \in [-r \ 0)$ for $j = n+1, \ldots, d$ and hence $z(s - \frac{jr}{d}) = 0,\ j = n+1, \ldots, d$.

If $n \geq d$ and $s \in [\frac{nr}{d} \ \frac{(n+1)r}{d})$, then $s - \frac{jr}{d} \geq 0$ for all $j = 1, \ldots, d$ and hence since $s \in [\frac{nr}{d} \ \frac{(n+1)r}{d})$ if and only if $[\frac{d}{r}(s - \frac{jr}{d})] = n - j$, it follows from (12) that for $n \geq d$, if $s \in [\frac{nr}{d} \ \frac{(n+1)r}{d})$ then for any $j \in \{1, \ldots, d\}$, $z(s - \frac{jr}{d}) = \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} h(s, x, p(x), (j + q(x)), d)$ otherwise.

Hence for $t \in [\frac{nr}{d} \ \frac{(n+1)r}{d})$, $z(t) = e^{a(t - \frac{nr}{d})} z(\frac{nr}{d}) + \int_{\frac{nr}{d}}^{t} e^{a(t-s)} \sum_{j=1}^{d \wedge n} b_j \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} h(s, x, p(x), (j + q(x)), d) ds =$
$e^{a(t - \frac{nr}{d})} \sum_{m=0}^{n} \sum_{\{x \in E_m\}} h(\frac{nr}{d}, x, p(x), q(x), d)$

$$+ \sum_{j=1}^{d \wedge n} \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} b_j x \int_{\frac{nr}{d}}^{t} \frac{(s - \frac{(j+q(x))r}{d})^{p(x)}}{p(x)!} ds \times$$
$$\exp\{a(t - \frac{(j+q(x))r}{d})\}$$

$$= \sum_{m=0}^{n} \sum_{\{x \in E_m\}} x \frac{(\frac{nr}{d} - \frac{q(x)r}{d})^{p(x)}}{p(x)!} e^{a(t - \frac{q(x)r}{d})} \tag{13}$$

$$+ \sum_{j=1}^{d \wedge n} \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} h(t, b_j x, (p(x)+1), (q(x)+j), d)$$

$$- \sum_{j=1}^{d \wedge n} \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} b_j x \frac{(\frac{nr}{d} - \frac{(j+q(x))r}{d})^{p(x)+1}}{(p(x)+1)!} e^{a(t - \frac{(j+q(x)r)}{d})}. \tag{14}$$

Letting $\psi(x) := x \frac{(\frac{nr}{d} - \frac{q(x)r}{d})^{p(x)}}{p(x)!} e^{a(t - \frac{q(x)r}{d})}$ in Lemma II.4 and using Lemma II.2, we obtain
$z(t) = e^{at} +$
$\sum_{j=1}^{d \wedge n} \sum_{m=0}^{n-j} \sum_{\{x \in E_m\}} h(t, b_j x, (p(x)+1), (q(x)+j), d)$.

Using Lemmas II.2 and II.4 again with $\psi(x) := h(t, x, p(x), q(x), d)$, we get

$$z(t) = \sum_{m=0}^{n} \sum_{\{x \in E_m\}} h(t, x, p(x), q(x), d) = y(t).$$

Therefore, $y(t) = z(t)$ for all $t \in [\frac{nr}{d} \ \frac{(n+1)r}{d})$.

Since z is continuous on $[0 \ \infty)$, $z\left(\frac{(n+1)r}{d}\right) = \lim_{u \uparrow \frac{(n+1)r}{d}} z(u) = \sum_{m=0}^{n} \sum_{\{x \in E_m\}} h(\frac{(n+1)r}{d}, x, p(x), q(x), d)$.

