Numerical solution of Hammerstein integral equations by using quasi-interpolation

M. Zarebnia and S. Khani

Abstract—In this paper first, a numerical method based on quasi-interpolation for solving nonlinear Fredholm integral equations of the Hammerstein-type is presented. Then, we approximate the solution of Hammerstein integral equations by Nystrom's method. Also, we compare the methods with some numerical examples.

Keywords—Hammerstein integral equations, quasi-interpolation, Nystrom's method.

I. INTRODUCTION

THE problem of finding numerical solution for Fredholm integral equations of the second kinds is one of the oldest problems in the applied mathematics and many computational methods are introduced in this filed [2], [3], [4]. Hammerstein integral equations are defined as follows:

$$y(x)=f(x)+\lambda\int_{-1}^{1}k(x,t)G(y(t))dt, \tag{1}$$

where, the function $f(x)$ is known, $k(x,t)$ is the kernel function which is known, continuous and $G(y(t))$ is known nonlinear function, the aim is to find the unknown function $y(x)$ which is solution of equation (1). Previously, some kinds of Fredholm integral equations had been solved numerically, by different methods that are indicated below. Borzabadi et al. [1], introduseed a numerical method for a class of nonlinear Fredholm integral equations of the second kind. In [5], Javidi et al. solved nonlinear Fredholm integral equations by using modified homotopy perurbation method.

The method of quasi-interpolation was 1991 introduced by Maz'ya [7] and became popular under the name approximate approximations. In the following years many applications of this method where presented by Maz'ya and Schmidt which are collected in the tex book [9].

We know that quasi-interpolations are defined by

$$\mu_{h,D}\,u(x)=\sum_{m=-\infty}^{\infty}\frac{u(mh)e^{-\frac{(x-mh)^2}{Dh^2}}}{\sqrt{\pi D}}, \tag{2}$$

where the function u is twice continuously differentiable with bounded derivatives [8], [10]. The Taylor expansion of u at the point mh has the form

$$u(mh)=u(x)+u^{'}(x)(mh-x)+u^{''}(x_m)\frac{(mh-x)^2}{2},$$

for some x_m between x and mh. We apply this method to solve the equation (1) and reduce it to system of equation.

The outline of the paper is as follows. First, in Section 2 we review some of the main properties of quasi function and quasi-interpolation that are necessary for the formulation of the discrete system. In Section 3, we illustrate how the quasi-interpolation and Nystrom's method may be used to replace Eq. (1) by an explicit system of nonlinear algebraic equations, which is solved by Newtons method. In Section 4, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical schemes by considering some numerical examples.

II. APPROXIMATION OF INTEGRAL OPERATORS

Let us consider the integral operator

$$T:C([-1,1])\to C([-1,1]),\quad (TGu)(s):=\int_{-1}^{1}k(s,t)Gu(t)\,dt, \tag{3}$$

where $k:[-1,1]^2\to\mathbf{R}$ is continuous kernel.

We approximate $TG(u(s))$ with the trapezoidal rule in point mh. Let $N\in\mathbf{N}$ and $h=\frac{1}{N}$. We obtain

$$\begin{aligned}TGu(s) &= \int_{-1}^{1}k(s,t)G(u(t))\,dt\\ &\approx \sum_{m=-N}^{N}\frac{h}{2}k(s,mh)G(u(mh))(2-\delta_{|m|N}),\end{aligned}$$

where

$$\delta_{ij}=\begin{cases}1 & i=j,\\ 0 & i\neq j.\end{cases}$$

We suppose the operator $T_N:C([-1,1])\to C([-1,1])$ as follow:

$$T_NGu(s):=\sum_{m=-N}^{N}\frac{h}{2}k(s,mh)G\Big(u(mh)\Big)(2-\delta_{|m|N}). \tag{4}$$

We obtain

$$TGu(s)\approx T_NGu(s).$$

definition 1. The error function is defined by

$$erf(a,b):=\frac{2}{\sqrt{\pi}}\int_a^b e^{-t^2}\,dt,$$

where $a,b\in\mathbf{R}$ and $a\leq b$.

The quasi-interpolation for $d>0$ is defined as follows:

$$Gu_{d,N}:[-1,1]\to\mathbf{R},\quad Gu_{d,N}:=\sum_{m=-N}^{N}Gu(mh)\frac{e^{-\frac{(t-mh)^2}{dh^2}}}{\sqrt{\pi d}}.$$

M. Zarebnia is with the Department of Mathematics, University of Mohaghegh Ardabili, P. 0. Box 179, Ardabil, Iran, Tel: (+98) 451-5520457; Fax: (+98) 451-5520458; e-mail: zarebnia@uma.ac.ir.

