

# State-Space PD Feedback Control

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**Abstract**—A challenged control problem is when the performance is pushed to the limit. The state-derivative feedback control strategy directly uses acceleration information for feedback and state estimation. The derivative part is concerned with the rate-of-change of the error with time. If the measured variable approaches the set point rapidly, then the actuator is backed off early to allow it to coast to the required level. Derivative action makes a control system behave much more intelligently. A sensor measures the variable to be controlled and the measured information is fed back to the controller to influence the controlled variable. A high gain problem can be also formulated for proportional plus derivative feedback transformation. Using MATLAB Simulink dynamic simulation tool this paper examines a system with a proportional plus derivative feedback and presents an automatic implementation of finding an acceptable controlled system. Using feedback transformations the system is transformed into another system.

**Keywords**—Feedback, PD, state-space, derivative.

## I. INTRODUCTION

FEEDBACK systems play an important role in control because the use of feedback can accomplish certain desirable results that would not be possible with an open-loop system. Consider the closed-loop control system of Fig. 1.

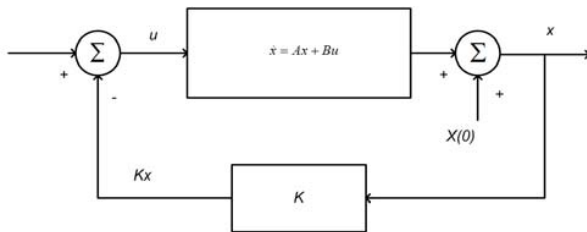


Fig. 1 State-space with proportional feedback

The feedback signal is considered to be proportional with the state variable  $x$ . Therefore the feedback control signal is:

$$u = -Kx \quad (1)$$

$$\dot{x} = Ax + Bu \quad (2)$$

$$y = Cx \quad (3)$$

If matrix  $F = A - BK$  then the condition for observability is that the matrix  $[C|CF|CF^2|\dots|CF^{n-1}]$  has rank  $n$ .

The system is controllable if matrix  $[B|BF|BF^2|\dots|BF^{n-1}]$  has rank  $n$ .

For an optimal control we can implement the methodology for the Linear Quadratic Regulator (LQR). By trying to minimize the cost function  $J$  we obtain the elements of the matrix  $K$ , [1].

LQR requires numerous iterations and is time consuming. In the case of implementing reconfiguration techniques may not be practical, a speedier methodology is required and the introduction of a PD feedback may be a reasonable approach.

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \quad (4)$$

$R$  represents the control effort or the control cost and  $Q$  is related with tracking error.  $Q$  and  $R$  are positive defined diagonal matrices. The elements of  $Q$  and  $R$  are chosen, and then the  $K$  matrix is calculated using Matlab. The process is repeated until an acceptable response is achieved.

## II. PROBLEM STATEMENT

If a derivative component is introduced in the feedback:

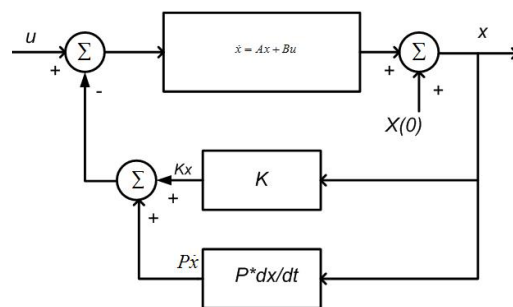


Fig. 2 State-space with derivative feedback

$$u = -Kx - P\dot{x} \quad (5)$$

$$\dot{x} = Ax - BKx - BP\dot{x} \quad (6)$$

The state space systems under proportional and derivative feedback are analyzed in depth in [2].

The problem is to find suitable  $K$  and  $P$  matrices. A series of transformations are performed that will lead to a better solution and keeping the controllability as well the efforts at a reasonable level.

$$(I + BP)\dot{x} = (A - BK)x \quad (7)$$

$$\dot{x} = (I + BP)^{-1} (A - BK)x \quad (8)$$

If we manage to make the expression similar to

$$\dot{x} = (A - BK)x \quad (9)$$

the problem is reduced to a classical state-space system control.

$$\dot{x} = [(I + BP)^{-1} A - (I + BP)^{-1} BK]x \quad (10)$$

By comparing the expressions:

$$a = (I + BP)^{-1} A \quad (11)$$

$$Bk = (I + BP)^{-1} BK \quad (12)$$

The last equation is multiplied by B transposed in order to find k.

$$B^T Bk = B^T (I + BP)^{-1} BK \quad (13)$$

$$k = (B^T B)^{-1} B^T (I + BP)^{-1} BK \quad (14)$$

The new transformed system is:

$$\dot{x} = (a - Bk)x \quad (15)$$

Note that if  $BP=0$  the new system will have the same state matrix A. This option is not taken into consideration.