Also,

$$y\left(\frac{(n+1)r}{d}\right) = \sum_{m=0}^{n+1} \sum_{\{x \in E_m\}} h(\frac{(n+1)r}{d}, x, p(x), q(x), d)$$

$= \sum_{m=0}^{n} \sum_{\{x \in E_m\}} h(\frac{(n+1)r}{d}, x, p(x), q(x), d)$, where the last equality results from Lemma II.3(i). Therefore, $z(t) = y(t)$ for all $t \in [\frac{nr}{d} \ \frac{(n+1)r}{d}]$ which completes the proof. ■

The following result which we will need is well known:

Corollary III.1. *The fundamental solution of the equation, $\acute{z}(t) = az(t) + bz(t - r),\ t \geq 0$ is given for $t \geq 0$, by* $z(t) = \sum_{m=0}^{[\frac{t}{r}]} \frac{b^m}{m!} (t - mr)^m \exp\{a(t - mr)\}$.

Proof: For the proof, let $d = 1$ in Proposition III.1 and note that in this case, $b = b_1$, hence $E_m = \{b^m\}$ and $q(b^m) = p(b^m) = m$, for all m. ∎

For the rest of this section, we assume that $d \geq 3$. By virtue of Lemma II.3(iv), we define the following sets M_k, for $k \in \mathbb{Z}$ with M_k non-empty, $k \geq 0$:

$$M_k := \begin{cases} \emptyset & : \quad k < 0 \\ \{\theta\} & : \quad k = 0 \\ \{b_1\} & : \quad k = 1 \\ \{x \in E_k : 1 \notin \beta(x)\} & : \quad k \geq 2. \end{cases}$$

Lemma III.1. *Let $m \geq 1$ and $x \in E$, then $x \in E_m$ if and only if*

(a) $\chi(x) = (1, m)$ *or*
(b) *there exists* $n \in \{1, \ldots, m-2\}$ *and* $y \in M_{m-n}$, *such that* $\chi(x) = ((1, n), \chi(y))$ *or*
(c) $x \in M_m,\ m \geq 2$.

Proof: Let $m \geq 1$ and $x \in E_m$, then $x = b_{j_1} \cdots b_{j_k}$ for some $k \in \{1, \ldots, m\}$, where $j_l \in \{1, \ldots, d\}$, $l \in \{1, \ldots, k\}$. By Lemma II.3(i), $q(x) = m$, hence, either $k = m$ in which case $j_l = 1$ for all $l \in \{1, \ldots, k\}$, i.e. $\chi(x) = (1, m)$ or $1 \notin \beta(x)$ in which case $x \in M_m, m \geq 2$ or there exists n such that $1 \leq n < k \leq m$ and $m_x(1) = n$. In this case, $x = b_1^n y$ for some $y \in E$ with $1 \notin \beta(y)$ and $y \neq \theta$ since $n < m$ and $q(x) = m$. Since $1 \notin \beta(y)$, it follows that $y \neq b_1$ and hence $y \notin \{\theta, b_1\}$. Since $\chi(x) = ((1, n), \chi(y))$ and $q(b_1^n) = n$, it follows from Lemma II.1(i) that $q(y) = m - n$. Lemma II.3(i) now implies that $y \in E_{m-n}$. Since $y \notin \{\theta,\ b_1\}$, it follows that $q(y) \geq 2$ i.e. $m - n \geq 2$ and hence $n \leq m - 2$. Since $y \in E_{m-n}$ and $1 \notin \beta(y)$, it follows that $y \in M_{m-n}$.

The other direction of the equivalence follows simply from the fact that if x satisfies (a) (b) or (c), then $q(x) = m$ and hence Lemma II.3(i) implies that $x \in E_m$. ∎

Let us define the following relation "$\sim$" on E; for $x,\ y \in E$,

$$x \sim y \text{ if } \chi(x) = \chi(y).$$

"$\sim$" is an equivalence relation on E. For $x \in E$, let $[x]$ denote its equivalence class relative to "$\sim$" and $E \backslash \sim$ be the quotient set of E relative to "$\sim$", then $E = \cup\{[x] : [x] \in E \backslash \sim\}$.