S. Khani is with the Department of Mathematics, University of Mohaghegh Ardabili, P. 0. Box 179, Ardabil, Iran, email: s.khani40@yahoo.com.

By replacing Gu with the quasi-interpolation, $TGu(s)$ is obtained as:

$$\begin{aligned} TGu(s) &= \int_{-1}^{1} k(s,t)Gu(t)\,dt \approx \int_{-1}^{1} k(s,t)Gu_{d,N}(t)\,dt \\ &= \sum_{m=-N}^{N} Gu(mh)\int_{-1}^{1} k(s,t)\frac{e^{-\frac{(t-mh)^2}{dh^2}}}{\sqrt{\pi d}}\,dt. \end{aligned}$$

We obtain estimate of the nonlinear integral operator by replacing $k(s,t)$ with $k(s,mh)$ as follows:

$$\begin{aligned} TGu(s) &\approx \sum_{m=-N}^{N} Gu(mh)\int_{-1}^{1} k(s,t)\frac{e^{-\frac{(t-m)^2}{dh^2}}}{\sqrt{\pi d}}\,dt \\ &\approx \sum_{m=-N}^{N} k(s,mh)Gu(mh)\int_{-1}^{1} \frac{e^{-\frac{(t-m)^2}{dh^2}}}{\sqrt{\pi d}}\,dt \\ &= \sum_{m=-N}^{N} k(s,mh)Gu(mh)\frac{h}{\sqrt{\pi}}\int_{\frac{m-N}{\sqrt{\pi}}}^{\frac{m+N}{\sqrt{\pi}}} e^{-t^2}\,dt \\ &= \sum_{m=-N}^{N} k(s,mh)Gu(mh)erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right). \end{aligned}$$

We define the operator

$$T_{d,N} : C([-1,1]) \to C([-1,1])$$

by

$$(T_{d,N}Gu)(s) := \sum_{m=-N}^{N} \frac{h}{2}k(s,mh)Gu(mh)erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right), \tag{5}$$

we have

$$(TGu)(s) \approx (T_{d,N}Gu)(s).$$

Lemma 1. It holds:

$$\lim_{d\to 0} erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right) = 2-\delta_{|m|N}.$$

proof. We have

$$\lim_{d\to 0} erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right) = \lim_{d\to 0}\frac{2}{\sqrt{\pi}}\int_{\frac{m-N}{\sqrt{d}}}^{\frac{m+N}{\sqrt{d}}} e^{-t^2}\,dt$$

and

$$\frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t^2}\,dt = 1,$$

then we obtain

$$\lim_{d\to 0} erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right) = \begin{cases} 1, & m=-N, \\ 2, & |m|\neq N, \\ 1. & m=N. \end{cases}$$

therefore

$$\lim_{d\to 0} erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right) = 2-\delta_{|m|N}.$$

Lemma 2. It holds:

$$\lim_{d\to 0} T_{d,N} = T_N$$

. **proof.** Let $Gu \in C([-1,1])$
and
$M := ||k||_\infty = \sup_{(s,t)\in[-1,1]^2}|k(s,t)|$.
We obtain

$$\begin{aligned} &||T_{d,N}Gu - T_NGu||_\infty = \sup_{s\in[-1,1]}|(T_{d,N}Gu)(s) - (T_NGu)(s)| \\ &\approx \sum_{m=-N}^{N}\frac{h}{2}M||Gu||_\infty|erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right) - (2-\delta_{|m|N})|, \end{aligned}$$

therefore from the above relation and lemma 1, we conclude that

$$\begin{aligned} ||T_{d,N}-T_N|| &= \sup_{||Gu||_\infty=1}||T_{d,N}Gu - T_NGu||_\infty \\ \le \sum_{m=-N}^{N}\frac{h}{2}M||Gu||_\infty|erf&\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right) - (2-\delta_{|m|N})| = 0. \end{aligned}$$

III. APPLICATION TO HAMMERSTEIN INTEGRAL EQUATIONS

Consider the Hammerstein integral equations

$$X - TGX = b, \tag{6}$$

where $b \in C[-1,1]$ and T is defined in (3).

A. Approximation with quasi-interpolation

We estimate solution of the nonlinear integral equation (6) using quasi-interpolation. By substituting T with T_N from (5), we have

$$X_{d,N} - T_{d,N}GX_{d,N} = b, \tag{7}$$

similarly we obtain the nonlinear system in the point jh as following:

$$u_d(jh) - \sum_{m=-N}^{N}\frac{h}{2}k(jh,mh)erf\left(\frac{m-N}{\sqrt{d}},\frac{m+N}{\sqrt{d}}\right)Gu_{d,m} = b(jh), \tag{8}$$

where $u_{d,m} = X_{d,N}(mh)$ are then approximate values for $u(mh)$.

