

# Positive Solutions for a Class of Semipositone Discrete Boundary Value Problems with Two Parameters

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**Abstract**— In this paper, the existence, multiplicity and noexistence of positive solutions for a class of semipositone discrete boundary value problems with two parameters is studied by applying nonsmooth critical point theory and sub-super solutions method.

**Keywords**— Discrete boundary value problems, nonsmooth critical point theory, positive solutions, semipositone, sub-super solutions method

## I. INTRODUCTION

LET  $\mathbf{Z}$  and  $\mathbf{R}$  be the set of all integers and real numbers respectively. For  $a, b \in \mathbf{Z}$ , define  $\mathbf{Z}(a) = \{a, a+1, \dots\}$ ,  $\mathbf{Z}(a, b) = \{a, a+1, \dots, b\}$  when  $a \leq b$ .

In this paper, a class of semipositone discrete boundary value problem with two parameters

$$\begin{cases} -\Delta^2 u(t-1) = \lambda f(u(t)) + \mu g(u(t)), t \in \mathbf{Z}(1, N), \\ u(0) = 0, u(N+1) = 0, \end{cases} \quad (1)$$

is considered, where  $\lambda, \mu > 0$  are parameters,  $N \geq 4$  is a positive integer,  $\Delta u(t) = u(t+1) - u(t)$  is the forward difference operator,  $\Delta^2 u(t) = \Delta(\Delta u(t))$ ,  $f: [0, +\infty) \rightarrow \mathbf{R}$  is a continuous positive function satisfying  $f(0) > 0$ , and  $g: [0, +\infty) \rightarrow \mathbf{R}$  is continuous and eventually strictly positive with  $g(0) < 0$ .

It is easy to see that for fixed  $\mu > 0$ ,  $\lambda f(0) + \mu g(0) < 0$  whenever  $\lambda > 0$  is sufficiently small. Then (1) is called a semipositone problem. Semipositone problems derive from [8], where Castro and Shivaji initially called them nonpositone problems, in contrast with the terminology positone problems, put forward by Cohen and Keller in [11], where the nonlinearity was positive and monotone. Semipositone problems arise in bulking of mechanical systems, design of suspension bridges, chemical reactions, astrophysics, combustion and management of natural resources. For example, see [1, 20, 22, 23].

In general, studying positive solutions for semipositone problems is more difficult than that for positone problems. The difficulty is due to the fact that in the semipositone case, solutions have to live in regions where the nonlinear term is negative as well as positive. However, many methods have been applied to deal with semipositone problems, the usual

approaches are quadrature method, fixed point theory, sub-super solutions method and degree theory. We refer the readers to the survey papers [9, 18] and references therein.

Due to its importance, in recent years, continuous semipositone problems have been widely studied by many authors, see [5, 6, 12–16, 19, 21, 24, 25]. However, there were only a few papers on discrete semipositone problems. One can refer to [2, 4, 7, 17]. In these papers, semipositone discrete boundary value problems with one parameter were discussed, and sub-super solutions method and fixed point theory were used to study them. To the author's best knowledge, there are no results established on semipositone discrete boundary value problems with two parameters. Here a different approach to deal with this topic will be presented. In [13], Costa, Tehrani and Yang applied the nonsmooth critical point theory developed by Chang [10] to study the existence and multiplicity results of a class of semipositone boundary value problems with one parameter. It is also an efficient tool in dealing with the semipositone discrete boundary value problems with two parameters. For knowledge about difference equation, one can see [3].

The main objective in this paper is to apply the nonsmooth critical point theory to deal with the positive solutions of semipositone problem (1). More precisely, the discontinuous nonlinear terms

$$f_1(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ f(s) & \text{if } s > 0, \end{cases}$$

and

$$g_1(s) = \begin{cases} 0 & \text{if } s \leq 0, \\ g(s) & \text{if } s > 0. \end{cases}$$

will be considered.

Next, consider the slightly modified problem

$$\begin{cases} -\Delta^2 u(t-1) = \lambda f_1(u(t)) + \mu g_1(u(t)), t \in \mathbf{Z}(1, N), \\ u(0) = 0, u(N+1) = 0. \end{cases} \quad (2)$$

Just to be on the convenient side, the following formulas  $h(s) = \lambda f(s) + \mu g(s)$ ,  $h_1(s) = \lambda f_1(s) + \mu g_1(s)$ ,  $H(s) = \lambda F(s) + \mu G(s)$ ,  $H_1(s) = \lambda F_1(s) + \mu G_1(s)$  are defined, where  $F(s) = \int_0^s f(\tau) d\tau$ ,  $G(s) = \int_0^s g(\tau) d\tau$ ,  $F_1(s) = \int_0^s f_1(\tau) d\tau = \begin{cases} 0 & \text{if } s \leq 0 \\ F(s) & \text{if } s > 0 \end{cases}$ ,  $G_1(s) = \int_0^s g_1(\tau) d\tau = \begin{cases} 0 & \text{if } s \leq 0 \\ G(s) & \text{if } s > 0 \end{cases}$ .

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In section 3, it will be proved that the sets of positive solutions of (1) and (2) do coincide. Moreover, any nonzero solution of (2) is nonnegative.

The following are the main results.

