

An approximate solution of the classical Van der Pol oscillator coupled gyroscopically to a linear oscillator using parameter-expansion method

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Abstract—In this article, we are dealing with a model consisting of a classical Van der Pol oscillator coupled gyroscopically to a linear oscillator. The major problem is analyzed. The regular dynamics of the system is considered using analytical methods. In this case, we provide an approximate solution for this system using parameter-expansion method. Also, we find approximate values for frequencies of the system. In parameter-expansion method the solution and unknown frequency of oscillation are expanded in a series by a bookkeeping parameter. By imposing the non-secularity condition at each order in the expansion the method provides different approximations to both the solution and the frequency of oscillation. One iteration step provides an approximate solution which is valid for the whole solution domain.

Keywords—Parameter-expansion method, classical Van der Pol oscillator.

I. INTRODUCTION

CONSIDER the following model of a classical Van der Pol oscillator coupled gyroscopically to a linear oscillator

$$\begin{cases} y'' + \varepsilon(y^2 - 1)y' + y + fx'' = E \cos(nt), \\ x'' + \lambda x' + x - dy = 0 \end{cases} \quad (1)$$

where a prime denotes time derivative. The Van der Pol oscillator is represented by the variable y while x stands for the linear oscillator, ε and λ are respectively the Van der Pol parameters and the damping coefficient of the linear oscillator. The quantities f and d are the coupling coefficients. E and n are the amplitude and frequency of the external excitation while t is the non-dimensional time. We have restricted our analysis to the case where the natural frequencies of both oscillators are identical (internal resonance). To solve non-linear evolution equations many effective methods have been introduced, such as the variational iteration method [1], [2], [3], the Adomian decomposition method [4], [5], the homotopy perturbation method [6], [7], [8], parameter expansion method [9], [10], [11], [12], spectral collocation method [13], [14], [15], [16], [17], homotopy analysis method [18], [19], [20], [21], three-wave method [22], [23], [24], extended homoclinic test approach [25], [26], [27], the $(\frac{G'}{G})$ -expansion method [28] and the Exp-function method [29], [30], [31], [32], [33], [34].

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In this article we apply the parameter-expansion method to obtain approximate solution of system (1), also we provide numerical approximations for frequencies of x and y .

II. PARAMETER EXPANSION METHOD

To solve (1) by parameter-expansion method we rewrite the system as

$$\begin{cases} y'' + 1 \cdot y = E \cos(nt) - fx'' + \varepsilon(1 - y^2)y', \\ x'' + 1 \cdot x = dy - \lambda x'. \end{cases} \quad (2)$$

According to the parameter-expansion method, all variables x and y can be expanded into a series of an artificial parameter p such as

$$x = x_0 + px_1 + p^2x_2 + \dots \quad (3)$$

$$y = y_0 + py_1 + p^2y_2 + \dots$$

where p is called a bookkeeping parameter [35]. We also expand all coefficients of the system (1) into a series of p in a similar way

$$\begin{aligned} 1 &= \alpha^2 + p\alpha_1 + p^2\alpha_2 + \dots \\ E &= pE_1 + p^2E_2 + \dots \\ \varepsilon &= p\varepsilon_1 + p^2\varepsilon_2 + \dots \\ f &= pf_1 + p^2f_2 + \dots \\ 1 &= \beta^2 + p\beta_1 + p^2\beta_2 + \dots \\ d &= pd_1 + p^2d_2 + \dots \\ \lambda &= p\lambda_1 + p^2\lambda_2 + \dots \end{aligned} \quad (4)$$

By substituting the above expansions (3) and (4) into the system (2), we have

$$\begin{aligned} (y_0'' + py_1'' + p^2y_2'' + \dots) + (\alpha^2 + p\alpha_1 + p^2\alpha_2 + \dots) \times \\ (y_0 + py_1 + p^2y_2 + \dots) = \\ (pE_1 + p^2E_2 + \dots) \cos(nt) - (pf_1 + p^2f_2 + \dots) \times \\ (x_0'' + px_1'' + p^2x_2'' + \dots) + (p\varepsilon_1 + p^2\varepsilon_2 + \dots) \times \\ [1 - (y_0'' + py_1'' + p^2y_2'' + \dots)^2](y_0 + py_1 + p^2y_2 + \dots), \end{aligned}$$

$$\begin{aligned} (x_0'' + px_1'' + p^2x_2'' + \dots) + (\beta^2 + p\beta_1 + p^2\beta_2 + \dots) \times \\ (x_0 + px_1 + p^2x_2 + \dots) = (pd_1 + p^2d_2 + \dots) \times \\ (y_0 + py_1 + p^2y_2 + \dots) - (p\lambda_1 + p^2\lambda_2 + \dots) \times \\ (x_0' + px_1' + p^2x_2' + \dots). \end{aligned} \quad (5)$$