B. Approximation with Nystrom's method

In this section we obtain approximate of integral equation by Nystrom's method [6].
Let u be the solution of (6) and $s \in [-1,1]$, we 'll have

$$u(s) - (TGu)(s) = b(s).$$

If we employ the trapezoidal rule for the quadrature procedure and approximate equation

$$X_N - T_NGX_N = b, \tag{9}$$

with T_N is defined in (4), we can obtain the nonlinear system of equation in the points jh.

$$u(jh) - \sum_{m=-N}^{N} \frac{h}{2} k(jh, mh) G u_m (2 - \delta_{|m|N}) = b(jh), \quad (10)$$

that $j = \{-N, ..., N\}$ and the values $u_m = X_N(mh)$ are then approximation for $u(mh)$.

IV. Numerical examples

In this section, we use the above proposed methods in examples with detailed explanations. we compare the results of numerical solution this method with the solution of the Nystrom's method.

Example 1 Consider the following nonlinear Fredholm integral equation:

$$u(s) = e^s - \frac{2(2+e^6)s}{9e^3} + \int_{-1}^{1} stu^3(t)\, dt,$$

with exact solution $u(s) = e^s$. Table I shows the solution u_d of the nonlinear system (8) with $d = 0.1$ and $d = 0.01$ and the solution u of the nonlinear system (10) for $N = 4$.

TABLE I
COMPARISON OF NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH $N = 4$, $d = 0.1$ AND $d = 0.01$.

u	$u_{0.1}$	$u_{0.01}$
0.4628229775674136	0.4628215310881807	0.4628215310881807
0.5435742050379936	0.5435731201785685	0.5435731201785685
0.6540024279106195	0.654001704671003	0.654001704671003
0.8025366671703977	0.8025363055505895	0.8025363055505895
1.0000000000000000	1.0000000000000000	1.0000000000000000
1.2602895325887487	1.2602898942085567	1.2602898942085567
1.6012495025021423	1.6012502257417587	1.6012502257417587
2.0457923643156954	2.0457934491751204	2.0457934491751204
2.6233382920630737	2.6233397385423065	2.6233397385423065

Example 2 As the second example consider the following nonlinear integral equation:

$$u(s) = s^2 - \frac{4}{15}s - 1 + \int_{-1}^{1} \frac{1}{4} s(t-1) u^2(t)\, dt,$$

with exact solution $u(s) = s^2 - 1$. The Table II illustrate the numerical results for $N = 4$ the solution u_d of the nonlinear system (8) with $d = 0.1$ and $d = 0.01$ and the solution u of the nonlinear system (10).

TABLE II
COMPARISON OF NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH $N = 4$, $d = 0.1$ AND $d = 0.01$.

u	$u_{0.1}$	$u_{0.01}$
0.515512338823494	0.515512294219369	0.515512294219369
-0.0508657458823796	-0.050865779335473	-0.050865779335473
-0.492243830588253	-0.492243852890315	-0.492243852890315
-0.808621915294126	-0.808621926445158	-0.808621926445158
-1.000000000000000	-1.000000000000000	-1.000000000000000
-1.066378084705874	-1.066378073554842	-1.066378073554842
-1.00775616941175	-1.007756147109684	-1.007756147109684
-0.824134254117620	-0.824134220664527	-0.824134220664527
-0.515512338823494	-0.515512294219369	-0.515512294219369

Example 3 Consider the nonlinear Fredholm integral equation

$$u(s) = \sin 2\pi s + \int_{-1}^{1} t \sin(2\pi s) u^2(t)\, dt,$$

with exact solution $u(s) = \sin 2\pi s$. The following table shows for $N = 4$ the solution of the nonlinear system (8) with $d = 0.1$ and $d = 0.01$ and the solution of the nonlinear system (10). The obtained solutions of Nystrom's and quasi-interpolation methods are exact for this example.

TABLE III
ABSOLUTE ERRORS ON THE NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH $N = 4$, $d = 0.1$ AND $d = 0.01$.

$\|u_{ex} - u\|$	$\|u_{ex} - u_{0.1}\|$	$\|u_{ex} - u_{0.01}\|$
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000
0.0000000000000	0.0000000000000	0.0000000000000

Example 4 Consider the Hammerstein integral equation

$$u(s) = s^2 - \frac{56}{15}s + 1 + \int_{-1}^{1} (s-t) u^2(t)\, dt.$$

In this example the exact solution of the nonlinear integral equation is $u(s) = s^2 + 1$. Errors results for the nonlinear system (8) and the nonlinear system (10) with $N = 7$, $d = 0.1$ and $d = 0.01$ are given in Table IV.