We choose

$$(I + BP)^{-1} = rI, \quad (16)$$

We are going to analyze the behavior of the system in relation to the parameter  $r$ .

$$a = rA \quad (17)$$

$$k = r(B^T B)^{-1} (B^T B)K \quad (18)$$

$$k = rK \quad (19)$$

The matrix P is:

$$P = \frac{1-r}{r} (B^T B)^{-1} B^T \quad (20)$$

The new system with a derivative feedback is transformed in a state space-state control system with the matrices:

$a = r^*A$ ,  $B$ ,  $C$ ,  $D$ . The feedback is

$$k = r^*K \quad (21)$$

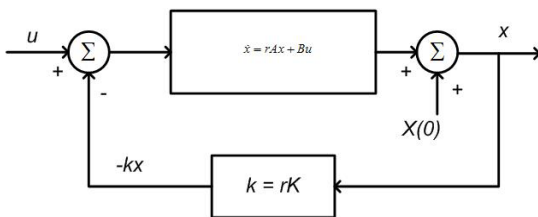


Fig. 3 Modified state-space control system

The observability matrix for the modified system has lines multiplied by the constants  $r$ ,  $r^2 \dots r^{n-1}$ , therefore the rank remains unchanged. For the controllability matrix the columns

are multiply by the factors  $r$ ,  $r^2 \dots r^{n-1}$ , therefore if the proportional feedback system is observable and controllable the PD feedback build as described would be observable and controllable as well.

### III. REFERENCE INPUT

A reference  $Ref$  is added to the control system. In a steady-state situation the output  $y$  is expected to be equal to the reference input  $Ref$ , [3].

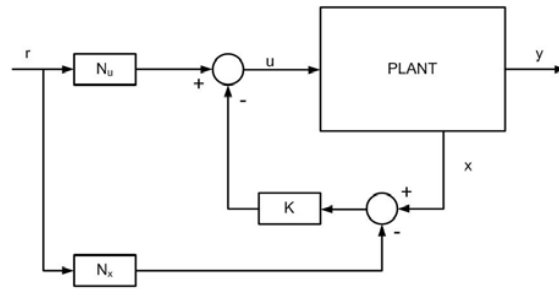


Fig. 4 Reference input with a single composite gain

The subscript  $ss$  stands for steady-state.

When the reference input reaches the steady level

$$Ref = Ref_{ss} \quad (22)$$

the system states become:

$$x = x_{ss} \quad (23)$$

The system output is expected to be

$$y = y_{ss} = Ref_{ss} \quad (24)$$

In steady-state

$$\dot{x} = 0 \quad (25)$$

The system equations become:

$$0 = Ax_{ss} + Bu_{ss} \quad (26)$$

$$Ref_{ss} = Cx_{ss} \quad (27)$$

The  $u_{ss}$  and  $x_{ss}$  are derived from the external reference signal  $Ref_{ss}$  as follows:

$$u_{ss} = N_u Ref_{ss} \quad (28)$$

$$x_{ss} = N_x \text{Ref}_{ss} \quad (29)$$

Substitute the  $u_{ss}$  and  $x_{ss}$  in the system equations:

$$0 = AN_x \text{Ref}_{ss} + BN_u \text{Ref}_{ss} \quad (30)$$

$$r_{ss} = CN_x \text{Ref}_{ss} \quad (31)$$

Solving the system of equations we find:

$$N_x = (C^T C)^{-1} \quad (32)$$

$$N_u = -(B^T B)^{-1} B^T A (C^T C)^{-1} \quad (33)$$

The control signal  $u$  is given by the following equation:

$$u = N_u \text{Ref} - K(x - N_x \text{Ref}) \quad (34)$$

Re-write the equation in another form:

$$u = (N_u + KN_x) \text{Ref} - Kx \quad (35)$$

$$u = N \text{Ref} - Kx \quad (36)$$

Where

$$N = N_u + KN_x \quad (37)$$

This leads to the following configuration:

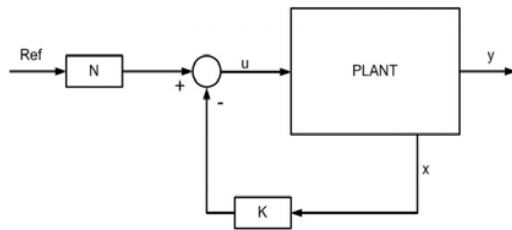


Fig. 5 Reference input with a single composite gain

The response of the new system can be plotted using the Matlab function `Impulse()`.

$$u = N \text{Ref} - Kx \quad (38)$$

The system equations are:

$$\dot{x} = ax + Bu \quad (39)$$

$$y = Cx + Du \quad (40)$$

Substitute  $u$  in the system equations:

$$\dot{x} = (a - Bk)x + BN \text{Ref} \quad (41)$$

$$y = (C - Dk)x + DN \text{Ref} \quad (42)$$

The matrices of the new system are:

$$A_{Ref} = a - Bk \quad (43)$$

$$B_{Ref} = BN \quad (44)$$

$$C_{Ref} = C - Dk \quad (45)$$

$$D_{Ref} = DN \quad (46)$$

The Matlab function `sys(ARef, BRef, CRef, DRef)` defines a system and the function `impulse(sys,20)` shows the evolution of the variable  $x$ .

The expected steady response would be  $x = 0$ .

#### IV. PRACTICAL EXAMPLE

An aircraft longitudinal control system described in [4] and [5] is used to illustrate the proposed feedback.

$$A = [-0.0582, 0.0651, 0, -0.171; -0.303, -0.685, 1.109, 0; \\ -0.0715, -0.658, -0.947, 0; 0, 0, 1, 0]$$

$$B = [0, 1; -0.0541, 1; -1.11, 0; 0, 0]$$

$$C = [1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1]$$

$$D = [0, 0; 0, 0; 0, 0; 0, 0]$$

The state variables are  $x(t) = [\alpha, \beta, \psi, \theta]^T$ , where  $\alpha$  is the forward speed,  $\beta$  is the vertical speed,  $\psi$  is the pitch rate and  $\theta$  is the pitch angle. The control signals are:  $\eta$  the elevator angle and  $\tau$  the throttle angle.

From the structure of the matrix  $C$  it is easy to see that the output variables are forward speed ( $y_1$ ), vertical speed ( $y_2$ ), pitch rate ( $y_3$ ) and pitch angle ( $y_4$ ).

The design starts by choosing a reasonable  $K$  matrix for the feedback. LQR is used to provide such a matrix. It is not essential to use LQR method to find the  $K$  matrix. Any other method can be implemented.

The next step is to find the derivative  $P$  matrix and the following procedure describes that by simple selecting a single parameter  $r$ . If we decide to use LQR to find  $K$  matrix we initialize the process by choosing the relative values.

$$J = \int_0^{\infty} \left[ \left( \frac{x_1}{x_{1\max}} \right)^2 + \left( \frac{x_2}{x_{2\max}} \right)^2 + \dots + \left( \frac{u_1}{u_{1\max}} \right)^2 + \left( \frac{u_2}{u_{2\max}} \right)^2 + \dots \right] dt$$

Where  $x_{1\max}$  is the maximum desired value of  $x_1$  and so forth.

We select the following diagonal matrices  $Q$  and  $R$ .

$$Q = [1/50^2, 0, 0, 0; 0, 1/0.25^2, 0, 0; 0, 0, 1/0.09, 0; 0, 0, 0, 1/0.3^2]$$

$$R = [1/0.2^2, 0; 0, 1/0.816^2];$$

The simulation is done using Simulink and Matlab.

Fig. 6 shows the Simulink diagram for the control system that cater for the three matrices  $K$ ,  $P$  and  $N$ .

The matrices of the state-space block are (43), (44), (45), (46).

To make the simulation easier I created a file in Matlab called AutoInput.m.

File: *AutoInput.m*

```
myopts=simset('MaxDataPoints',1000,'Refine',1);
```

```
r=input('Enter the first value for r, r1 = ');
```

```
CalcR
```

```
sim('Der_20Feb_N',20,myopts)
```

```
display41;
```

```
r=input('Enter the second value for r, r2 = ');
```

```
CalcR
```

```
sim('Der_20Feb_N',20,myopts)
```

```
display42
```

Other script files are *CalcR.m*, *display41* and *desplay42*.

*CalcR.m* calculates the matrices  $K$ ,  $N$  and the response of of the system.

