

(R, S) -Modules and $(1, k)$ -Jointly Prime (R, S) -Submodules

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Abstract—We introduced the notions of $(1, k)$ -prime ideal and $(1, k)$ -jointly prime (R, S) -submodule as a generalization of prime ideal and jointly prime (R, S) -submodule, respectively. We provide a relationship between $(1, k)$ -prime ideal and $(1, k)$ -jointly prime (R, S) -submodule. Characterizations of $(1, k)$ -jointly prime (R, S) -submodules are also given.

Keywords— (R, S) -module, $(1, k)$ -prime ideal, $(1, k)$ -jointly prime (R, S) -submodule.

I. INTRODUCTION

THROUGHOUT this paper, let R and S be rings and M an abelian group.

Definition 1.1: [1] Let R and S be rings and M an abelian group under addition. We say that M is an (R, S) -**module** if there is a function $\cdot : R \times M \times S \rightarrow M$ satisfying the following properties: for all $r, r_1, r_2 \in R$, $m, n \in M$ and $s, s_1, s_2 \in S$,

- (i) $r \cdot (m + n) \cdot s = r \cdot m \cdot s + r \cdot n \cdot s$
- (ii) $(r_1 + r_2) \cdot m \cdot s = r_1 \cdot m \cdot s + r_2 \cdot m \cdot s$
- (iii) $r \cdot m \cdot (s_1 + s_2) = r \cdot m \cdot s_1 + r \cdot m \cdot s_2$
- (iv) $r_1 \cdot (r_2 \cdot m \cdot s_1) \cdot s_2 = (r_1 r_2) \cdot m \cdot (s_1 s_2)$.

We usually abbreviate $r \cdot m \cdot s$ by rms . We may also say that M is an (R, S) -module under $+$ and \cdot .

An (R, S) -**submodule** of an (R, S) -module M is a subgroup N of M such that $rns \in N$ for all $r \in R$, $n \in N$ and $s \in S$.

Definition 1.2: [1] Let M be an (R, S) -module. A proper (R, S) -submodule P of M is called **jointly prime** if for each left ideal I of R , right ideal J of S and (R, S) -submodule N of M ,

$$INJ \subseteq P \text{ implies } IMJ \subseteq P \text{ or } N \subseteq P.$$

The structure of an (R, S) -module was created as a generalization of a module structure. The basic results of an (R, S) -module structure have been given by [1] and [2]. Almost all of those results was studied analogous to a module structure such as the primalities of (R, S) -submodules of (R, S) -modules and left multiplication (R, S) -modules; see [1] and [2].

In this paper, we introduce the notions of $(1, 2)$ -prime ideal, $(1, k)$ -prime ideal, $(1, 2)$ -jointly prime (R, S) -submodule and $(1, k)$ -jointly prime (R, S) -submodule and obtain equivalent conditions for an (R, S) -submodule to be $(1, k)$ -jointly prime (R, S) -submodule.

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II. $(1, 2)$ -JOINTLY PRIME (R, S) -SUBMODULES

In this research, we modify the structure of a jointly prime (R, S) -submodules for more general. Now, we start this section by giving the definition of $(1, 2)$ -jointly prime (R, S) -submodules.

Definition 2.1: A proper (R, S) -submodule P of M is called $(1, 2)$ -**jointly prime** if for each left ideal I of R , right ideal J of S and (R, S) -submodule N of M ,

$$INJ^2 \subseteq P \text{ implies } IMJ^2 \subseteq P \text{ or } N \subseteq P.$$

By the dual of $(1, 2)$ -jointly prime, we define $(2, 1)$ -jointly prime as follow.

A proper (R, S) -submodule P of M is called $(2, 1)$ -**jointly prime** if for each left ideal I of R , right ideal J of S and (R, S) -submodule N of M ,

$$I^2NJ \subseteq P \text{ implies } I^2MJ \subseteq P \text{ or } N \subseteq P.$$

It is clear that a jointly prime (R, S) -submodule is $(1, 2)$ -jointly prime and $(2, 1)$ -jointly prime. Next, we give a characterization of $(1, 2)$ -jointly prime and $(2, 1)$ -jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -submodule of \mathbb{Z} where $r, s \in \mathbb{Z}^+$.

