

A new implementation of Miura-Arita algorithm for Miura curves

A. Basiri, S. Rahmany, D. Khatibi

Abstract—The aim of this paper is to review some of standard fact on Miura curves. We give some easy theorem in number theory to define Miura curves, then we present a new implementation of Arita algorithm for Miura curves.

Keywords—Miura curve, discrete logarithm problem, algebraic curve cryptography, Jacobian group.

I. INTRODUCTION

THE The goal of this paper is to describe a practical and efficient algorithm for computing in the Jacobian of a C_A curves over a finite field. Authors in [6] proposed an algorithm to complete the arithmetic in the base field for superelliptic curves, and the authors in [2], [7], generalise the algorithm to the class of C_{ab} curves and in [3] generalise the algorithm to the class of C_A curves, which includes superelliptic and C_{ab} curves as a special case. Furthermore, in [4], [5], [1], for the case of C_{34} curves, has presented some faster method to compute the addition of two point on the curve.

II. NUMERICAL SEMIGROUP

In this paper we denote by \mathbb{N}_0 , the set of all non negative integers numbers, so \mathbb{N}_0 is an additive semigroup. In addition we suppose that M be a proper sub semigroup of \mathbb{N}_0 such that $0 \in M \neq 0$.

Theorem 1: There is an integer number t and there exist some members a_1, a_2, \dots, a_t in M such that

$$M = \langle a_1, a_2, \dots, a_t \rangle, \quad a_1 < a_2 < \dots < a_t, \quad t \leq a_1.$$

In other words, M is a finitely generated semigroup in \mathbb{N}_0 .

Proof: Since $<$ is a well-ordering order in \mathbb{N}_0 , then there exists a minimal member, say a_1 , in $M - \{0\}$. On the other hand since M is a proper semigroup, then $1 \neq a_1$, so $1 < a_1$. Now let T_2 be the set of all members $a \in M$ such that $a \equiv 1 \pmod{a_1}$, so there are two cases: if T_2 is the empty set then $M = \langle a_1 \rangle$ and the proof is completed, else if $T_2 \neq \emptyset$ then the minimum of T_2 , denoted a_2 , exists. we then suppose T_3 be the set of all members $a \in M$ such that $a \equiv 2 \pmod{a_1}$, so if $T_3 \neq \emptyset$ then the minimum of T_3 , denoted a_3 , exists. Here suppose that the T_2, T_3, \dots, T_l and the a_2, a_3, \dots, a_l are chosen, we claim that $M = \langle a_1, a_2, \dots, a_t \rangle$. The inclusion $M \supseteq \langle a_1, a_2, \dots, a_t \rangle$ follow directly from the definition. Going the other way, note that, $w \in M$, by division algorithm, there exist $q \in \mathbb{N}_0$ and $0 \leq r \leq a_1 - 1$ such that $w = a_1q + r$.

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Hence T_{r+1} is a non empty set and has a minimum denoted by a_{r+1} and so $a_{r+1} = a_1q' + r$ with $q' \leq q$ and so

$$w = a_1(q-q') + a_1q' + r = a_1(q-q') + a_{r+1} \in \langle a_1, a_2, \dots, a_t \rangle$$

Example 2: If $M = \{0, 7, 8, 14, 15, 16, 19, 21, 22, 23, \dots\}$ then $a_1 = 7, a_2 = 8, a_3 = 16, a_4 = 24, a_5 = 25, a_6 = 19$ and $a_7 = 27$.

The following theorem express whenever the complement of any semigroup with identity of \mathbb{N}_0 is finite?

Theorem 3: The set $\bar{M} = \mathbb{N}_0 - M$ is finite if and only if $\gcd(a_1, a_2, \dots, a_t) = 1$, and in this case, $|\bar{M}| = \sum_{i=1}^{a_1-1} \lfloor \frac{b_i}{a_1} \rfloor$, where b_i is the minimum amount of members a in M with $a \equiv i \pmod{a_1}$.