File: *CalcR.m*

$$A = [-0.0582, 0.0651, 0, -0.171; -0.303, -0.685, 1.109, 0; -0.0715, -0.658, -0.947, 0; 0, 0, 1, 0];$$

$$B = [0, 1; -0.0541, 1; -1.11, 0; 0, 0];$$

$$C = [1, 0, 0, 0; 0, 1, 0, 0; 0, 0, 1, 0; 0, 0, 0, 1];$$

$$D = [0, 0; 0, 0; 0, 0; 0, 0];$$

```
states={'alpha' 'beta' 'psi' 'theta'};
```

```
inputs={'elevator' 'throttle'};
```

```
outputs={'forward speed' 'vertical speed' 'pitch rate' 'pitch angle'};
```

$$Q = [1/50^2, 0, 0, 0; 0, 1/0.25^2, 0, 0; 0, 0, 1/0.3^2, 0; 0, 0, 0, 1/0.3^2];$$

```
;
```

$$R = [1/0.2^2, 0; 0, 1/0.816^2];$$

$$K = \text{lqr}(A, B, Q, R);$$

$$Acl = A - B * K;$$

```
sys=ss(Acl,B,C,D,'statename',states,'inputname',inputs,'outputname',outputs);
```

```
a=r*A;
```

```
k=r*K;
```

```
Nx=inv(C'*C);
```

```
Nu=-(inv(B'*B))*B'*A*Nx;
```

```
N=Nu+k*Nx;
```

```
Ar=a-B*k;
```

```
Br=B*N;
```

```
Cr=C-D*k;
```

```
Dr=D*N;
```

```
inputs={'alpha_Ref' 'beta_Ref' 'psi_Ref' 'theta_Ref'};
```

```
sys3=ss(Ar,Br,Cr,Dr,'statename',states,'inputname',inputs,'outputname',outputs);
```

*display41.m* display the four outputs after the parameter  $r=r1$  was entered.

File: *display41.m*

```
subplot(4,1,1);plot(simout(:,1),simout(:,2),
```

```
'b--'),xlabel('t'),ylabel('alpha'),hold
```

```
subplot(4,1,2);plot(simout(:,1),simout(:,3),
```

```
'b--'),xlabel('t'),ylabel('beta'),hold
```

```
subplot(4,1,3);plot(simout(:,1),simout(:,4),
```

```
'b--'),xlabel('t'),ylabel('psi'),hold
```

```
subplot(4,1,4);plot(simout(:,1),simout(:,5),
```

```
'b--'),xlabel('t'),ylabel('theta'),hold
```

*display42.m* display the four outputs after the parameter  $r=r2$  was entered.

File: *display42.m*

```
subplot(4,1,1);plot(simout(:,1),simout(:,2),'r'),xlabel('t'),
```

```
ylabel('alpha')
```

```
subplot(4,1,2);plot(simout(:,1),simout(:,3),'r'),xlabel('t'),
```

```
ylabel('beta')
```

```
subplot(4,1,3);plot(simout(:,1),simout(:,4),'r'),xlabel('t'),
```

```
ylabel('psi'),legend('r1','r2')
```

```
subplot(4,1,4);plot(simout(:,1),simout(:,5),'r'),xlabel('t'),
```

```
ylabel('theta')
```

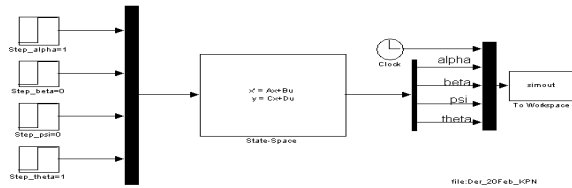


Fig. 6 Simulink diagram for longitudinal control system

If  $r1=1$  and  $r2=10$  the response is:

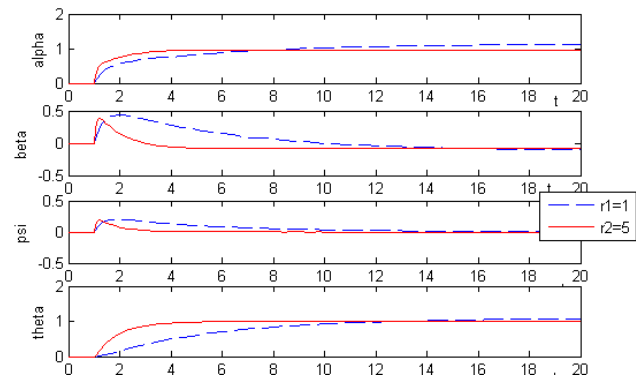


Fig. 7 System response for  $r1$  and  $r5$

A higher value of  $r$  provides a better response.