If x is such that $\chi(x) = ((i_1, m_x(i_1)), \ldots, (i_k, m_x(i_k)))$, define

$$\overline{x} := b_{i_1}^{m_x(i_1)} b_{i_2}^{m_x(i_2)} \cdots b_{i_k}^{m_x(i_k)}.$$

We will chose $\overline{x}$ to represent $[x]$. θ shall represent its class, i.e. $\overline{\theta} = \theta$.

We note that if $x \in E_k,\ k \geq 1$, then since $\theta x = x\theta = x$ for all $x \in \mathbb{R}$, $\overline{x} = b_1^{m_x(1)} \overline{y}$, where $y \in M_{k - m_x(1)}$, $m_x(1) \geq 0$ and $b_1^0 = \theta$.

Let $\overline{M}_k := \begin{cases} \emptyset & : \quad k < 0 \\ \{\overline{x} : x \in M_k\} & : \quad k \geq 0, \end{cases}$

then it follows that $x \in M_k \iff \overline{x} \in \overline{M}_k$ and hence by Lemma III.1, for $m \geq 1$,

$$\begin{aligned} x \in E_m \quad &\iff \quad \overline{x} \in \{b_1^m\} \cup \cup_{n=1}^{m-2} \{b_1^n y : y \in \overline{M}_{m-n}\} \cup \overline{M}_m \\ &\iff \quad \overline{x} \in \{b_1^m\} \cup \cup_{n=0}^{m-2} b_1^n \{y : y \in \overline{M}_{m-n}\} \\ &\iff \quad \overline{x} \in \{b_1^m\} \cup \cup_{n=0}^{m-2} b_1^n \overline{M}_{m-n} \\ &\iff \quad \overline{x} \in \{b_1^m\} \cup \cup_{k=2}^{m} b_1^{m-k} \overline{M}_k. \end{aligned}$$

Since $\overline{M}_1 = \{b_1\}$ and $b_1^m = b_1^{m-1} b_1$, it follows that $x \in E_m \iff \overline{x} \in \cup_{k=1}^m b_1^{m-k} \overline{M}_k$. Let

$$\Gamma_m := \cup_{k=1}^m b_1^{m-k} \overline{M}_k,\ m \geq 1,$$

then by Proposition III.1, for $t \geq 0$,

$$z(t) = e^{at} + \sum_{m=1}^{[\frac{dt}{r}]} \sum_{\{x \in E_m\}} h(t, x, p(x), q(x), d)$$

$$= e^{at} + \sum_{m=1}^{[\frac{dt}{r}]} \sum_{\{\overline{x} \in \Gamma_m\}} \sum_{\{y \in E_m : y \in [\overline{x}]\}} h(t, y, p(y), q(y), d).$$

If $y \in [\overline{x}]$, then $p(\overline{x}) = p(y)$, $q(\overline{x}) = q(y)$ and further $\overline{x}$ and y have the same numerical value.

Therefore

$$\begin{aligned} z(t) \quad &= \quad e^{at} + \sum_{m=1}^{[\frac{dt}{r}]} \sum_{\{\overline{x} \in \Gamma_m\}} (\sharp[\overline{x}]) h(t, \overline{x}, p(\overline{x}), q(\overline{x}), d) \\ &= \quad e^{at} + \sum_{m=1}^{[\frac{dt}{r}]} \sum_{\{\overline{x} \in \Gamma_m\}} \frac{h(t, \overline{x}, p(\overline{x}), q(\overline{x}), d) p(\overline{x})!}{v(\overline{x})}, \quad (15) \end{aligned}$$

where (15) follows from the following Lemma:

Lemma III.2. *Let $x \in E$, then $\sharp[x] = \dfrac{p(x)!}{v(x)}$.*

Proof: The assertion is easily verified for $x \in E_0$. We will assume henceforth that $x \in E_m,\ m \geq 1$. Let $\chi(x) = ((i_1, m_x(i_1)), \ldots, (i_k, m_x(i_k)))$. In view of Remark II.2$(i)$, to determine $\sharp[x]$, we determine the number of possible ways of dividing $m_x(i_1) + \cdots + m_x(i_k)$ distinct objects into k groups of sizes $m_x(i_1), \ldots, m_x(i_k)$. Each of these ways corresponds to $\varepsilon(y)$ for some y for which $\chi(x) = \chi(y)$, i.e. $y \in [x]$. This number is given by $\dfrac{(m_x(i_1) + \cdots + m_x(i_k))!}{m_x(i_1)! \cdots m_x(i_k)!} = \dfrac{p(x)!}{v(x)}$. ∎

Let

$$H(t, x, d) := \frac{h(t, x, p(x), q(x), d) p(x)!}{v(x)}$$

$$= x \frac{(t - \frac{q(x)r}{d})^{p(x)} e^{a(t - \frac{q(x)r}{d})}}{v(x)},$$

then it follows that

$$\begin{aligned} Z(t) \quad &= \quad e^{at} + \sum_{m=1}^{[\frac{dt}{r}]} \sum_{k=1}^{m} \sum_{\{\overline{x} \in \overline{M}_k\}} H(t, b_1^{m-k} \overline{x}, d) \\ &= \quad e^{at} + \sum_{k=1}^{[\frac{dt}{r}]} \sum_{m=k}^{[\frac{dt}{r}]} \sum_{\{\overline{x} \in \overline{M}_k\}} H(t, b_1^{m-k} \overline{x}, d). \end{aligned}$$

We have thus proven the following Theorem:

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:2, 2013

Theorem III.1. *For $d \geq 3$, let $z(t)$ be the fundamental solution of (1), then for $t \geq 0$,*

$$z(t) = e^{at} + \sum_{k=1}^{[\frac{dt}{r}]} \sum_{m=k}^{[\frac{dt}{r}]} \sum_{\{x \in \overline{M}_k\}} H(t, b_1^{m-k} x, d).$$

IV. The Equation $\acute{z}(t) = az(t) + bz(t - \frac{r}{2}) + cz(t - r)$

When $d = 2$, it follows from Lemma II.3(iii) that $M_k = \emptyset$ for odd $k \geq 2$. The sets M_{2k} which are not empty have a very simple structure. We will derive a more explicit representation of the fundamental solution in this case to use in our study of stability.

Lemma IV.1. *Let $d = 4$ and for $m \in \mathbb{N}_*$ let*

$$N_m := \{x \in E_m : \beta(x) \cap \{1, 3\} = \emptyset\},\ \overline{N}_m := \{\overline{x} : x \in N_m\}$$

and

$$\Lambda_m := \{b_2^{m-2j} b_4^j : j = 0, \ldots, [\frac{m}{2}]\}. \tag{16}$$

(i) *If $m \in \mathbb{N}_*$ is odd then $N_m = \emptyset$.*
(ii) *For all $m \in \mathbb{N}_*$, it holds that*

$$\overline{N}_{2m} = \Lambda_m. \tag{17}$$

Proof: (i) The assertion is clear when $m \in \{0, 1\}$. We will now show that if $m \geq 2$, $b_1 = b_3 = 0$, $b_2 \neq 0$, $b_4 \neq 0$ and there exists $x \in E_m$ which has a numerical value different from 0, i.e., $\beta(x) \cap \{1, 3\} = \emptyset$, then m is even. Since $x \in E_m$, $m \geq 2$, $b_1 = b_3 = 0$ and x has a numerical value different from 0, $\overline{x} = b_2^{\alpha_2} b_4^{\alpha_4}$. Therefore, $q(x) = m = 2\alpha_2 + 4\alpha_4 = 2(\alpha_2 + 2\alpha_4)$ and hence m is even.

(ii) We will prove (17) for $m = 2k+1$, $k \in \mathbb{N}_*$, the proof for $m = 2k$ being similar. Let $m = 2k+1$, with $k \in \mathbb{N}_*$. It is easy to see that $\Lambda_m \subseteq \overline{N}_{2m}$. On the other hand, let $x \in \overline{N}_{2m}$, then $x = b_2^{\alpha} b_4^j$, with $j \geq 0$, $\alpha \geq 0$ and $q(x) = 2m = 2\alpha + 4j$.