Proposition 2.2: Let $r, s \in \mathbb{Z}^+$ and $p \in \mathbb{Z}_0^+ \setminus \{1\}$. Then

- (i) $p\mathbb{Z}$ is an $(1, 2)$ -jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -module of \mathbb{Z} if and only if $p = 0$, p is a prime integer or $p | rs^2$.
- (ii) $p\mathbb{Z}$ is a $(2, 1)$ -jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -module of \mathbb{Z} if and only if $p = 0$, p is a prime integer or $p | r^2s$.

Proof: (i) (\Rightarrow) Assume that $p\mathbb{Z}$ is a $(1, 2)$ -jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -module of \mathbb{Z} . Suppose that $p \neq 0$ and p is not a prime integer. Then $p = mn$ for some integer $m, n > 1$. It implies that $(rm\mathbb{Z})(n\mathbb{Z})(s^2\mathbb{Z}) = (rmns^2)\mathbb{Z} \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is $(1, 2)$ -jointly prime and $p \nmid n$, $(rm\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$. Note that $(r\mathbb{Z})(m\mathbb{Z})(s^2\mathbb{Z}) = (rm\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is $(1, 2)$ -jointly prime and $p \nmid m$, $(r\mathbb{Z})\mathbb{Z}(s^2\mathbb{Z}) \subseteq p\mathbb{Z}$. Hence $p | rs^2$.

(\Leftarrow) If $p = 0$ or p is a prime integer or $p | rs^2$, then it is clear that $p\mathbb{Z}$ is $(1, 2)$ -jointly prime. ■

Now, we already have an example of $(1, 2)$ -jointly prime but is not jointly prime.

Example 2.3: It is clear that \mathbb{Z} is a $(2\mathbb{Z}, 3\mathbb{Z})$ -module. Then $9\mathbb{Z}$ is a $(1, 2)$ -jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} but $9\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} .

The following is an example showing that $(1, 2)$ -jointly prime and $(2, 1)$ -jointly prime are exactly different.

Example 2.4: Recall that \mathbb{Z} is a $(2\mathbb{Z}, 3\mathbb{Z})$ -module. Then $4\mathbb{Z}$ is a $(2, 1)$ -jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} but $4\mathbb{Z}$ is not a $(1, 2)$ -jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} and $9\mathbb{Z}$

is a (1, 2)-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} but $9\mathbb{Z}$ is not a (2, 1)-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} .

Moreover, $p\mathbb{Z}$ is both a (1, 2)-jointly prime and (2, 1)-jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} if and only if $p\mathbb{Z}$ is a jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} .

Note that (1, 2)-jointly prime and (2, 1)-jointly prime may be different even if ring R and S are commutative.

We have a question from Example 2.4 that for general, if P is (1, 2)-jointly prime and (2, 1)-jointly prime, then can P be a jointly prime (R, S) -submodule? The following is an answers.

Example 2.5: It easy to see that \mathbb{Z} is a $(2\mathbb{Z}, 4\mathbb{Z})$ -module. Then $16\mathbb{Z}$ is both a (1, 2)-jointly prime and (2, 1)-jointly prime $(2\mathbb{Z}, 4\mathbb{Z})$ -submodule of \mathbb{Z} but $16\mathbb{Z}$ is not a jointly prime $(2\mathbb{Z}, 4\mathbb{Z})$ -submodule of \mathbb{Z} .

Example 2.6: Let \mathbb{Z} be a ring of integer and let

$$R = \left\{ \left[\begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right] \mid x, y \in \mathbb{Z} \right\},$$

$$S = \left\{ \left[\begin{array}{ccc} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{array} \right] \mid x, y \in \mathbb{Z} \right\} \text{ and}$$

M is the set of all 3×3 matrices on integer. Then M is an (R, S) -module. Since $R^2 = 0 = S^2$, all proper (R, S) -submodules of M are both (1, 2)-jointly prime and (2, 1)-jointly prime (R, S) -submodule of M . However, 0 is not a jointly prime (R, S) -submodule of M .