**Theorem 1.** Suppose that there are constants  $C_1 > 0$ ,  $\alpha > 1$  and  $\beta > 2$  such that when  $s > 0$  is large enough,

$$f(s) < C_1 s^\alpha, \quad (3)$$

$$sf(s) \geq \beta F(s) > 0, \quad (4)$$

and

$$\lim_{s \rightarrow +\infty} \frac{g(s)}{s} = 0. \quad (5)$$

Then for fixed  $\mu > 0$ , there is a  $\bar{\lambda} > 0$  such that for  $\lambda \in (0, \bar{\lambda})$ , problem (2) has a nontrivial nonnegative solution. Hence problem (1) has a positive solution.

**Remark 1.** By (4), there are constants  $C_2, C_3 > 0$  such that for any  $s \geq 0$ ,

$$F(s) \geq C_2 s^\beta - C_3. \quad (6)$$

(4) and (6) imply that

$$\lim_{s \rightarrow +\infty} \frac{f(s)}{s} = +\infty,$$

which shows that  $f$  is superlinear at infinity.

**Remark 2.** (5) implies that  $g$  is sublinear at infinity. Moreover, it is easy to know that

$$\lim_{s \rightarrow +\infty} \frac{G(s)}{s^2} = 0.$$

Hence  $G$  is subquadratic at infinity.

**Theorem 2.** Suppose that the conditions of Theorem 1 hold. Moreover,  $g$  is increasing on  $[0, +\infty)$ . Then there is a  $\mu^* > 0$  such that for  $\mu > \mu^*$ , problem (1) has at least two positive solutions for sufficiently small  $\lambda$ .

**Theorem 3.** Suppose that the conditions of Theorem 1 hold. Moreover,  $f$  is nondecreasing on  $[0, +\infty)$ . Then for fixed  $\mu > 0$ , problem (1) has no positive solution for sufficiently large  $\lambda$ .

## II. PRELIMINARIES

In this section, some basic results on variational method for locally Lipschitz functional  $I : X \rightarrow \mathbf{R}$  defined on a real Banach space  $X$  with norm  $\|\cdot\|$  are recalled.  $I$  is called locally Lipschitzian if for each  $u \in X$ , there is a neighborhood  $V = V(u)$  of  $u$  and a constant  $B = B(u)$  such that

$$|I(x) - I(y)| \leq B\|x - y\|, \forall x, y \in V.$$

The following abstract theory has been developed by Chang [10].

**Definition 1.** For given  $u, z \in X$ , the generalized directional derivative of the functional  $I$  at  $u$  in the direction  $z$  is defined by

$$I^0(u; z) = \limsup_{k \rightarrow 0} \frac{1}{t} [I(u + k + tz) - I(u + k)].$$

The following properties are known:

- (i)  $z \rightarrow I^0(u; z)$  is sub-additive, positively homogeneous, continuous and convex;
- (ii)  $|I^0(u; z)| \leq B\|z\|$ ;
- (iii)  $I^0(u; -z) = (-I)^0(u; z)$ .

**Definition 2.** The generalized gradient of  $I$  at  $u$ , denoted by  $\partial I(u)$ , is defined to be the subdifferential of the convex function  $I^0(u; z)$  at  $z = 0$ , that is,

$$w \in \partial I(u) \subset X^* \iff \langle w, z \rangle \leq I^0(u; z), \forall z \in X.$$

The generalized gradient  $\partial I(u)$  has the following main properties:

- (1) For all  $u \in X$ ,  $\partial I(u)$  is a non-empty convex and  $w^*$ -compact subset of  $X^*$ ;
- (2)  $\|w\|_{X^*} \leq B$  for all  $w \in \partial I(u)$ ;
- (3) If  $I, J : X \rightarrow \mathbf{R}$  are locally Lipschitz functional, then

$$\partial(I + J)(u) \subset \partial I(u) + \partial J(u);$$

- (4) For any  $\lambda > 0$ ,  $\partial(\lambda I)(u) = \lambda \partial I(u)$ ;
- (5) If  $I$  is a convex functional, then  $\partial I(u)$  coincides with the usual subdifferential of  $I$  in the sense of convex analysis;
- (6) If  $I$  is Gâteaux differential at every point of  $v$  of a neighborhood  $V$  of  $u$  and the Gâteaux derivative is continuous, then  $\partial I(u) = \{I'(u)\}$ ;
- (7) The function

$$\zeta(u) = \min_{w \in \partial I(u)} \|w\|_{X^*}$$

exists, that is, there is a  $w_0 \in \partial I(u)$  such that  $\|w_0\|_{X^*} = \min_{w \in \partial I(u)} \|w\|_{X^*}$ ;

$$(8) I^0(u; z) = \max\{\langle w, z \rangle | w \in \partial I(u)\};$$

$$(9) \text{ If } I \text{ has a minimum at } u_0 \in X, \text{ then } 0 \in \partial I(u_0).$$

**Definition 3.**  $u \in X$  is a critical point of the locally Lipschitz functional  $I$  if  $0 \in \partial I(u)$ .

**Definition 4.**  $I$  is said to satisfy Palais-Smale condition ((PS) condition for short), if any sequence  $\{u_n\}$  such that  $I(u_n)$  is bounded and  $\zeta(u_n) = \min_{w \in \partial I(u_n)} \|w\|_{X^*} \rightarrow 0$  has a convergent subsequence.