Therefore $m - 2j = \alpha$, $j \geq 0$, $\alpha \geq 0$. This is equivalent to saying that $x = b_2^{m-2j} b_4^j$, where $0 \leq j \leq \frac{m}{2}$. Since m is odd, $0 \leq j \leq [\frac{m}{2}]$, showing that $x \in \Lambda_m$. ∎

We can now prove the following theorem:

Theorem IV.1. *Let $z(t)$ be the fundamental solution of $\acute{z}(t) = az(t) + bz(t - \frac{r}{2}) + cz(t - r)$, then for $t \geq 0$,*

$$z(t) = \sum_{k=0}^{[\frac{t}{r}]} \sum_{m=0}^{[\frac{2t}{r}]-2k} \frac{b^m c^k}{m! k!} (t - \frac{(m+2k)r}{2})^{m+k} \times \exp\{a(t - \frac{(m+2k)r}{2})\}. \tag{18}$$

Proof: Let z be the fundamental solution of the equation

$$\acute{z}(t) = az(t) + b_1 z(t - \frac{r}{4}) + b_2 z(t - \frac{2r}{4}) + b_3 z(t - \frac{3r}{4}) + b_4 z(t - \frac{4r}{4}). \tag{19}$$

We shall set $b_1 = b_3 = 0$, i.e., the solution we obtain is the fundamental solution of the equation, $\acute{z}(t) = az(t) + b_2 z(t - \frac{r}{2}) + b_4 z(t - r)$, and then we show that for $t \geq 0$, $z(t)$ is given by $z(t) = \sum_{k=0}^{[\frac{t}{r}]} \sum_{m=0}^{[\frac{2t}{r}]-2k} H(t, b_2^m b_4^k, 4)$. By Theorem III.1, the fundamental solution of (19) is given for $t \geq 0$ by

$$z(t) = e^{at} + \sum_{m=1}^{[\frac{4t}{r}]} \sum_{k=m}^{[\frac{4t}{r}]} \sum_{\{x \in \overline{M}_m\}} H(t, b_1^{k-m} x, 4)$$

$$= e^{at} + \sum_{m=1}^{[\frac{4t}{r}]} \sum_{\{x \in \overline{M}_m\}} H(t, x, 4) \tag{20}$$

$$= \sum_{m=0}^{[\frac{4t}{r}]} \sum_{\{x \in \overline{M}_m\}} H(t, x, 4)$$

$$= \sum_{m=0}^{[\frac{1}{2}[\frac{4t}{r}]]} \sum_{\{x \in \overline{M}_{2m}\}} H(t, x, 4) \tag{21}$$

$$= \sum_{m=0}^{[\frac{1}{2}[\frac{4t}{r}]]} \sum_{\{x \in \overline{N}_{2m}\}} H(t, x, 4). \tag{22}$$

(20) holds because $b_1 = 0$, while (21) holds because of Lemma IV.1(i) and (22) holds because $b_3 = 0$. By lemma IV.1 (ii),

$$\begin{aligned} z(t) &= \sum_{m=0}^{[\frac{1}{2}[\frac{4t}{r}]]} \sum_{k=0}^{[\frac{m}{2}]} H(t, b_2^{m-2k} b_4^k, 4) \\ &= \sum_{k=0}^{[\frac{1}{2}[\frac{1}{2}[\frac{4t}{r}]]]} \sum_{m=2k}^{[\frac{1}{2}[\frac{4t}{r}]]} H(t, b_2^{m-2k} b_4^k, 4) \\ &= \sum_{k=0}^{[\frac{t}{r}]} \sum_{m=2k}^{[\frac{2t}{r}]} H(t, b_2^{m-2k} b_4^k, 4) \\ &= \sum_{k=0}^{[\frac{t}{r}]} \sum_{m=0}^{[\frac{2t}{r}]-2k} H(t, b_2^m b_4^k, 4). \end{aligned}$$