Proof: Firstly, suppose that \bar{M} is a finite set, to have a contrast let there exists a prime number p such that $p|a_i$ for all $1 \leq i \leq t$. We claim that for all non negative integer q , $a_1q + 1 \notin M$, if it is not the case then there exists $q \in \mathbb{N}_0$ such that $a_1q + 1 \in M$ and so the $T = \{a_1u + 1 : u \in \mathbb{N}_0, a_1u + 1 \in M\}$ is a non empty set and so has a minimum, denoted by a_2 . Hence there exists $r \in \mathbb{N}_0$ such that $a_2 = a_1r + 1$, but $p|a_1$ and $p|a_2$, and this implies that p divides 1 and this contradicts the fact that p is a prime number. A consequence of all this is that the set $\{a_1q + 1 : q \in \mathbb{N}_0\}$ is a subset of \bar{M} and so \bar{M} is infinite which contradicts the hypothesis. To get the opposite direction, let $\gcd(a_1, a_2, \dots, a_t) = 1$. Note that for $0 \leq i \leq a_1 - 1$,

$$b_i = \min\{\lambda a_1 + i : \lambda \in \mathbb{N}_0, \lambda a_1 + i \in M\}$$

, let $s = a_1 - 1$, $b_i = w_i a_1 + i$ and for $1 \leq i \leq s$ put

$$A_i = \{i, a_1 + i, 2a_1 + i, \dots, (w_i - 1)a_1 + i\},$$

we claim that A_1, A_2, \dots, A_s are a partition of \bar{M} . We show first that for $i \neq j$, $A_i \cap A_j = \emptyset$, if this is not the case then there are r, r' such that

$$ra_1 + i = r'a_1 + j \Leftrightarrow (r - r')a_1 = j - i \Leftrightarrow a_1|j - i,$$

but $1 \leq i, j \leq s = a_1 - 1 < a_1$, hence $j - i = 0$ which is a contradiction and so $A_i \cap A_j = \emptyset$. we now show that $\bigcup_{i=1}^s A_i = \bar{M}$. To establish the desired equality, we use the usual strategy of proving containment in both directions. The inclusion $\bigcup_{i=1}^s A_i \subseteq \bar{M}$ follow directly from the fact that $A_i \subseteq \bar{M}$ for all $1 \leq i \leq s$. To get the opposite inclusion, suppose $x \in \bar{M}$ so there are $\lambda \in \mathbb{N}_0$ and $1 \leq j \leq s$ such that $x = \lambda a_1 + j$. We claim that $\lambda \leq w_j - 1$ and this implies that $x \in A_j \subseteq \bigcup_{i=1}^s A_i \subseteq \bar{M}$. If it is not the case, then $w_j \leq \lambda$, hence

$$x = (w_j + (\lambda - w_j))a_1 + j = b_j + (\lambda - w_j)a_1 \in M$$

which is a contradiction. Hence A_1, A_2, \dots, A_s are a partition of \bar{M} , and so

$$|\bar{M}| = \left| \bigcup_{i=1}^s A_i \right| = \sum_{i=1}^s |A_i| = \sum_{i=1}^s w_i$$

but since $a_1 > 1$ we have

$$w_i = \left\lfloor w_i + \frac{1}{a_1} \right\rfloor = \left\lfloor \frac{w_i a_1 + 1}{a_1} \right\rfloor = \left\lfloor \frac{b_i}{a_1} \right\rfloor.$$

A semigroup M of \mathbb{N}_0 with $0 \in M \neq 0$ is called a numerical semigroup if its complement in \mathbb{N}_0 be a finite set.