**Lemma 1.** (Mountain Pass Theorem [10]) Let  $X$  be a real Hilbert space and  $I$  be a locally Lipschitz functional satisfying (PS) condition. Suppose  $I(0) = 0$  and

- (i) There exist constants  $\rho > 0$  and  $a > 0$  such that  $I(u) \geq a$  if  $\|u\| = \rho$ ;
- (ii) There is an  $e \in X$  such that  $\|e\| > \rho$  and  $I(e) \leq 0$ .

Then  $I$  possesses a critical value  $c \geq a$ . Moreover,  $c$  can be characterized as

$$c = \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} I(\gamma(s)),$$

where

$$\Gamma = \{g \in C([0, 1], X) | \gamma(0) = 0, \gamma(1) = e\}.$$

Next the definitions of the subsolution and the supersolution of the following boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = \mu g(u(t)), t \in \mathbf{Z}(1, N), \\ u(0) = 0, u(N+1) = 0. \end{cases} \quad (7)$$

are given.

**Definition 5.** If  $u_1(t), t \in \mathbf{Z}(0, N+1)$  satisfies the following conditions

$$\begin{cases} -\Delta^2 u_1(t-1) \leq \mu g(u_1(t)), t \in \mathbf{Z}(1, N), \\ u_1(0) \leq 0, u_1(N+1) \leq 0, \end{cases}$$

then  $u_1$  is called a subsolution of problem (7).

**Definition 6.** If  $u_2(t), t \in \mathbf{Z}(0, N+1)$  satisfies the following conditions

$$\begin{cases} -\Delta^2 u_2(t-1) \geq \mu g(u_2(t)), t \in \mathbf{Z}(1, N), \\ u_2(0) \geq 0, u_2(N+1) \geq 0, \end{cases}$$

then  $u_2$  is called a supersolution of problem (7).

**Lemma 2.** Suppose that there exist a subsolution  $u_1$  and a supersolution  $u_2$  of problem (7) such that  $u_1(t) \leq u_2(t)$  in  $\mathbf{Z}(1, N)$ . Then there is a solution  $\tilde{u}$  of problem (7) such that  $u_1(t) \leq \tilde{u}(t) \leq u_2(t)$  in  $\mathbf{Z}(1, N)$ .

**Remark 3.** If (7) is replaced by (1), then similar definitions and result as definitions 5, 6 and Lemma 2 can be obtained.

### III. PROOF OF MAIN RESULTS

Let  $E$  be the class of the functions  $u : \mathbf{Z}(0, N+1) \rightarrow \mathbf{R}$  such that  $u(0) = u(N+1) = 0$ . Equipped with the usual inner product and the usual norm

$$(u, v) = \sum_{t=1}^N (u(t), v(t)), \|u\| = \left( \sum_{t=1}^N u^2(t) \right)^{1/2},$$

$E$  is a  $N$ -dimensional Hilbert space. Define the functional  $J$  on  $E$  as

$$\begin{aligned} J(u) &= \frac{1}{2} \sum_{t=1}^{N+1} [(\Delta u(t-1))^2 - 2H_1(u(t))] \\ &= \frac{1}{2} u^T A u - \sum_{t=1}^N H_1(u(t)) = K(u) - \sum_{t=1}^N H_1(u(t)), \end{aligned}$$

where  $u = \{u(1), u(2), \dots, u(N)\}$ ,  $K(u) = \frac{1}{2} u^T A u$  and

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{N \times N}.$$

Clearly  $H_1$  is a locally Lipschitz function and  $J(u)$  is a locally Lipschitz functional on  $E$ . By a simple computation, the following result

$$\frac{\partial}{\partial u(t)} K(u) = 2u(t) - u(t+1) - u(t-1) = -\Delta^2 u(t-1).$$

holds. By Theorem 2.2 of [10], the critical point of the functional  $J(u)$  is a solution of the inclusion

$$-\Delta^2 u(t-1) \in [h_1(u(t)), \bar{h}_1(u(t))], t \in \mathbf{Z}(1, N),$$

where  $h_1(s) = \min[h_1(s+0), h_1(s-0)]$ ,  $\bar{h}_1(s) = \max[h_1(s+0), h_1(s-0)]$ .

**Remark 4.** It is easy to see that  $\underline{h}_1(s) = \overline{h}_1(s) = \lambda f(s) + \mu g(s)$  for  $s > 0$ ,  $\underline{h}_1(s) = \overline{h}_1(s) = 0$  for  $s < 0$ . For fixed  $\mu$  and sufficiently small  $\lambda$ ,  $\lambda f(0) + \mu g(0) < 0$ . Then  $\underline{h}_1(0) = \lambda f(0) + \mu g(0)$ ,  $\bar{h}_1(0) = 0$ .

