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Large Vibration Amplitude of Circular Functionally Graded Plates Resting on Pasternak Foundations

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Abstract**—**In the present study, the problem of geometrically nonlinear free vibrations of functionally graded circular plates (FGCP) resting on Pasternak elastic foundation with immovable ends was studied. The material properties of the functionally graded composites examined were assumed to be graded in the thickness direction and estimated through the rule of mixture. The theoretical model is based on the classical Plate theory and the Von Kármán geometrical nonlinearity assumptions. Hamilton's principle is applied and a multimode approach is derived to calculate the fundamental nonlinear frequency parameters, which are found to be in a good agreement with the published results dealing with the problem of functionally graded plates. On the other hand, the influence of the foundation parameters on the nonlinear frequency to the linear frequency ratio of the FGCP has been studied. The effect of the linear and shearing foundations is to decrease the frequency ratio, where it increases with the effect of the nonlinear foundation stiffness.

Keywords**—**Non-linear vibrations, Circular plates, Pasternak foundation, functionally graded materials.

I. Introduction

In recent years, functionally graded materials (FGMs) have gained much popularity as materials to be used in structural components exposed to extremely high-temperature environments such as nuclear reactors and high-speed spacecraft industries. FGMs are composite materials that are microscopically inhomogeneous, and their mechanical properties vary smoothly or continuously from one surface to the other. Typically, these materials are made from a mixture of ceramic and metal, or a combination of different materials. The concept of FGMs was first introduced in Japan in 1984 [1], [2]. Since then, it has gained considerable attention. FGMs have various available or potential applications in many fields such as aerospace engineering, electrical engineering, biomedical engineering, and architecture engineering [3], [4]. Thin plate structures are commonly used in these engineering applications, and they are often subjected to severe dynamic loading, which may result in large vibration amplitudes. When the amplitude of vibration is of the same order of the plate thickness, a significant geometrical nonlinearity is induced and linear models are not sufficient to predict the dynamic behavior of the plate which may exhibit many new features, such as the amplitude dependence of the frequency and mode shapes on the amplitude of vibration and the jump phenomenon.

Geometrically nonlinear vibration of plates has long been a subject receiving numerous research efforts. Numerous studies have been reported in open literature, such as those of

Haterbouch and Benamar [5] presented a more complete study for the effects of large vibration amplitudes on the axisymmetric mode shapes and natural frequencies of clamped thin isotropic circular plates. Allahverdizadeh et al. [6] who investigated the nonlinear free and forced vibration of thin circular functionally graded plates by using assumed-time-mode method and Kantorovich time averaging technique. After that, Zhou et al. [7] analyzed the Natural vibration of circular and annular thin plates by Hamiltonian approach, and the nonlinear theories of axisymmetric bending of functionally graded circular plates with modified couple stress are developed by Reddy and Jessica Berry [8]. The method of differential quadrature, which has been successfully used in solving boundary value problems, has also been extended to solve initial value problems of plates and was used to discretize the time domain [9], [10], Civalek [11] also studied the geometrically nonlinear dynamic problem of thin rectangular plates resting on Winkler–Pasternak two parameter elastic foundation by discretizing the governing nonlinear partial differential equations of the plate in space and time domains using the discrete singular convolution and harmonic differential quadrature methods. Recently, Zerkane et al [12] solved a homogenization procedure for nonlinear free vibration analysis of functionally graded beams resting on nonlinear elastic foundations.

In the present paper, the problem of geometrically nonlinear free vibrations of clamped FGCP with immovable ends resting on linear and nonlinear Pasternak elastic foundation is investigated using Hamilton's principle and spectral analysis. Based on the governing axial equation of the circular plate in which the axial inertia and damping are ignored. The spectral expansion used in the model is discussed here for the first non-linear axisymmetric mode shape.

II. Functionally Graded Materials

In this section, we consider a clamped-clamped FGCP having the geometrical characteristics shown in Fig. 1. It is assumed that the FGCP is made of ceramic and metal, and the effective material properties of the FGCP, Young's modulus E and mass density ρ, are functionally graded in the thickness direction according to a function of the volume fractions V of the constituents.