∎

Remark IV.1. Note that the expression obtained when $c = 0$ in (18) is $y(t) := \sum_{m=0}^{[\frac{2t}{r}]} \frac{b^m}{m!} (t - \frac{mr}{2})^m e^{a(t - \frac{mr}{2})}$, which by Corollary III.1 is the fundamental solution of $\acute{y}(t) = ay(t) + by(t - \frac{r}{2})$ on $[0\ \infty)$ and the expression obtained when $b = 0$ in that formula is $w(t) := \sum_{k=0}^{[\frac{t}{r}]} \frac{c^k}{k!} (t - kr)^k e^{a(t - kr)}$, which is the fundamental solution of $\acute{w}(t) = aw(t) + cw(t - r)$ on $[0\ \infty)$. This is what we would expect, but it suggests more: it suggests that some properties of solutions of equations with multiple delays may be deduced from properties of solutions of equations with single delays. This is indeed the case as seen for example in what follows.

Lemma IV.2. *For $t \geq 0$, consider the equation*

$$\acute{z}(t) = az(t) + bz(t - \frac{r}{2}) + cz(t - r). \tag{23}$$

Let G denote its fundamental solution and x^* be the fundamental solution of the equation

$$\acute{x}(t) = \frac{a}{2}x(t) + (|b|e^{-\frac{ar}{4}})x(t-\frac{r}{2}), \quad (24)$$

then

$$|G(t)| \le \sum_{m=0}^{[\frac{t}{r}]} \frac{|c|^m}{m!}(t-mr)^m x^*(t-mr)e^{\frac{a}{2}(t-mr)}. \quad (25)$$

Proof: Since $[\frac{2t}{r}] - 2m = [\frac{2(t-mr)}{r}]$, it follows from theorem IV.1 that for $t \ge 0$,

$$|G(t)| = \left| \sum_{m=0}^{[\frac{t}{r}]} \sum_{k=0}^{[\frac{2(t-mr)}{r}]} \frac{b^k c^m}{k!m!}(t - \frac{(k+2m)r}{2})^{k+m} e^{a(t-\frac{(k+2m)r}{2})} \right|$$

$$= |\sum_{m=0}^{[\frac{t}{r}]} \frac{c^m}{m!}e^{\frac{a}{2}(t-mr)} \sum_{k=0}^{[\frac{2(t-mr)}{r}]} \frac{b^k(e^{-\frac{ar}{4}})^k}{k!} \times (t-mr-\frac{kr}{2})^{k+m}e^{\frac{a}{2}(t-mr-\frac{kr}{2})}|$$

$$\le \sum_{m=0}^{[\frac{t}{r}]} \frac{|c|^m}{m!}(t-mr)^m e^{\frac{a}{2}(t-mr)} \sum_{k=0}^{[\frac{2(t-mr)}{r}]} \frac{(|b|e^{-\frac{ar}{4}})^k}{k!} \times (t-mr-\frac{kr}{2})^k e^{\frac{a}{2}(t-mr-\frac{kr}{2})}.$$

By Corollary III.1, it follows that for $t \ge 0$,

$$|G(t)| \le \sum_{m=0}^{[\frac{t}{r}]} \frac{|c|^m}{m!}e^{\frac{a}{2}(t-mr)}(t-mr)^m x^*(t-mr). \quad \blacksquare$$

Using the notation of Lemma IV.2 we have the following corollary:

Corollary IV.1. (i) *There exists $\lambda_0 \in \mathbb{R}$ for which the following holds: For all $\lambda > \lambda_0$, there exists a constant $M(\lambda) > 0$ such that $|G(t)| \le M(\lambda)w_\lambda(t),\ t \ge 0$, where w_λ is the fundamental solution of $\acute{w}(t) = (\frac{a}{2}+\lambda)w(t) + |c|w(t-r)$.*