**Remark 5.** If  $u > 0$ , then the above inclusion becomes

$$-\Delta^2 u(t-1) = \lambda f(u(t)) + \mu g(u(t)), t \in \mathbf{Z}(1, N).$$

It is clear that  $A$  is a positive definite matrix. Let  $\eta_{max} > 0, \eta_{min} > 0$  be the largest and smallest eigenvalue of  $A$  respectively. Denote by  $u^- = \max\{-u, 0\}$ . Let  $P_1 = \{t \in \mathbf{Z}(1, N) | u(t) \leq 0\}$ ,  $P_2 = \{t \in \mathbf{Z}(1, N) | u(t) > 0\}$ . Notice that  $u^-(t) = 0$  for  $t \in P_2$  and  $f_1(u(t)) = 0$  for  $t \in P_1$ . Then

$$\begin{aligned} & \sum_{t=1}^N f_1(u(t)) u^-(t) \\ &= \sum_{t \in P_1} f_1(u(t)) u^-(t) + \sum_{t \in P_2} f_1(u(t)) u^-(t) = 0. \end{aligned}$$

Similarly,  $g_1(u(t)) = 0$  for  $t \in P_1$ . Hence

$$\begin{aligned} & \sum_{t=1}^N g_1(u(t)) u^-(t) \\ &= \sum_{t \in P_1} g_1(u(t)) u^-(t) + \sum_{t \in P_2} g_1(u(t)) u^-(t) = 0. \end{aligned}$$

**Lemma 3.** If  $u$  is a solution of (2), then  $u \geq 0$ . Moreover, either  $u > 0$  in  $\mathbf{Z}(1, N)$ , or  $u = 0$  everywhere.

**Proof.** It is not difficult to see that  $(\Delta u^-(t) + \Delta u(t)) \Delta u^-(t) \leq 0$  for  $t \in \mathbf{Z}(0, N)$ . In fact, no matter that  $\Delta u(t) \geq 0$  or  $\Delta u(t) < 0$ , the former inequality holds. Hence  $\Delta u^-(t) \cdot \Delta u(t) \leq -(\Delta u^-(t))^2$ .

If  $u$  is a solution of (2), then

$$\begin{aligned} 0 &= \sum_{t=1}^N [\Delta^2 u(t-1) + \lambda f_1(u(t)) + \mu g_1(u(t))] u^-(t) \\ &= -\sum_{t=1}^{N+1} \Delta u(t-1) \Delta u^-(t-1) \\ &\quad + \sum_{t=1}^N [\lambda f_1(u(t)) + \mu g_1(u(t))] u^-(t) \\ &\geq \sum_{t=1}^{N+1} (\Delta u^-(t-1))^2 = (u^-)^T A u^- \geq \eta_{min} \|u^-\|^2. \end{aligned}$$

So  $u^- = 0$ . Hence  $u \geq 0$ . If  $u(t) = 0$ , then

$$\begin{aligned} & u(t+1) + u(t-1) \\ &= \Delta^2 u(t-1) = -\lambda f_1(u(t)) - \mu g_1(u(t)) \\ &= -\lambda f_1(0) - \mu g_1(0) = 0. \end{aligned}$$

Therefore  $u(t+1) = u(t-1) = 0$ . It follows that  $u = 0$  everywhere.

**Lemma 4.** If (4) and (5) hold, then  $h_1(s)s \geq \beta_0 H_1(s)$  for large  $s > 0$ , where  $\beta_0 \in (2, \beta)$ .

**Proof.** Notice that  $h_1(s)s \geq \beta_0 H_1(s)$  is equivalent to  $h(s)s \geq \beta_0 H(s)$  if  $s > 0$ . To prove that  $h(s)s \geq \beta_0 H(s)$

for large  $s > 0$ , it suffices to show that

$$\lim_{s \rightarrow +\infty} \frac{h(s)s}{\beta_0 H(s)} > 1.$$

By (4), for large  $s > 0$ , we have

$$\frac{\beta_0 F(s)}{f(s)s} \leq \frac{\beta_0}{\beta}.$$

Hence, if  $s > 0$  is large, then

$$\begin{aligned} \frac{h(s)s}{\beta_0 H(s)} &= \frac{\lambda f(s)s + \mu g(s)s}{\beta_0(\lambda f(s) + \mu g(s))} = \frac{1 + \frac{\mu g(s)}{\lambda f(s)}}{\frac{\beta_0 F(s)}{f(s)s} + \frac{\beta_0 \mu G(s)}{\lambda f(s)s}} \\ &\geq \frac{1 + \frac{\mu g(s)}{\lambda f(s)}}{\frac{\beta_0}{\beta} + \frac{\beta_0 \mu G(s)}{\lambda f(s)s}}. \end{aligned}$$

Taking inferior limit on both side of the above inequality, then

$$\begin{aligned} \lim_{s \rightarrow +\infty} \frac{h(s)s}{\beta_0 H(s)} &\geq \lim_{s \rightarrow +\infty} \frac{1 + \frac{\mu g(s)}{\lambda f(s)}}{\frac{\beta_0}{\beta} + \frac{\beta_0 \mu G(s)}{\lambda f(s)s}} \\ &\geq \frac{\lim_{s \rightarrow +\infty} (1 + \frac{\mu g(s)}{\lambda f(s)})}{\lim_{s \rightarrow +\infty} (\frac{\beta_0}{\beta} + \frac{\beta_0 \mu G(s)}{\lambda f(s)s})}. \end{aligned}$$