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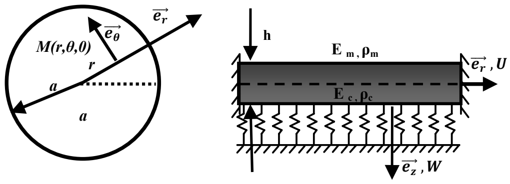

Fig. 1 Geometry of a FG clamped circular plate with Pasternak elastic foundation

According to the rule of mixture, the effective material properties P can be expressed as:

$$P = P_m V_m + P_c V_c \quad (1)$$

where subscripts "m" and "c" refer to the metal and ceramic constituents, respectively. A simple power law is considered here to describe the variation of the volume fraction of the metal and the ceramic constituents as follows:

$$V_m = \left(z/(h) + 1/(2)\right)^n \quad (2)$$

where h is the structure thickness, and n ($0 \le n \le \infty$) is a volume fraction exponent,

With $V_m + V_c = 1$ n is a non-negative parameter (power-law exponent) which dictates the material variation profile through the thickness of the plate.

Effective material properties P of the CFGP such as Young's modulus (E) and mass density (ρ) can be determined by substituting (2) into (1), which gives:

$$P(z) = P_m + (P_c - P_m)\left(z/(h) + 1/(2)\right)^n \quad (3)$$

III. Nonlinear Free Vibration Analysis

Consider a fully clamped thin circular plate of a uniform thickness h and a radius a. The co-ordinate system is chosen such that the middle plane of the plate coincides with the polar coordinates (r, θ), the origin of the co-ordinate system being at the centre of the plate with the z-axis downward, as depicted in Fig. 1. The plate is made of a mixture of ceramic and metal. Considering axisymmetric vibrations of the circular plate, the displacements are given in accordance with classical plate theory by [13]:

$$u_r(r,z,t) = U(r,t) - z\,\partial w(r,t)/(\partial r)\,, u_\theta(r,t) = 0\,, u_z(r,t) = W(r,t) \quad (4)$$

where U and W are the in-plane and out-of-plane displacements of the middle plane point (r, θ, 0) respectively, and u_r,u_θ and u_z are the displacements along r , θ and z directions, respectively. The non-vanishing components of the strain tensor in the case of large displacements are given by Von-Karman relationships:

$$\{\varepsilon\} = \{\varepsilon^0\} + z\{K\} + \{\lambda^0\} \quad (5)$$

In which $\{\varepsilon^0\}$, {K} and $\{\lambda^0\}$ are given by:

$$\{\varepsilon^0\} = \begin{bmatrix} \varepsilon_r^0 \\ \varepsilon_\theta^0 \end{bmatrix} = \begin{bmatrix} \partial u/(\partial r) \\ u/(r) \end{bmatrix} \quad (6)$$

$$\{K\} = \begin{bmatrix} K_r \\ K_\theta \end{bmatrix} = \begin{bmatrix} -\partial w^2/(\partial r^2) \\ -1/(r)\partial w/(\partial r) \end{bmatrix} \quad (7)$$

$$\{\lambda^0\} = \begin{bmatrix} \lambda_r \\ \lambda_\theta \end{bmatrix} = \begin{bmatrix} 1/(2)(\partial w/(\partial r))^2 \\ 0 \end{bmatrix} \quad (8)$$

For the FGM circular plate shown in Fig. 1, the stress can be expressed as:

$$\{\sigma\} = [Q]\{\varepsilon\} \quad (9)$$

In which $\{\sigma\} = [\sigma_r \; \sigma_\theta]^T$ and the terms of the matrix [Q] can be obtained by the relationships given in [8]. The force and moment resultants are defined by:

$$(N_r, N_\theta) = \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta)\; dz \quad (10)$$

$$(M_r, M_\theta) = \int_{-h/2}^{h/2} (\sigma_r, \sigma_\theta)\; dz \quad (11)$$

The in-plane forces and bending moments in the plate are given by:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \{\varepsilon^0\} + \{\lambda^0\} \\ \{K\} \end{bmatrix} \quad (12)$$

A, B and D are symmetric matrices given by the following equation:

$$\left(A_{ij}, B_{ij}, D_{ij}\right) = \int_{-h/2}^{h/2} Q_{ij}\,(1, z, z^2)\, dz \quad (13)$$