For each (R, S) -submodule P of M and $k \in \mathbb{Z}^+$, let

$$(P : M)_{R;S^k} = \{r \in R \mid rMS^k \subseteq P\}.$$

Proposition 2.7: Let P be an (R, S) -submodule of an (R, S) -module M and $k \in \mathbb{Z}^+$. The followings hold.

- (i) $(P : M)_{R;S^k}$ is a subgroup of R under addition.
- (ii) $(P : M)_{R;S^k} \subseteq (P : M)_{R;S^{k+1}}$.
- (iii) If $S^2 = S$, then $(P : M)_{R;S^k}$ is an ideal of R .

Proof: The proof is straightforward. ■

Next, we introduced a particular nonempty subset of R which play a role in this research.

Let R be a ring and T a proper ideal of R . Then T is said to be a (1, 2)-**prime ideal** of R if for each ideal A and B of R , if $AB^2 \subseteq T$, then $A \subseteq T$ or $B^2 \subseteq T$. A prime ideal of R is a (1, 2)-prime ideal of R but the converse is not true. We show by observing the following example.

Example 2.8: Let p be an integer. If $p = 0$ or p is a prime integer or $p = q^2$ where q is a prime integer, then $p\mathbb{Z}$ is a (1, 2)-prime ideal of \mathbb{Z} .

It clear that $4\mathbb{Z}$ is a (1, 2)-prime ideal of \mathbb{Z} but is not a prime ideal of \mathbb{Z} .

Proposition 2.9: Let P be an (R, S) -submodule of an (R, S) -module M such that $(P : M)_{R;S^2}$ is a proper ideal of R . If P is a (1, 2)-jointly prime (R, S) -submodule of M , then $(P : M)_{R;S^2}$ is a (1, 2)-prime ideal of R .

Proof: Assume that P is a (1, 2)-jointly prime (R, S) -submodule of M . Let A and B be ideals of R such that $AB^2 \subseteq (P : M)_{R;S^2}$. Hence $(AB^2)MS^2S^2 \subseteq (AB^2)MS^2 \subseteq P$. Thus $A(B^2MS^2)S^2 \subseteq P$. Since P is (1, 2)-jointly prime, $AMS^2 \subseteq P$ or $B^2MS^2 \subseteq P$. Therefore $A \subseteq (P : M)_{R;S^2}$

or $B^2 \subseteq (P : M)_{R;S^2}$. This means that $(P : M)_{R;S^2}$ is a (1, 2)-prime ideal of R . ■

The converse of Proposition 2.9 is invalid. For example, $6\mathbb{Z}$ is a $(\mathbb{Z}, 2\mathbb{Z})$ -submodule of \mathbb{Z} . We see that $(6\mathbb{Z} : \mathbb{Z})_{\mathbb{Z};(2\mathbb{Z})^2} = 3\mathbb{Z}$ is a prime ideal of \mathbb{Z} , of course, $3\mathbb{Z}$ is a (1, 2)-prime ideal of \mathbb{Z} but $6\mathbb{Z}$ is not a (1, 2)-jointly prime $(\mathbb{Z}, 2\mathbb{Z})$ -submodule of \mathbb{Z} .

III. (1, k)-JOINTLY PRIME (R, S)-SUBMODULES

In this section, we extend the notion of (1, 2)-jointly prime to (1, k)-jointly prime where $k \in \mathbb{Z}^+$. Similarly, we also extend the notion of (2, 1)-jointly prime to (k, 1)-jointly prime where $k \in \mathbb{Z}^+$.

Definition 3.1: Let $k \in \mathbb{Z}^+$ and M be an (R, S) -module. A proper (R, S) -submodule P of M is called (1, k)-**jointly prime** if for each left ideal I of R , right ideal J of S and (R, S) -submodule N of M ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

Dually, a proper (R, S) -submodule P of M is called (k, 1)-**jointly prime** if for each left ideal I of R , right ideal J of S and (R, S) -submodule N of M ,

$$I^kNJ \subseteq P \text{ implies } I^kMJ \subseteq P \text{ or } N \subseteq P.$$

Note here that jointly prime and (1, 1)-jointly prime are identical.