Since  $f$  is superlinear and  $g$  is sublinear,  $\lim_{s \rightarrow +\infty} \frac{\mu g(s)}{\lambda f(s)} = 0$ . Then  $\lim_{s \rightarrow +\infty} (1 + \frac{\mu g(s)}{\lambda f(s)}) = \lim_{u \rightarrow +\infty} (1 + \frac{\mu g(s)}{\lambda f(s)}) = 1$ . Moreover, since  $G$  is subquadratic and  $f$  is superlinear,

$$\begin{aligned} \lim_{u \rightarrow +\infty} \frac{G(s)}{f(s)s} &= \lim_{s \rightarrow +\infty} \frac{\frac{G(s)}{s^2}}{\frac{f(s)}{s^2}} = 0. \text{ Therefore,} \\ \lim_{s \rightarrow +\infty} (\frac{\beta_0}{\beta} + \frac{\beta_0 \mu G(s)}{\lambda f(s)s}) &= \lim_{s \rightarrow +\infty} (\frac{\beta_0}{\beta} + \frac{\beta_0 \mu G(s)}{\lambda f(s)s}) = \frac{\beta_0}{\beta}. \end{aligned}$$

From the above results, it follows that  $\lim_{s \rightarrow +\infty} \frac{h(s)s}{\beta_0 H(s)} \geq \frac{\beta}{\beta_0} > 1$ .

**Lemma 5.** If (4) and (5) hold, then  $J$  satisfies (PS) condition.

**Proof.** Notice that  $E^* = E$ . Let  $L(u) = \sum_{t=1}^N H_1(u(t))$ . From Theorem 2.2 of [10], for any given  $w \in \partial L(u) \subset E^*$ , we have  $w(t) \in [\underline{h}_1(u(t)), \overline{h}_1(u(t))]$ . Then

$$w(t) = \lambda f_1(u(t)) + \mu g_1(u(t)) \text{ if } u(t) \neq 0,$$

$$w(t) \in [\lambda f(0) + \mu g(0), 0] \text{ if } u(t) = 0.$$

Therefore

$$\langle w, u \rangle = \sum_{t=1}^N h_1(u(t))u(t) \text{ for all } w \in \partial L(u).$$

By Lemma 4, there is a constant  $M > 0$  such that  $L(u) \leq \frac{1}{\beta_0} \langle w, u \rangle + M$  for  $u \in \mathbf{R}^N$ . Suppose that  $\{u_n\}$  is a sequence such that  $J(u_n)$  is bounded and  $\zeta(u_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then by properties 3 and 7, there are  $C > 0$  and  $w_n \in \partial L(u_n)$  such that  $|J(u_n)| \leq C$  and

$$|\langle \partial K(u_n) - w_n, u_n \rangle| \leq \|u_n\| \text{ for sufficiently large } n.$$

It implies that

$$u_n^T A u_n - \langle w_n, u_n \rangle \geq -\|u_n\|.$$

Hence

$$\begin{aligned} C &\geq \frac{1}{2} u_n^T A u_n - L(u_n) \\ &\geq \frac{1}{2} u_n^T A u_n - \frac{1}{\beta_0} \langle w_n, u_n \rangle - M \\ &= (\frac{1}{2} - \frac{1}{\beta_0}) u_n^T A u_n + \frac{1}{\beta_0} [u_n^T A u_n - \langle w_n, u_n \rangle] - M \\ &\geq (\frac{1}{2} - \frac{1}{\beta_0}) \eta_{\min} \|u_n\|^2 - \frac{1}{\beta_0} \|u_n\| - M. \end{aligned}$$

This implies that  $\{u_n\}$  is bounded. Since  $E$  is finite dimensional,  $\{u_n\}$  has a convergent subsequence in  $E$ .

**Lemma 6.** For fixed  $\mu > 0$ , there exist  $\rho > 0$  and  $\bar{\lambda} > 0$  such that if  $\lambda \in (0, \bar{\lambda})$ , then  $J(u) \geq \frac{\eta_{\min} M_1^2}{16} \lambda^{-\frac{2}{\alpha-1}}$  for  $\|u\| = \rho$ .

**Proof.** By (3) and (5), there are  $C_4, C_5 > 0$  such that

$$F_1(s) \leq \frac{C_1 |s|^{\alpha+1}}{\alpha+1} + C_4 \text{ for all } s \in \mathbf{R}, \quad (8)$$

$$G_1(s) \leq \frac{\eta_{\min}}{4\mu} |s|^2 + C_5 \text{ for all } s \in \mathbf{R}. \quad (9)$$

The equivalence of norm on  $E$  implies that there exists  $C_6 > 0$  such that  $\|u\|_{\alpha+1} \leq C_6 \|u\|$ , where  $\|u\|_{\alpha+1} = (\sum_{t=1}^N |u(t)|^{\alpha+1})^{\frac{1}{\alpha+1}}$ . Let  $M_1 = (\frac{\eta_{\min}(\alpha+1)}{8C_1 C_6^{\alpha+1}})^{\frac{1}{\alpha-1}}$  and  $\rho = M_1 \lambda^{-\frac{1}{\alpha-1}}$ . Let  $\|u\| = \rho$ . It follows from (8) and (9) that there is  $\bar{\lambda} > 0$  such that if  $\lambda \in (0, \bar{\lambda})$ , then