Here, the Q_{ij}'s are the reduced stiffness coefficients of the plate. The expression for the bending strain energy V_b, the membrane strain energy V_m, the coupling strain energy V_c and the kinetic energy T are given by:

$$V_b = \pi \int_0^a D_{11}\Big[(\partial w^2/(\partial r^2))^2 + 1/(r)^2\,(\partial w/(\partial r))^2 + 2\nu/(r)\;\partial w/(\partial r)\;\partial w^2/(\partial r^2)\Big]\, r dr \quad (14)$$

$$V_m = \pi \int_0^a A_{11}\left[(\partial u/(\partial r))^2 + \partial u/(\partial r)\,(\partial w/(\partial r))^2\right] r dr + \pi \int_0^a A_{11}\left[\,2\nu U/(r)\;\partial u/(\partial r) + \nu U/(r)\;(\partial w/(\partial r))^2\right] r dr + \pi \int_0^a A_{11}\left[\frac{1}{4}\,(\partial w/(\partial r))^4 + u^2/(r)^2\right] r dr \quad (15)$$

$$V_c = \pi \int_0^a -B_{11}\left[\partial w^2/(\partial r^2)\;(\partial w/(\partial r))^2 + \nu/(r)\;\partial w/(\partial r)\;(\partial w/(\partial r))^2\right]\, r dr \quad (16)$$

and

$$T = \pi\, I_0 \int_0^a (\partial w/(\partial r))^2\; r dr \quad (17)$$

where I_0 is the inertial term given by:

$$I_0 = \int_{-h/2}^{h/2} \rho(z)\ dz \tag{18}$$

An approximation has been adopted in the present work consisting on neglecting the contribution of the in-plane displacement U in the membrane strain energy expression. Such an assumption of neglecting the in-plane displacements in the non-linear plate strain energy has been made in Refs. [14], [15] when calculating the first two non-linear mode shapes of fully clamped rectangular plates. For the first non-linear mode shape, the range of validity of this assumption has been discussed in the light of the experimental and numerical results obtained for the non-linear frequency–amplitude dependence and the non-linear bending stress estimates obtained at large vibration amplitude [15], [16]. In order to examine the effects of large vibration amplitudes on the membrane stress patterns for clamped circular plates. The assumption introduced above leads to:

$$V_m = \pi\,(A_{11}/(4)) \int_0^a (\partial w/(\partial r))^4\ rdr \tag{19}$$

The strain energy of the elastic foundation V_f of the CFGP is given by:

$$V_f = \frac{1}{2}\int_0^a\int_0^{2\pi} K_L\, w^2 rdrd\theta + \frac{1}{4}\int_0^a\int_0^{2\pi} K_{NL}\, w^4\ rdrd\theta + \frac{1}{2}\int_0^a\int_0^{2\pi} K_S\left(\frac{\partial w}{\partial r}\right)^2 rdrd\theta \tag{20}$$

where K_L and K_{NL} are the linear and the non linear foundation stiffness respectively. K_S is the shear modulus of Pasternak foundation .For a general parametric study; we use the following non dimensional formulation by putting:

$$r^* = r/(a) \tag{21}$$

$$w_i^* = w_i/(h) \tag{22}$$

Applying Hamilton's principle and expanding the displacement W in the form of a finite series, the following set of nonlinear algebraic equations is obtained:

$$2a_i k_{ir}^* + 3a_i a_j a_k b_{ijkr}^* + (8/(\pi)) a_i a_j C_{ijr}^{s*} - 2\omega^{*2} a_i m_{ir}^* = 0 \tag{23}$$

where m_{ij}^*, k_{ij}^*, b_{ijkl}^* and c_{ijk}^{s*} stand for the non dimensional mass tensor, the linear rigidity tensor, the fourth order non-linear rigidity tensor and the third order non-linear coupling tensor, respectively, which are defined as:

$$k_{ij}^* = \int_0^1 \begin{bmatrix} (\partial w_i^{2*}/(\partial r^{*2}))(\partial w_j^{2*}/(\partial r^{*2})) \\ +(1/(r^{*2}))(\partial w_i^*/(\partial r^*))(\partial w_j^*/(\partial r^*)) \end{bmatrix} r^* dr^* + \int_0^1 2(\nu/(r^*))(\partial w_i^*/(\partial r^*))(\partial w_j^{2*}/(\partial r^{*2}))\ r^* dr^* + K_L^* \int_0^1 w_i^* w_j^*\ r^* dr^* + K_S^* \int_0^1 \left(\frac{\partial w_i^*}{\partial r^*}\right)\left(\frac{\partial w_j^*}{\partial r^*}\right) r^* dr^* \tag{24}$$