Example 4: The semigroup introduced in example 2 is a numerical semigroup because

$$\gcd(7, 8, 16, 24, 25, 19, 27) = 1$$

and

$$|\bar{M}| = \left\lfloor \frac{8}{7} \right\rfloor + \left\lfloor \frac{16}{7} \right\rfloor + \left\lfloor \frac{24}{7} \right\rfloor + \left\lfloor \frac{25}{7} \right\rfloor + \left\lfloor \frac{19}{7} \right\rfloor + \left\lfloor \frac{27}{7} \right\rfloor = 14,$$

in this case we have

$$M = \{1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 17, 18, 20\}.$$

In the rest of this article we suppose that M is a numerical semigroup which is generated by the set $\{a_1, a_2, \dots, a_t\}$ and $t \leq a_1$. For a numerical semigroup M there is a unique surjective map

$$\psi : \mathbb{N}_0^t \rightarrow M$$

where

$$\psi(n_1, n_2, \dots, n_t) = \sum_{i=1}^t n_i a_i$$

Definition 5: Every numerical semigroup M with the above notations introduced a C_A order as follow:

For $\alpha, \beta \in \mathbb{N}_0^t$ we say that $\alpha < \beta$ if $\psi(\alpha) < \psi(\beta)$ or $\psi(\alpha) = \psi(\beta)$ and there exists $1 \leq i \leq t-1$ such that $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \dots, \alpha_i = \beta_i$ and $\alpha_{i+1} > \beta_{i+1}$.

Note that if K is a field then the C_A order defined a monomial order in the polynomial ring $K[x_1, x_2, \dots, x_t]$.

Definition 6: For $a \in M$ we define

$$\mu(a) = \min\{\alpha \in \mathbb{N}_0^t : \alpha \in \psi^{-1}(a)\}$$

and

$$B(A) = \{\mu(a) : a \in M\},$$

$$T(A) = \{\mu(b_i) \in B(A) : 0 \leq i \leq a_1 - 1\},$$

at last we denote by $V(A)$, the set of all $\gamma \in \mathbb{N}_0^t - B(A)$ such that for all $\alpha \in \mathbb{N}_0^t - B(A)$ and $\beta \in \mathbb{N}_0^t$, the equality $\gamma = \alpha + \beta$ implies that $\beta = 0$.

III. MIURA C_A CURVES

In this section we denote by K , a finite field with q elements. For $m \in V(A)$, suppose that the polynomial $F_m \in K[x_1, x_2, \dots, x_t]$ has two following conditions:

i) for all $m \in V(A)$,

$$F_m = X^m + a_l X^l + \sum_{l \neq n < m} a_n X^n$$

where $l = \mu(\psi(m))$, $a_l \neq 0$.

ii) $\text{Span}\{X^n : n \in B(A)\} \cap \langle F_m : m \in V(A) \rangle = \langle 0 \rangle$.

In the above conditions $\text{Span}\{X^n : n \in B(A)\}$ means the set of all polynomials generated by X^n 's with coefficients in K and $\langle F_m : m \in V(A) \rangle$ is the ideal generated by F_m 's in $K[x_1, x_2, \dots, x_t]$.

Definition 7: Let M be a numerical semigroup of \mathbb{N}_0 which is generated by $A = \{a_1, a_2, \dots, a_t\}$ and let I be an ideal in $K[x] := K[x_1, x_2, \dots, x_t]$ which is generated by some polynomials which satisfy in the above two conditions. In this case $\text{spec}\left(\frac{K[x]}{I}\right)$ is called a Miura curve or a C_A curve over the field of fractions $R = \frac{K[x]}{I}$.

Using Arita algorithm we can compute the addition of two points on a C_A curve, in Appendix A we give an another implementation of this algorithm on Maple 11.

IV. CONCLUSION

By the implementation presented in Appendix A we can compute the addition of two distinct point on a C_A curve or compute the n^{it} power of a point on the curve.