$$\begin{aligned} J(u) &= \frac{1}{2} u^T A u - \sum_{t=1}^N H_1(u(t)) \\ &\geq \frac{1}{2} \eta_{\min} \|u\|^2 - \frac{\lambda C_1}{\alpha+1} \sum_{t=1}^N |u(t)|^{\alpha+1} - \lambda C_4 N \\ &\quad - \frac{\eta_{\min}}{4\mu} \cdot \mu \sum_{t=1}^N |u(t)|^2 - \mu C_5 N \\ &\geq \frac{1}{4} \eta_{\min} \|u\|^2 - \frac{\lambda C_1 C_6^{\alpha+1}}{\alpha+1} \|u\|^{\alpha+1} - \lambda C_4 N - \mu C_5 N \\ &= \lambda^{-\frac{2}{\alpha-1}} \left( \frac{\eta_{\min} M_1^2}{8} - \lambda^{\frac{\alpha+1}{\alpha-1}} C_4 N - \lambda^{\frac{2}{\alpha-1}} \mu C_5 N \right) \\ &\geq \frac{\eta_{\min} M_1^2}{16} \lambda^{-\frac{2}{\alpha-1}}. \end{aligned}$$

**Lemma 7.** There is an  $e \in E$  such that  $\|e\| > \rho$  and  $J(e) < 0$ .

**Proof.** It follows from Remark 1 that  $F(s) \geq C_2 s^\beta - C_3$  for  $s > 0$ . By the equivalence of the norms on  $E$ , there exists  $C_7 > 0$  such that  $\|u\|_\beta \geq C_7 \|u\|$ , where  $\|u\|_\beta = (\sum_{t=1}^N |u(t)|^\beta)^{\frac{1}{\beta}}$ . Let  $v_1$  be the eigenfunction to the principal eigenvalue  $\eta_1$  of

$$-\Delta^2 u(t-1) = \eta u(t), t \in \mathbf{Z}(1, N),$$

$$u(0) = 0, u(N+1) = 0$$

with  $v_1 > 0$  and  $\|v_1\| = 1$ . Let

$$G_m = \min\{G(u) | u \in [0, +\infty)\}.$$

Clearly  $G_m < 0$ . Since  $\beta > 2$ , for  $k > 0$ ,

$$\begin{aligned} J(kv_1) &= \frac{1}{2}k^2 v_1^T A v_1 - \lambda \sum_{t=1}^N F(kv_1(t)) - \mu \sum_{t=1}^N G(kv_1(t)) \\ &\leq \frac{\eta_{max}}{2} k^2 - \lambda C_2 (C_7 k)^\beta + \lambda C_3 N - \mu G_m N \\ &\rightarrow -\infty \text{ as } k \rightarrow +\infty \end{aligned}$$

Hence there is a  $k_1 > \rho$  such that  $J(k_1 v_1) < 0$ . Let  $e = k_1 v_1$ . Then  $\|e\| > \rho$  and  $J(e) < 0$ . The second condition of Mountain Pass Theorem is verified.

**Proof of Theorem 1.** Clearly  $J(0) = 0$ . Lemma 5 implies that  $J$  satisfies (PS) condition. It follows from Lemma 6, Lemma 7 and Lemma 1,  $J$  has a nontrivial critical point  $\hat{u}$  such that  $J(\hat{u}) \geq \frac{\eta_{min} M_1^2}{16} \lambda^{-\frac{2}{\alpha-1}}$ . By Lemma 3 and Remark 5,  $\hat{u}$  is a positive solution of (1). The proof is complete.

**Proof of Theorem 2.** The sub-super solutions method will be applied to prove the multiplicity results.

Firstly it will be proved that there exists  $\mu^* > 0$  such that if  $\mu > \mu^*$ , then the following boundary value problem

$$\begin{cases} -\Delta^2 u(t-1) = \mu g(u(t)), t \in \mathbf{Z}(1, N), \\ u(0) = 0, u(N+1) = 0 \end{cases} \quad (10)$$

has a positive solution  $\underline{u}$ . In fact, since  $g(u)$  is increasing on  $[0, +\infty)$  and eventually strictly positive,  $g(u) \geq -C_8$  for  $u \geq 0$  and some  $C_8 > 0$ . Let  $r_1$  be the eigenfunction to the principal eigenvalue  $\mu_1$  of

$$-\Delta^2 u(t-1) = \mu u(t), t \in \mathbf{Z}(1, N),$$

$$u(0) = 0, u(N+1) = 0$$

with  $r_1 > 0$  and  $\|r_1\| = 1$ .

Notice that  $\mu_1 = 2 - 2 \cos \frac{\pi}{N+1}$  and  $r_1(t) = \sin \frac{\pi t}{N+1}$  (see [3]). Let  $C_9 > 0$  be a constant such that  $C_9 \leq 2 \sin^2 \frac{\pi}{N+1} \cos \frac{2\pi}{N+1}$ . For  $t \in Q_1 = \{t \in \mathbf{Z}(1, N) | t = 1 \text{ or } t = N\}$ ,  $N \geq 4$ , we have  $(\Delta r_1(t))^2 + (\Delta r_1(t-1))^2 - 2\mu_1 r_1^2(t) = 2 \sin^2 \frac{\pi}{N+1} \cos \frac{2\pi}{N+1} \geq C_9 > 0$ .