(ii) *Let λ_0 be as in (i). For each $\lambda > \lambda_0$, there exists $\mu_0(\lambda) \in \mathbb{R}$ such that for all $\mu_\lambda > \mu_0(\lambda)$ we have $|G(t)| \le M(\lambda,\mu_\lambda)e^{\mu_\lambda t}$ for some constant $M(\lambda,\mu_\lambda)$.*

Proof: (i) Let h denote the characteristic function of (24). By [1], Theorem 5.2, if $\lambda_0 := \max\{Re\ \lambda : h(\lambda) = 0\}$, then for all $\lambda > \lambda_0$, there exists a constant $M(\lambda)$ such that $x^*(t) \le M(\lambda)e^{\lambda t},\ t \ge 0$. From this and Lemma IV.2, it follows that for $\lambda > \lambda_0$,

$$|G(t)| \le M(\lambda)\sum_{m=0}^{[\frac{t}{r}]} \frac{|c|^m}{m!}(t-mr)^m e^{(\frac{a}{2}+\lambda)(t-mr)} \quad (26)$$

$$= M(\lambda)w_\lambda(t). \quad (27)$$

(ii) Let λ_0 be chosen such that (i) is satisfied and $\lambda > \lambda_0$. Again by [1], Theorem 5.2, there exists $\mu_0(\lambda)$ such that if $\mu_\lambda > \mu_0(\lambda)$, then $w_\lambda(t) \le M(\mu_\lambda)e^{\mu_\lambda t}$ for some constant $M(\mu_\lambda)$. We can now set $M(\lambda,\mu_\lambda) := M(\lambda)M(\mu_\lambda)$. $\blacksquare$

Corollary IV.2. *Let $\varphi : [-r\ 0] \to \mathbb{R}$ be bounded and integrable and consider the equation (23) with the initial condition $z(t) = \varphi(t),\ t \in [-r\ 0]$. Let z^φ denote its solution.*

(i) *There exists a constant $\lambda_0 \in \mathbb{R}$ for which the following holds: If $\lambda > \lambda_0$, then there exists $\mu_0(\lambda)$ such that $\limsup_{t\to\infty} \frac{1}{t}\ln|z^\varphi(t)| \le \mu_\lambda$ and $\limsup_{t\to\infty} \frac{1}{t}\ln|\acute{z^\varphi}(t)| \le \mu_\lambda$ for all $\mu_\lambda > \mu_0(\lambda)$.*

(ii) *Let S be the region of the plane $\mathbb{R}^2$ defined by $S := \{(u,v) \in \mathbb{R}^2 : u < 1, u+v < 0, -v < \gamma\sin\gamma + u\cos\gamma\}$, where $\gamma = \gamma(u)$ is the root of*

$$\gamma = \begin{cases} u\tan\gamma & : \quad 0 < \gamma < \pi,\ u \ne 0 \\ \frac{\pi}{2} & : \quad u = 0. \end{cases}$$

If $(\frac{ar}{4},\ |br|e^{\frac{-ar}{4}}) \in S$ and $(\frac{ar}{2},\ |c|r) \in S$, then $\limsup_{t\to\infty} \frac{1}{t}\ln|z^\varphi(t)| < 0$ and $\limsup_{t\to\infty} \frac{1}{t}\ln|\acute{z^\varphi}(t)| < 0$.

Proof: From Corollary IV.1(ii), there exists a constant $\lambda_0 \in \mathbb{R}$, such that for all $\lambda > \lambda_0$, there exists $\mu_0(\lambda)$ such that if $\mu_\lambda > \mu_0(\lambda)$, then $|G(t)| \le M(\lambda,\mu_\lambda)e^{\mu_\lambda t},\ t \ge 0$ for some constant $M(\lambda,\mu_\lambda) > 0$. For $x \in [-r\ 0]$, let $f(t,x,\varphi) := \int_{-x}^{0} |G(t-s-x)||\varphi(s)|ds$, then from (10), it holds that for $t \ge 0$,

$$|z^\varphi(t)| \le |G(t)||\varphi(0)| + |b|f(t,\frac{r}{2},\varphi) + |c|f(t,r,\varphi)$$
$$\le |G(t)||\varphi(0)| + |b|\ \|\varphi\|\ f(t,\frac{r}{2},1) + |c|\ \|\varphi\|\ f(t,r,1).$$