Proposition 3.2: Let $r, s, k \in \mathbb{Z}^+$ and $p \in \mathbb{Z}_0^+ \setminus \{1\}$. Then

- (i) $p\mathbb{Z}$ is a (1, k)-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -module of \mathbb{Z} if and only if $p = 0$, p is a prime integer or $p \mid rs^k$.
- (ii) $p\mathbb{Z}$ is a (k, 1)-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -module of \mathbb{Z} if and only if $p = 0$, p is a prime integer or $p \mid r^ks$.

Proof: (\Rightarrow) Assume that $p\mathbb{Z}$ is a (1, k)-jointly prime $(r\mathbb{Z}, s\mathbb{Z})$ -module of \mathbb{Z} . Suppose that $p \neq 0$ and p is not a prime integer. Then $p = mn$ for some integer $m, n > 1$. It implies that $(rm\mathbb{Z})(n\mathbb{Z})(s^k\mathbb{Z}) = (rmns^k)\mathbb{Z} \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is (1, k)-jointly prime and $p \nmid n$, $(rm\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z}) \subseteq p\mathbb{Z}$. Note that $(r\mathbb{Z})(m\mathbb{Z})(s^k\mathbb{Z}) = (rm\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z}) \subseteq p\mathbb{Z}$. Since $p\mathbb{Z}$ is (1, k)-jointly prime and $p \nmid m$, $(r\mathbb{Z})\mathbb{Z}(s^k\mathbb{Z}) \subseteq p\mathbb{Z}$. Hence $p \mid rs^k$.

(\Leftarrow) If $p = 0$ or p is a prime integer or $p \mid rs^k$, then it is clear that $p\mathbb{Z}$ is (1, k)-jointly prime. ■

Proposition 3.3: Let $k \in \mathbb{Z}^+$ and M be an (R, S) -module. Then

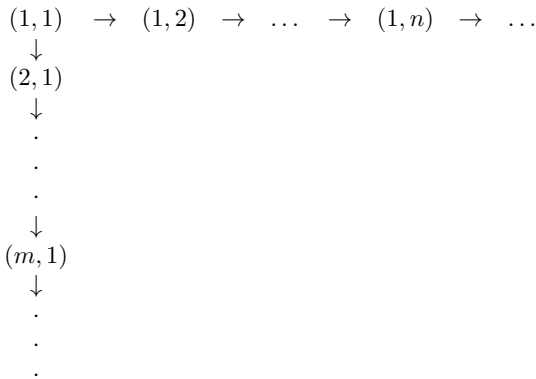
- (i) If P is (1, k)-jointly prime, then P is (1, k + 1)-jointly prime.
- (ii) If P is (k, 1)-jointly prime, then P is (k + 1, 1)-jointly prime.

Proof: Assume that P is a (1, k)-jointly prime (R, S) -submodule of M . Let I be a left ideal of R , N an (R, S) -submodule of M and J be a right ideal of S such that $INJ^{k+1} \subseteq P$. Note that $I(INJ)J^k \subseteq I^2NJ^{k+1} \subseteq INJ^{k+1} \subseteq P$. Since P is (1, k)-jointly prime, $IMJ^k \subseteq P$ or $INJ \subseteq P$. Note that $J^{k+1} \subseteq J^k \subseteq J$. If $IMJ^k \subseteq P$, then $IMJ^{k+1} \subseteq P$. Assume that $INJ \subseteq P$. Then $INJ^k \subseteq P$. Since P is (1, k)-jointly prime, $IMJ^k \subseteq P$ or $N \subseteq P$. ■

The following example shows that the converse of Proposition 3.3 is false in general.

Example 3.4: Recall that \mathbb{Z} is a $(2\mathbb{Z}, 3\mathbb{Z})$ -module. Then $27\mathbb{Z}$ and $54\mathbb{Z}$ are $(1, 3)$ -jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} but $27\mathbb{Z}$ and $54\mathbb{Z}$ are not $(1, 2)$ -jointly prime $(2\mathbb{Z}, 3\mathbb{Z})$ -submodule of \mathbb{Z} .

We obtain the following diagram from Proposition 3.3. Note that the order pair (m, n) means P is a (m, n) -jointly prime (R, S) -submodule of M .