TABLE IV
ABSOLUTE ERRORS ON THE NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH $N = 7$, $d = 0.1$ AND $d = 0.01$.

$\|u_{ex} - U\|$	$\|u_{ex} - u_{0.1}\|$	$\|u_{ex} - u_{0.01}\|$
0.006850882568	0.006850048150	0.006850048150
0.006541928759	0.006541131773	0.006541131773
0.006232974951	0.006232215395	0.006232215395
0.005924021143	0.005923299017	0.005923299017
0.005615067335	0.005614382640	0.005614382640
0.005306113526	0.005305466262	0.005305466262
0.004997159718	0.004996549884	0.004996549884
0.004688205910	0.004687633507	0.004687633507
0.004379252102	0.004378717129	0.004378717129
0.004070298293	0.004069800751	0.004069800751
0.003761344485	0.003760884374	0.003760884374
0.003451967996	0.003451967996	0.003451967996
0.003143051619	0.003143051619	0.003143051619
0.002834483060	0.002834135241	0.002834135241
0.002525529252	0.002525218863	0.002525218863

Example 5 Consider the nonlinear Fredholm integral equation

$$u(s) = s - \frac{\pi}{4} + \frac{1}{2}\int_{-1}^{1} \frac{1}{1+u^2(t)}\,dt$$

with exact solution $u(s) = s$. The following table shows the errors nonlinear systems (8) and (10) for $N = 4$, $d = 0.1$ and $d = 0.01$.

TABLE V
ABSOLUTE ERRORS ON THE NYSTRROM'S MTHOD AND QUASI-INTERPOLATION METHOD WITH $N = 4$, $d = 0.1$ AND $d = 0.01$.

$\|u_{ex} - U\|$	$\|u_{ex} - u_{0.1}\|$	$\|u_{ex} - u_{0.01}\|$
0.00260574350092058	0.00260636384788038	0.00260636384788038
0.00260574350092069	0.00260636384788027	0.00260636384788027
0.00260574350092069	0.00260636384788027	0.00260636384788027
0.00260574350092063	0.00260636384788027	0.00260636384788027
0.00260574350092065	0.00260636384788027	0.00260636384788027
0.00260574350092066	0.00260636384788027	0.00260636384788027
0.00260574350092069	0.00260636384788032	0.00260636384788032
0.00260574350092069	0.00260636384788027	0.00260636384788027
0.00260574350092069	0.00260636384788027	0.00260636384788027

V. CONCLUSION

In this paper, we approximate solution of Hamnmerstein integral equation (1) by using quasi-interpolation. We show that the approximation of the nonlinear integral equation, gained with this method, lead to the same numerical results as Nystrom's method with the trapezoidal rule. Also approximate solutions of quasi-interpolation method and Nystrom's method are convergence to exact solution.

REFERENCES

[1] Akbar H.Borzabadi and Omid S. Fard,A Numerical scheme for a class of nonlinear Fredholm integral equation of the second kind, Journal of Computational and Applied Mathematics $232(2009)449-454$.

[2] K.E. Atkinson, The numerical solution of integral equation of the second kind, Cambridge University press, 1997.

[3] L.M. Delves and J.L. Mohamed, Computational methods for integral equation, Cambridge University press, 1985.

[4] L.M. Delves and J. Wash, Numerical solution of integral equation, Oxford University press, 1974.

[5] M. Javidi and A. Galbabai, Modified homotopy perturbation method for solving non-linear Fredholm integral equations, Chaos, Solitons and Fractals. $40(2009)1408-1412$.

[6] R. Kress, Linear Integral Equations, 2nd ed., Springer, Berlin, 1999.

[7] V. Maz'ya, A new approximation method and its applications to the calculation of volume potentials, boundary point method. in: 3. DFG-Kolloquium des DFG-Forschungsschwerpunktes Randelementmethoden, 1991.

[8] V. Maz'ya, G. Schmidt, On approximate approximations using Gaussian Kernels, IMA J. Num. Anal. $16(1996)13-29$.

[9] V. Maz'ya, G. Schmidt, Approximate Approximations, Mathematical Surveys and Monographs, vol. 141, AMS, 2007.

[10] F. Muller and W. Varhorn, On approximation and numerical solution of Fredholm interal equation of second kind using quasi-interpolation, Applied Mathematics and Computation $217(2011)6409-6416$.