$$C_{ijk}^{s*} = \beta \int_0^1 \left[(\partial w_i^{2*}/(\partial r^{*2}))(\partial w_j^*/(\partial r^*))(\partial w_k^*/(\partial r^*))\right] r^* dr^* + \beta \int_0^1 \left[(\nu/(r^*))(\partial w_i^*/(\partial r^*))(\partial w_j^*/(\partial r^*))(\partial w_k^*/(\partial r^*))\right] r^* dr^* \tag{25}$$

$$m_{ij}^* = \int_0^1 w_i^* w_j^*\ r^* dr^* \tag{26}$$

$$b_{ijkl}^* = \alpha \int_0^1 (\partial w_i^*/\partial r^*)(\partial w_j^*/\partial r^*)(\partial w_k^*/\partial r^*)(\partial w_l^*/\partial r^*) r^* dr^* + K_{NL}^* \int_0^1 w_i^* w_j^* w_k^* w_l^*\ r^* dr^* \tag{27}$$

where α, β, K^*_L, K^*_{NL} and K^*_S are given by:

$$\alpha = (A_{11}h^2/(4D_{11})) \tag{28}$$
$$\beta = (-\,B_{11}h/(D_{11})) \tag{29}$$
$$K_L^* = (a^4/(D_{11}))K_L \tag{30}$$
$$K_{NL}^* = (2a^4/(A_{11}))K_{NL} \tag{31}$$
$$K_S^* = \frac{a^2}{D_{11}} K_S \tag{32}$$

To obtain the nonlinear free response of a clamped-clamped FGCP in the neighborhood of its first resonant frequency, the values of the linear rigidity matrix K^*_{ij} and the nonlinear geometrical rigidity tensor b^*_{ijkl} have been calculated using the first six normalized symmetric linear circular plate function, w^*_1, w^*_2 ,....., w^*_6 .The functions have been normalized in such a manner that the obtained mass matrix equals the identity matrix.

IV. Numerical Results and Discussions

In the problem considered herein, the top surface of the FGCP is ceramic rich (E_c=384.43e9GPa, υ_c=0.3, ρ_c=2370Kg/m^3), whereas the bottom surface of the FGCP is metal rich (E_m=201.04e9GPa, υ_m=0.3, ρ_m=8166 Kg/m^3).

TABLE I
Frequency Ratio Ω^*_{NL}/Ω^*_L of a Clamped Circular Isotropic Plate

W^*_{Max}	Present Work 2013	[6] 2008	[5] 2003
0.2	1.0108	1.0075	1.0072
0.4	1.0421	1.0296	1.0284
0.5	1.0648	1.0459	1.0439
0.6	1.0916	1.0654	1.0623
0.8	1.1560	1.1135	1.1073
1.0	1.2318	1.1724	1.1615

In Table I, the first nonlinear frequency ratios ω^*_{nl}/ω^*_l, calculated in the present work at various vibration amplitudes, is compared with the results obtained in [5], [6]. It is noted that the solution given in the present work overestimates the frequencies of the clamped circular isotropic plate, especially for high values of dimensionless amplitude. This discrepancy is mainly due to the fact that the axial displacements have been neglected in the expression of the axial strain energy.

TABLE II
FREQUENCY RATIO $\Omega^*_{NL}/\Omega^*_{L}$ OF A CLAMPED CIRCULAR FG PLATE AT T=300(K)

$T(K)$	n	W^*_{Max}	Present Work 2013	From graph [6] 2008
300	0.5	0.2	1.0110	1.0068
		0.4	1.0430	1.0275
		0.5	1.0663	1.0413
		0.6	1.0935	1.0586
		0.8	1.1593	1.1034
		1.0	1.2363	1.1586

The same comparison has been conducted in the case of circular functionally graded plate. As expected, the frequency ratios obtained with the present model are higher than those obtained in [6]. Especially for large vibration amplitudes for which the contribution of axial displacement becomes significant.