In this point, we present a generalization of a $(1, 2)$ -prime ideal of R which is called $(1, k)$ -prime ideal of R where $k \in \mathbb{Z}^+$. Let R be a ring and T a proper ideal of R and $k \in \mathbb{Z}^+$. Then T is said to be a $(1, k)$ -**prime ideal** of R if for each ideal A and B of R , if $AB^k \subseteq T$, then $A \subseteq T$ or $B^k \subseteq T$.

Proposition 3.5: Let $k \in \mathbb{Z}^+$ and P be an (R, S) -submodule of an (R, S) -module M such that $(P : M)_{R;S^k}$ is a proper ideal of R . If P is a $(1, k)$ -jointly prime (R, S) -submodule of M , then $(P : M)_{R;S^k}$ is a $(1, k)$ -prime ideal of R .

Note that $(X)_l$ and $(X)_r$ is the left ideal generated by X and the right ideal generated by X , respectively, for any subset X of a ring R . $\langle Y \rangle$ is the (R, S) -submodule generated by Y for any (R, S) -submodule Y of an (R, S) -module M Next result needs the following lemma.

Lemma 3.6: Let M be an (R, S) -module and $k \in \mathbb{Z}^+$. The following statements hold:

- (i) For all left ideal I of R , left ideal J of S and (R, S) -submodule N and P of M ,

$$INJ^k \subseteq P \text{ implies } I^2N(J)_r^k \subseteq P.$$

- (ii) For all right ideal I of R , left ideal J of S and (R, S) -submodule N and P of M ,

$$INJ^k \subseteq P \text{ implies } (I)_lN(J^2)^k \subseteq P.$$

- (iii) For all right ideal I of R , left ideal J of S and (R, S) -submodule P of M ,

$$(I)_lM(J^2)^k \subseteq P \text{ implies } (I)_l\langle IMJ^k \rangle J^k \subseteq P.$$

- (iv) For all right ideal I of R , right ideal J of S and (R, S) -submodule N and P of M ,

$$INJ^k \subseteq P \text{ implies } I^2N(J)_l^k \subseteq P.$$

- (v) For all right ideal I of R , right ideal J of S and (R, S) -submodule P of M ,

$$I^2M(J)_l^k \subseteq P \text{ implies } I\langle IMJ^k \rangle (J)_l^k \subseteq P.$$

- (vi) For all left ideal I of R , right ideal J of S and (R, S) -submodule N and P of M ,

$$INJ^k \subseteq P \text{ implies } (I)_rN(J^2)^k \subseteq P.$$

- (vii) For all left ideal I of R , right ideal J of S and (R, S) -submodule P of M ,

$$(I)_rM(J^2)^k \subseteq P \text{ implies } (I)_r\langle IMJ^k \rangle J^k \subseteq P.$$

Proof: (i) Let I be a left ideal of R , J a left ideal of S , N and P be (R, S) -submodules of M . Assume that $INJ^k \subseteq P$. Then

$$\begin{aligned}
 I^2N(J)_r^k &= I^2N(J + JS)^k \\
 &\subseteq I^2N(J^k + J^kS) \\
 &\subseteq INJ^k + I(INJ^k)S \\
 &\subseteq P.
 \end{aligned}$$

- (ii) Let I be a right ideal of R , J a left ideal of S , N and P be (R, S) -submodules of M . Assume that $INJ^k \subseteq P$. Then

$$\begin{aligned}
 (I)_lN(J^2)^k &= (I + RI)N(J^k)^2 \\
 &= (I + RI)N(J^k)J^k \\
 &\subseteq INJ^k J^k + RINJ^k J^k \\
 &\subseteq INJ^k + R(INJ^k)J^k \\
 &\subseteq P.
 \end{aligned}$$

- (iii) Let I be a right ideal of R , J a left ideal of S , N and P be (R, S) -submodules of M . Assume that $(I)_lM(J^2)^k \subseteq P$. Then

$$\begin{aligned}
 (I)_l\langle IMJ^k \rangle J^k &= (I)_l(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)J^k \\
 &\subseteq (I)_l(\mathbb{Z}(IMJ^k))J^k + (I)_l(R(IMJ^k)S)J^k \\
 &\subseteq \mathbb{Z}(I)_lIM(J^k)^2 + (I)_lRIMJ^kS J^k \\
 &\subseteq \mathbb{Z}(I)_lM(J^2)^k + (I)_lM(J^2)^k \\
 &\subseteq P.
 \end{aligned}$$