It will be verified that  $\psi = \frac{\mu C_8}{C_9} r_1^2$  is a subsolution of (10) for  $\mu$  large. Notice that

$$\begin{aligned} -\Delta^2 r_1^2(t-1) &= 2r_1^2(t) - r_1^2(t+1) - r_1^2(t-1) \\ &= 2r_1^2(t) - (r_1(t) + \Delta r_1(t))^2 \\ &\quad - (r_1(t) - \Delta r_1(t-1))^2 \\ &= 2\mu_1 r_1^2(t) - (\Delta r_1(t))^2 - (\Delta r_1(t-1))^2. \end{aligned}$$

On the other hand, for  $t \in Q_1$ , we have  $(\Delta r_1(t))^2 + (\Delta r_1(t-1))^2 - 2\mu_1 r_1^2(t) \geq C_9$ , which implies that

$$\frac{C_8}{C_9} [2\mu_1 r_1^2(t) - (\Delta r_1(t))^2 - (\Delta r_1(t-1))^2] - g(\psi(t)) \leq 0.$$

Then for  $t \in Q_1$ ,  $-\Delta^2 \psi(t-1) \leq \mu g(\psi(t))$ . Next, for  $t \in \mathbf{Z}(1, N) \setminus Q_1$ , we have  $r_1(t) \geq \bar{r}$  for some  $\bar{r} > 0$  and  $\frac{C_8}{C_9} r_1^2(t) \geq C_{10}$  for some  $C_{10} = \frac{C_8}{C_9} \bar{r}^2 > 0$ . Hence  $\psi(t) = \frac{\mu C_8}{C_9} r_1^2(t) \geq \mu C_{10}$ . Since  $g$  is increasing and eventually

strictly positive, there is a  $\mu^* > 0$  such that if  $\mu > \mu^*$  and  $t \in \mathbf{Z}(1, N) \setminus Q_1$ ,

$$g(\psi(t)) \geq \frac{C_8}{C_9} \cdot 2\mu_1 \geq \frac{C_8}{C_9} [2\mu_1 r_1^2(t) - (\Delta r_1(t))^2 - (\Delta r_1(t-1))^2].$$

Hence for  $t \in \mathbf{Z}(1, N) \setminus Q_1$ ,  $-\Delta^2 \psi(t-1) \leq \mu g(\psi(t))$ . Notice that  $r_1(0) = 0, r_1(N+1) = 0$ . Then  $\psi(0) = 0, \psi(N+1) = 0$ . So we have

$$-\Delta^2 \psi(t-1) \leq \mu g(\psi(t)), t \in \mathbf{Z}(1, N),$$

$$\psi(0) \leq 0, \psi(N+1) \leq 0,$$

i.e.,  $\psi$  is a subsolution of (10).

Now it is necessary to look for the supersolution of (10).

Let  $z$  be a solution of

$$\begin{cases} -\Delta^2 u(t-1) = 1, t \in \mathbf{Z}(1, N), \\ u(0) = 0, u(N+1) = 0. \end{cases} \quad (11)$$

Then

$$\begin{aligned} z(s) &= \sum_{t=1}^N G(s, t) \\ &= \frac{1}{N+1} \left\{ \sum_{t=1}^{s-1} [(N+1) - s]t + \sum_{t=s}^N s[(N+1) - t] \right\} \\ &= \frac{s[(N+1) - s]}{2}, \end{aligned}$$

where

$$G(s, t) = \begin{cases} \frac{t[(N+1)-s]}{\frac{N+1}{N+1}} & 0 \leq t \leq s-1, \\ \frac{s[(N+1)-t]}{\frac{N+1}{N+1}} & s \leq t \leq N+1. \end{cases}$$

Clearly  $z(s) > 0$  for  $s \in \mathbf{Z}(1, N)$ ,  $z(0) = 0, z(N+1) = 0$ . Define  $\phi = \mu \sigma z$ , where  $\sigma > 0$  is large enough so  $\phi > \psi$  in  $\mathbf{Z}(1, N)$  and

$$\frac{g(\mu \sigma z)}{\sigma} < 1.$$

This is possible since  $g$  is a sublinear function. So

$$-\Delta^2 \phi(t-1) \geq \mu g(\phi(t)), t \in \mathbf{Z}(1, N),$$

$$\phi(0) \geq 0, \phi(N+1) \geq 0,$$

which shows that  $\phi$  is a supersolution of (10). Therefore, by Lemma 2, there is a solution  $\underline{u}$  of (10) such that  $\psi \leq \underline{u} \leq \phi$ .

Secondly it will be proved that  $\underline{u}$  is a subsolution of (1). Since  $\lambda > 0$  and  $f > 0$ , it follows that

$$-\Delta^2 \underline{u}(t-1) \leq \lambda f(\underline{u}(t)) + \mu g(\underline{u}(t)), t \in \mathbf{Z}(1, N),$$

$$\underline{u}(0) \leq 0, \underline{u}(N+1) \leq 0,$$

which implies that  $\underline{u}$  is a subsolution of (1).