Let $\lambda > \lambda_0$ and $\mu_\lambda > \mu_0(\lambda)$, then

$$\begin{aligned} f(t,x,1) &= \int_{-x}^{0} |G(t-s-x)|ds \\ &\le M(\lambda,\mu_\lambda)\int_{-x}^{0} e^{\mu_\lambda(t-s-x)}ds \\ &\le M(\lambda,\mu_\lambda)e^{\mu_\lambda t}\int_{-r}^{0} e^{-\mu_\lambda(s+x)}ds \le K(\lambda,x)e^{\mu_\lambda t}, \end{aligned}$$

where $K(\lambda,x) := M(\lambda,\mu_\lambda)\int_{-r}^{0} e^{-\mu_\lambda(s+x)}ds$. Let $K(\lambda) := \max\{K(\lambda,\frac{r}{2}),K(\lambda,r),1\}$, then $f(t,x,1) \le K(\lambda)e^{\mu_\lambda t}$.

If $C(\lambda,\mu_\lambda) := K(\lambda)\ \|\varphi\|\ (M(\lambda,\mu_\lambda)+|b|+|c|)$, then $|z^\varphi(t)| \le C(\lambda,\mu_\lambda)e^{\mu_\lambda t}$ and hence $\limsup_{t\to\infty} \frac{1}{t}|z^\varphi(t)| \le \mu_\lambda$.

The second assertion simply follows from the observation that $|\acute{z^\varphi}(t)| \le |a||z^\varphi(t)| + |b||z^\varphi(t-\frac{r}{2})| + |c||z^\varphi(t-r)|$ and using what we have just shown.

To complete the proof, we will show that if $(\frac{ar}{4}, |br|e^{\frac{-ar}{4}}) \in S$ and $(\frac{ar}{2}, |c|r) \in S$, then we can choose μ_λ such that $\mu_\lambda < 0$. By [2] Proposition 2.7, if $(\frac{ar}{4}, \frac{|br|}{2}e^{\frac{-ar}{4}}) \in S$, then $\lambda_0 < 0$ and hence we can choose λ such that $\lambda < 0$. By (26), it follows that for this λ, $w_\lambda(t) \le \sum_{m=0}^{[\frac{t}{r}]} \frac{|c|^m}{m!}(t-mr)^m e^{\frac{a}{2}(t-mr)}$. If $(\frac{ar}{2}, |c|r) \in S$, then applying [2] Proposition 2.7 again, we can choose $\mu_\lambda < 0$ and $M(\lambda,\mu_\lambda) \ge 0$ such that $w_\lambda(t) \le \left|\sum_{m=0}^{[\frac{t}{r}]} \frac{|c|^m}{m!}(t-mr)^m e^{\frac{a}{2}(t-mr)}\right| \le M(\lambda,\mu_\lambda)e^{\mu_\lambda t}$. $\blacksquare$

References

[1] J. K. Hale and S. M. V. Lunel, *Theory of Functional Differential Equations*, Berlin: Springer, 1993.
[2] U. Küchler and B. Mensch, "Langevins stochastic differential equation extended by a time delayed term," Stochastics and stochastic reports, vol. 40, 1992, pp. 23–42.
[3] F. C. Shu, "Explicit solutions of irreducible linear systems of delay differential equations of dimension 2," Applied Mathematics E-Notes, vol. 11, 2011, pp. 261–273.
[4] F. C. Shu, "Stability of a 2 dimensional irreducible linear system of delay differential equations," Applied Mathematics E-Notes, vol. 12, 2012, pp. 36–43.