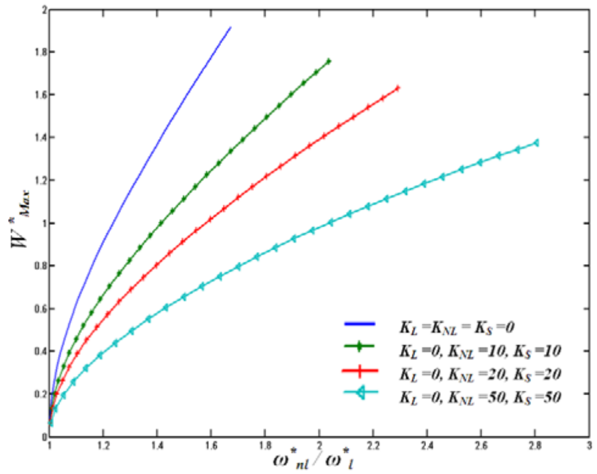

Fig. 2 Effect of the linear elastic foundation stiffness on the fundamental frequency ratio, case of n=0.5

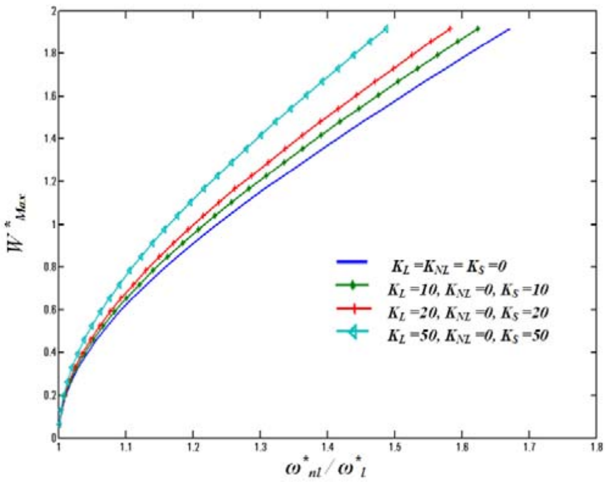

Fig. 3 Effect of the non linear elastic foundation stiffness on the fundamental frequency ratio, case of n=0.5

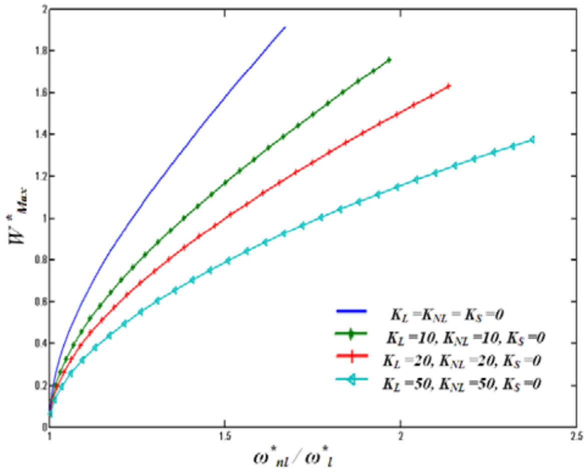

Fig. 4 Effect of the shearing elastic foundation stiffness on the fundamental frequency ratio, case of n=0.5

It can be shown from Figs. 2-4 that an increase in the value of linear elastic foundation stiffness leads to a decrease in the nonlinear to linear frequency ratio. On the other hand, this ratio enhances with an increase in nonlinear elastic foundation stiffness.

V. CONCLUSION

The present study deals with the problem of geometrically nonlinear free vibrations of a clamped-clamped FGCP resting on Pasternak elastic foundations. The main feature of the present contribution is the fact that the existing analytical solutions, numerical techniques and software developed over the years for the nonlinear analysis of isotropic circular plates can be easily used for FGCP case. On the other hand, the influence of the foundation parameters on the nonlinear fundamental frequency has been studied. The effect of the linear and the shearing foundation is to decrease the nonlinear frequency ratio of the FGCP, whereas the effect of the nonlinear foundation stiffness is to stiffen the nonlinear response. It's expected in future work to complete the present model by taking into account the contribution of the axial displacement in the membrane strain energy expression in order to improve the frequency precision and to determine the membrane stresses which cannot be obtained with the present formulation.

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