- (iv) Let I be a right ideal of R , J a right ideal of S , N and P be (R, S) -submodules of M . Assume that $INJ^k \subseteq P$. Then

$$\begin{aligned}
 I^2N(J)_l^k &= I^2N(J + SJ)^k \\
 &\subseteq I^2N(J^k + SJ^k) \\
 &\subseteq INJ^k + I(INS)J^k \\
 &\subseteq INJ^k + INJ^k \\
 &\subseteq P + P \\
 &\subseteq P.
 \end{aligned}$$

- (v) Let I be a right ideal of R , J a right ideal of S , N and P be (R, S) -submodules of M . Assume that $I^2M(J)_l^k \subseteq P$. Then

$$\begin{aligned}
 I\langle IMJ^k \rangle (J)_l^k &= I(\mathbb{Z}(IMJ^k) + R(IMJ^k)S)(J)_l^k \\
 &\subseteq \mathbb{Z}(I^2MJ^k(J)_l^k) + IRIMJ^kS(J)_l^k \\
 &\subseteq \mathbb{Z}(I^2M(J)_l^k) + I^2M(J)_l^k \\
 &\subseteq P.
 \end{aligned}$$

(vi) Let I be a left ideal of R , J a right ideal of S , N and P be (R, S) -submodules of M . Assume that $INJ^k \subseteq P$. Then

$$\begin{aligned}(I)_r N(J^2)^k &= (I + IR)NJ^k J^k \\ &\subseteq INJ^k J^k + (IR)NJ^k J^k \\ &\subseteq INJ^k + I(RNJ^k)J^k \\ &\subseteq INJ^k \\ &\subseteq P.\end{aligned}$$

(vii) Let I be a left ideal of R , J a right ideal of S , N and P be (R, S) -submodules of M . Assume that $(I)_r M(J^2)^k \subseteq P$. Then

$$\begin{aligned}(I)_r \langle IMJ^k \rangle J^k &= (I)_r (\mathbb{Z}(IMJ^k) + R(IMJ^k)S)J^k \\ &\subseteq \mathbb{Z}((I)_r IMJ^k J^k) + (I)_r RIMJ^k S J^k \\ &\subseteq \mathbb{Z}(I)_r M(J^2)^k + (I)_r M(J^2)^k \\ &\subseteq P + P. \\ &\subseteq P.\end{aligned}$$

■

Next, we obtain equivalent conditions for an (R, S) -submodule to be $(1, k)$ -jointly prime (R, S) -submodules.

Theorem 3.7: Let M be an (R, S) -module and P a proper (R, S) -submodule of M and $k \in \mathbb{Z}^+$. The following statements are equivalent:

- (i) P is an $(1, k)$ -jointly prime (R, S) -submodule of M .
- (ii) For all left ideal I of R , left ideal J of S and (R, S) -submodule N of M ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

- (iii) For all right ideal I of R , left ideal J of S and (R, S) -submodule N of M ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

- (iv) For all right ideal I of R , right ideal J of S and (R, S) -submodule N of M ,

$$INJ^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } N \subseteq P.$$

- (v) For all left ideal I of R , right ideal J of S and $m \in M$,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

- (vi) For all left ideal I of R , left ideal J of S and $m \in M$,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

- (vii) For all right ideal I of R , left ideal J of S and $m \in M$,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

- (viii) For all right ideal I of R , right ideal J of S and $m \in M$,

$$I\langle m \rangle J^k \subseteq P \text{ implies } IMJ^k \subseteq P \text{ or } m \in P.$$

Proof: This follows from Lemma 3.6. ■

REFERENCES

- [1] T. Khumprapussorn, S. Pianskool and M. Hall, *(R, S)-modules and Their Fully prime and Jointly prime Submodules*, IME, 7 (2012), 1631 - 1643.
- [2] T. Khumprapussorn, *Left R-prime (R, S)-submodules*, IME, 8 (2013), 619 - 626.