The matrices  $K$ ,  $P$  and  $N$  are:

$K =$

$$\begin{bmatrix} 0.0003 & 0.0528 & -0.5770 & -0.5783 \\ 1.5430 & 1.4587 & -0.8664 & -0.3051 \end{bmatrix}$$

```
>> P=((1-r)/r)*(inv(B'*B))*B'
P =
    -0.0175    0.0175    0.7199     0
    -0.4005   -0.3995    0.0195     0
N =
    -0.0683   -0.3448   -3.7127   -2.8879
     7.8936    7.5870   -4.9088   -1.4400
```

## V. CONCLUSION

In this paper a simplified method to design a multivariable control system using state-space is presented. A fictitious state-space matrix  $a$  is introduced and the problem is reduced to select a parameter  $r$  and after calculate the two matrices that represent the proportional and derivative matrices.

A derivative component in the feedback has a positive effect on the system response. In [6] is proven that the derivative feedback has a net stabilizing effect.

The exponential solution for the original system ( $r=1$ ) is:

$$x(t) = e^{At}x(t_0) + e^{At} \int_{t_0}^t e^{-B\tau} bu(\tau) d\tau \quad (47)$$

The exponential solution for a system with a derivative component in the feedback is:

$$x(t) = e^{at}x(t_0) + e^{at} \int_{t_0}^t e^{-B\tau} bu(\tau) d\tau = e^{A(rt)}x(t_0) + e^{A(rt)} \int_{t_0}^t e^{-B\tau} bu(\tau) d\tau \quad (48)$$

We notice that the new system is achieving the final value much faster ( $e^{At} \rightarrow e^{A(rt)}$ ).

The eigenvalues for the original system are:

```
>> damp(sys1)
Eigenvalue      Damping      Freq. (rad/s)
-2.20e-002 + 1.34e-001i  1.62e-001  1.35e-001
-2.20e-002 - 1.34e-001i  1.62e-001  1.35e-001
-8.23e-001 + 8.51e-001i  6.95e-001  1.18e+000
-8.23e-001 - 8.51e-001i  6.95e-001  1.18e+000
```

For  $r=5$  the eigenvalues are:

```
Eigenvalue      Damping      Freq. (rad/s)
-1.10e-001 + 6.68e-001i  1.62e-001  6.77e-001
-1.10e-001 - 6.68e-001i  1.62e-001  6.77e-001
-4.12e+000 + 4.26e+000i  6.95e-001  5.92e+000
-4.12e+000 - 4.26e+000i  6.95e-001  5.92e+000
```

The damping values are the same but the frequencies are five times higher. The values of eigenvalues have a linear dependence of the values of the matrix  $A$  elements. It is easy to demonstrate that the damping values remain unchanged but the natural frequencies are multiplied by the parameter  $r$ .

This is an easy method of increasing the natural frequency.

The new system is controllable and observable as long as the original system is observable and controllable.

The method is easy to implement and doesn't require extensive calculations.

The practical example used a complete state measurement, but comparable results are achieved by using an observer.

A similar methodology may be found to control the dampings without changing the natural frequencies.

In the case of an automatic reconfiguration process a simple algorithm is needed to change some parameter values ad-hoc since no simulation or design tools are available. We need something that improves the response when no human can intervene.

## REFERENCES

- [1] William J. Palm III, "Modeling, Analysis, and Control of Dynamic Systems", John Wiley & Sons, Inc., Second Edition
- [2] D. Hinrichsend, J. O'Halloran, "Limits of Generalized State Space Systems under Proportional and Dervative Feedback", *Mathematics of control signals and systems ISSN 0932-4194*, 1997, vol.10, pp 97-124.
- [3] Gene F. Franklin, J. David Powell, Abbas Emami-Naeini, "Feedback Control of Dynamic Systems", Prentice Hall", Fourth edition.I. K..
- [4] Jin Jiang, "Design of reconfigurable control systems using eigenstructure assignments," *Int. J. Control*, vol. 59, no.2, pp.359-410.
- [5] I. K.. Konstantinopoulos, P.J. Antsaklis, *An optimization strategy for reconfigurable control systems*, Technical report of the ISIS Group at the University of Notre Dame, September, 1995.
- [6] "DAPL Commands as State Observers – A Hydraulic Control Application", <http://www.mstarlabs.com/control/hydocs.htm>.