Lastly the question is to look for the supersolution of (1) and prove the existence of positive solution of (1). Let  $z$  be as in (11). Notice that  $g$  is sublinear. Define  $\bar{u} = \xi z$ , where  $\xi > 0$  is independent of  $\lambda$  and large enough so  $\bar{u} \geq \underline{u}$  in  $\mathbf{Z}(1, N)$  and

$$\mu \frac{g(\xi z(t))}{\xi} < \frac{1}{2}.$$

Let  $\lambda > 0$  be so small that

$$\lambda \frac{f(\xi z(t))}{\xi} < \frac{1}{2}.$$

Then

$$-\Delta^2 \bar{u}(t-1) = \xi \geq \lambda f(\bar{u}(t)) + \mu g(\bar{u}(t)), t \in \mathbf{Z}(1, N),$$

$$\bar{u}(0) \geq 0, \bar{u}(N+1) \geq 0.$$

Hence  $\bar{u}$  is a supersolution of (1). Thus, by Remark 3, problem (1) has a solution  $\hat{u}$  such that  $\underline{u} \leq \hat{u} \leq \bar{u}$  for  $\mu > \mu^*$  and  $\lambda$  small, which is positive for  $t \in \mathbf{Z}(1, N)$ .

Now it is time to find the second positive solution of problem (1). Notice that  $\underline{u}$  and  $\bar{u}$  are independent of  $\lambda$ . Since  $f$  is positive on  $[0, +\infty)$ , by the definition of  $f_1$   $\sum_{t=1}^N F_1(u(t)) \geq 0$ . Then for  $u \in [\underline{u}, \bar{u}]$ ,

$$\begin{aligned} J(u) &= \frac{1}{2} u^T A u - \lambda \sum_{t=1}^N F_1(u(t)) - \mu \sum_{t=1}^N G_1(u(t)) \\ &\leq \frac{1}{2} u^T A u - \mu \sum_{t=1}^N G_1(u(t)) \leq J_0, \end{aligned}$$

where  $J_0 = \max_{u \in [\underline{u}, \bar{u}]} \left( \frac{1}{2} u^T A u - \mu \sum_{t=1}^N G_1(u(t)) \right)$ . On the other hand, by Lemma 6, one can take appropriate  $\bar{\lambda}$  such that if  $\lambda \in (0, \bar{\lambda})$ , then  $J(u) \geq \frac{\eta_{\min} M^2}{16} \lambda^{-\frac{2}{\alpha-1}} > J_0 + 1$  for  $\|u\| = \rho$ . Hence by Theorem 1,  $J(\hat{u}) > J_0$ . So  $\hat{u} \notin [\underline{u}, \bar{u}]$  and  $\hat{u} \neq \bar{u}$ , which shows that  $\hat{u}$  and  $\bar{u}$  are two different positive solutions of (1). The proof is complete.

**Proof of Theorem 3.** Just to be on the contradiction side, let  $u$  be a positive solution of (1). Since  $f$  is superlinear and increasing,  $f(0) > 0$ , there are  $C_{11}, C_{12} > 0$  such that for  $s \geq 0$ ,  $f(s) \geq C_{11}s + C_{12}$ . Hence for  $\lambda > 0$  and  $s \geq 0$ ,  $\lambda f(s) + \mu g(s) \geq \lambda(C_{11}s + C_{12}) + \mu G_m$ , where  $G_m$  is the same as that of the proof of Lemma 7. If  $\lambda$  is large enough, then  $\lambda C_{12} + \mu G_m \geq \frac{1}{2} \lambda C_{12}$ . Therefore  $\lambda f(s) + \mu g(s) \geq \lambda C_{11}s + \frac{1}{2} \lambda C_{12}$  for large  $\lambda > 0$  and  $s \geq 0$ . Multiplying both side of

$$-\Delta^2 y_1(t-1) = \lambda_1 y_1(t)$$

by  $u(t)$  and summing it from 1 to  $N$ , we get

$$\sum_{t=1}^N (-\Delta^2 y_1(t-1)) u(t) = \sum_{t=1}^N \lambda_1 y_1(t) u(t).$$

Multiplying both side of (1) by  $y_1(t)$  and summing it from 1 to  $N$ , we have

$$\sum_{t=1}^N (-\Delta^2 u(t-1)) y_1(t) = \sum_{t=1}^N (\lambda f(u(t)) + \mu g(u(t))) y_1(t).$$

It is easy to see that

$$\sum_{t=1}^N (-\Delta^2 u(t-1)) y_1(t) = \sum_{t=1}^N (-\Delta^2 y_1(t-1)) u(t).$$

Hence

$$\sum_{t=1}^N \lambda_1 y_1(t) u(t) = \sum_{t=1}^N (\lambda f(u(t)) + \mu g(u(t))) y_1(t),$$

$$\sum_{t=1}^N \lambda_1 u(t) y_1(t) \geq \sum_{t=1}^N (\lambda C_{11} u(t) + \frac{1}{2} \lambda C_{12}) y_1(t),$$

$$\sum_{t=1}^N (\lambda_1 - \lambda C_{12}) u(t) y_1(t) \geq \sum_{t=1}^N \frac{1}{2} \lambda C_{12} y_1(t).$$

For  $\lambda > \frac{\lambda_1}{C_{12}}$ , a contradiction exists. So for a given  $\mu > 0$ , (1) has no positive solution if  $\lambda$  is large. The proof is complete.

**Example.** An example is given to illustrate the result of Theorem 1. Let  $f(u) = u^3 + 1$  and  $g(u) = (u-1)^{\frac{2}{3}} - 2$ . Clearly  $f$  and  $g$  satisfy the conditions of Theorem 1. Then problem (1) has at least a positive solution.

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