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APPENDIX A

IMPLEMENTATION OF THE ALGORITHM IN MAPLE 11

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> with(Ore_algebra):
> with(PolynomialIdeals):
> with(Groebner):
> Initial:=proc(n1,p1)
> global p,nn,Tlex,C_A,A:
> local Jabr,i,xInput;
> nn:=n1;p:=p1;
> Jabr:=poly_algebra(t,seq(x[i],i=1..nn),characteristic=p):
> for i from 1 to nn do
> xInput:=scanf("%d",a);
> A[i]:=xInput[1];
> end do;
> Tlex:=MonomialOrder(Jabr,'matrix'([[1,seq(0,i=1..nn)],
seq([seq(0,j=1..nn-i),1,seq(0,j=0..i-1)],i=0..
> nn-1)], [t,seq(x[i],i=1..nn)])):
> C_A:=MonomialOrder(Jabr,'matrix'([[1,seq(0,i=1..nn)],
seq([0,seq(0,j=1..i),seq(A[j],j=i+1..nn)],i=0..
> nn-1)], [t,seq(x[i],i=1..nn)])):
> end:
> #[J:g]
> xQuotient:=proc(J,g,TT)
> local h,G,res,i:
> G:=Basis(expand([seq(t*h,h=J),(1-t)*g]),TT):
> res:=[]:
> for i from 1 to nops(G) do
> if (not member(t,indets(LeadingMonomial(G[i],TT))))
then
> res:=[op(res),Normal(G[i]/g) mod p]:
> fi:
> end do:
> return res:
> end:
> #I1 Intersect I2
> IntersectId:=proc(I1,I2,TT)
> local i,G,res:
> G:=Basis(expand([seq(t*i,i=I1),seq((1-t)*i,i=I2)]),TT):
> res:=[]:
> for i from 1 to nops(G) do
> if (not member(t,indets(LeadingMonomial(G[i],TT))))
then
> res:=[op(res),G[i]]:
> fi:
> end do:
> return res:
> end:
> #[J:K]
> QuotientId:=proc(J,K,TT)
> local i,G:
> G:=xQuotient(J,K[1],TT):
> for i from 2 to nops(K) do
> G:=IntersectId(G,xQuotient(J,K[i],TT),TT):
> end do:
> return G:
> end:
> #J1*J2
> ProductId:=proc(J1,J2,TT)
> local i,j:
> Basis([op(F),seq(seq(modp(expand(J1[i]*J2[j]),p),j=1..nops(J2)),i=1..nops(J1))],TT):
> end:

```

```

> #Arita's Algorithm
> AritaAlg:=proc(J12,Tlex,C_A)
> local J,fff,J3,J4,J5,h,i3:
> fff:=J12[1]:# step 2 of algorithm
> J3:=QuotientId([fff,op(F)],J12,C_A):#step
3
> J3:=Basis(J3,C_A):#step 3
> h:=modp(expand(op(1,J3)/lcoeff(op(1,J3))),p):#step
3
> # if modp(h-(coeff(h,y,3)*F),p)=0
then h:=J3[2] fi:
> i3:=1:
> while NormalForm(h,[op(F)],C_A)=0
and i3 < nops(J3) do
> i3:=i3+1:
> h:=J3[i3]:
> end do:
> if nops(J3)<i3 then print("Error"):
fi:
> J4:=Basis([op(F),seq(h*J12[i],i=1..nops(J12))],C_A):
> J5:=xQuotient(J4,fff,C_A):
> end:
> SumId:=proc(I1,I2)
> local Multi,Ans;
> Multi:=ProductId(I1,I2,C_A);
> Ans:=AritaAlg(Multi,Tlex,C_A);
> return Ans:
> end:
> Powern:=proc(n,II)
> local r,e,i,J12;
> r:=[1];
> e:=II;
> i:=n;
> while(i>0) do
> if(i mod 2)=1 then
> J12:=ProductId(r,e,C_A);r:=AritaAlg(J12,Tlex,C_A):
> print(r);
> i:=((i-1)/2);
> else
> i:=(i/2);
> fi;
> if(i>0) then
> J12:=ProductId(e,e,C_A);
e:=AritaAlg(J12,Tlex,C_A);
> print(e);
> fi;
> end do;
> return r;
> end:

```