Equating in the powers of p , we have

$$p^0 : \begin{cases} y_0'' + \alpha^2 y_0 = 0 \\ x_0'' + \beta^2 x_0 = 0 \end{cases} \quad (6)$$

and

$$p^1 : \begin{cases} y_1'' + \alpha^2 y_1 = E_1 \cos(nt) + \varepsilon_1 y_0' - \varepsilon_1 y_0^2 y_0' - \alpha_1 y_0 - f_1 x_0'', \\ x_1'' + \beta^2 x_1 = d_1 y_0 - \beta_1 x_0 - \lambda_1 x_0'. \end{cases} \quad (7)$$

Solving the equation (6), we obtain

$$\begin{cases} y_0 = A_1 \cos(\alpha t) + A_2 \sin(\alpha t) \\ x_0 = B_1 \cos(\beta t) + B_2 \sin(\beta t). \end{cases} \quad (8)$$

where A_1, A_2, B_1 and B_2 are arbitrary constants. Substituting (8) into (7), we obtain

$$\begin{cases} y_1'' + \alpha^2 y_1 = E_1 \cos(nt) + f_1 [B_1 \beta^2 \cos(\beta t) - B_2 \beta^2 \sin(\beta t)] \\ \quad + \sin(\alpha t) [\frac{1}{4} A_1 A_2^2 \alpha \varepsilon_1 + \frac{1}{4} A_1^3 \alpha \varepsilon_1 - A_1 \alpha \varepsilon_1 - \alpha_1 A_2] \\ \quad + \cos(\alpha t) [-\frac{1}{4} A_2^3 \alpha \varepsilon_1 - \frac{1}{4} A_1^2 A_2 \alpha \varepsilon_1 + A_2 \alpha \varepsilon_1 - \alpha_1 A_1] \\ \quad + \sin(3\alpha t) [\frac{1}{4} A_1^3 \alpha \varepsilon_1 - \frac{3}{4} A_1 A_2^2 \alpha \varepsilon_1] + \\ \quad \cos(3\alpha t) [\frac{3}{4} A_1^2 A_2 \alpha \varepsilon_1 - \frac{1}{4} A_2^3 \alpha \varepsilon_1], \\ x_1'' + \beta^2 x_1 = d_1 A_1 \cos(\alpha t) + d_1 A_2 \sin(\alpha t) \\ \quad + \sin(\beta t) [\lambda_1 B_1 \beta - \beta_1 B_2] + \cos(\beta t) [-\lambda_1 B_2 \beta - \beta_1 B_1]. \end{cases} \quad (9)$$

If the first-order approximation is enough, then, setting $p = 1$ in both equations (3) and (4), we have

$$\begin{aligned} y &= y_0 + y_1, \quad \alpha^2 + \alpha_1 = 1, \quad E = E_1, \quad \varepsilon = \varepsilon_1, \quad f = f_1 \\ x &= x_0 + x_1, \quad \beta^2 + \beta_1 = 1, \quad d = d_1, \quad \lambda = \lambda_1. \end{aligned} \quad (10)$$

Now substituting (10) into (9) yields:

$$\begin{cases} y_1'' + \alpha^2 y_1 = E \cos(nt) + f [B_1 \beta^2 \cos(\beta t) - B_2 \beta^2 \sin(\beta t)] \\ \quad + \sin(\alpha t) [\frac{1}{4} A_1 A_2^2 \alpha \varepsilon + \frac{1}{4} A_1^3 \alpha \varepsilon - A_1 \alpha \varepsilon - \alpha_1 A_2] \\ \quad + \cos(\alpha t) [-\frac{1}{4} A_2^3 \alpha \varepsilon - \frac{1}{4} A_1^2 A_2 \alpha \varepsilon + A_2 \alpha \varepsilon - \alpha_1 A_1] \\ \quad + \sin(3\alpha t) [\frac{1}{4} A_1^3 \alpha \varepsilon - \frac{3}{4} A_1 A_2^2 \alpha \varepsilon] \\ \quad + \cos(3\alpha t) [\frac{3}{4} A_1^2 A_2 \alpha \varepsilon - \frac{1}{4} A_2^3 \alpha \varepsilon], \\ x_1'' + \beta^2 x_1 = d A_1 \cos(\alpha t) + d A_2 \sin(\alpha t) \\ \quad + \sin(\beta t) [\lambda B_1 \beta - \beta_1 B_2] + \cos(\beta t) [-\lambda B_2 \beta - \beta_1 B_1] \end{cases} \quad (11)$$

No secular term in y_1 and x_1 requires that

$$\begin{cases} \frac{1}{4} A_1 A_2^2 \alpha \varepsilon + \frac{1}{4} A_1^3 \alpha \varepsilon - A_1 \alpha \varepsilon - \alpha_1 A_2 = 0, \\ -\frac{1}{4} A_2^3 \alpha \varepsilon - \frac{1}{4} A_1^2 A_2 \alpha \varepsilon + A_2 \alpha \varepsilon - \alpha_1 A_1 = 0 \end{cases} \quad (12)$$

and

$$\begin{cases} \lambda B_1 \beta - \beta_1 B_2 = 0, \\ -\lambda B_2 \beta - \beta_1 B_1 = 0 \end{cases} \quad (13)$$

and using (12), (13) and (10), we have

$$\alpha = \beta = 1. \quad (14)$$

and the frequencies of equations are

$$\begin{cases} T_y = \frac{2\pi}{\alpha} = 2\pi, \\ T_x = \frac{2\pi}{\beta} = 2\pi. \end{cases} \quad (15)$$

Furthermore, equation (11) can be simplified as

$$\begin{cases} y_1'' + y_1 = E \cos(nt) + f [B_1 \cos t + B_2 \sin t] \\ \quad + \sin 3t [\frac{1}{4} A_1^3 \varepsilon - \frac{3}{4} A_1 A_2^2 \varepsilon] + \cos 3t [\frac{3}{4} A_1^2 A_2 \varepsilon - \frac{1}{4} A_2^3 \varepsilon], \\ x_1'' + x_1 = d A_1 \cos t + d A_2 \sin t \end{cases} \quad (16)$$

Solving equation (16) yields

$$\begin{cases} y_1 = \frac{E}{1-n^2} \cos(nt) + \frac{t}{2} (B_1 f \sin t - B_2 f \cos t) \\ \quad + \sin 3t [\frac{3}{32} A_1 A_2^2 \varepsilon - \frac{1}{32} A_1^3 \varepsilon] + \cos 3t [\frac{1}{32} A_2^3 \varepsilon - \frac{3}{32} A_1^2 A_2 \varepsilon], \\ x_1 = \frac{t}{2} (d A_1 \sin t - d A_2 f \cos t). \end{cases} \quad (17)$$

Now using (17), (8) and (10) we obtain the first order solution for x and y as follows

$$\begin{cases} y = y_0 + y_1 = A_1 \cos t + A_2 \sin t \\ \quad + \frac{E}{1-n^2} \cos(nt) + \frac{t}{2} (B_1 f \sin t - B_2 f \cos t) \\ \quad + \sin 3t [\frac{3}{32} A_1 A_2^2 \varepsilon - \frac{1}{32} A_1^3 \varepsilon] + \cos 3t [\frac{1}{32} A_2^3 \varepsilon - \frac{3}{32} A_1^2 A_2 \varepsilon], \\ x = x_0 + x_1 = B_1 \cos t + B_2 \sin t + \frac{t}{2} (d A_1 \sin t - d A_2 f \cos t). \end{cases} \quad (18)$$

III. CONCLUSIONS

In this study, we have applied the parameter-expansion method to solve the classical Van der Pol oscillator coupled gyroscopically to a linear oscillator equation. The method more efficient than perturbation method [36] for this problem because the method is independent of perturbation parameter assumption and one iteration step provide an approximate solution which is valid for the whole solution domain. One can apply the parameter-expansion method on another nonlinear oscillators, easily. Because, the method can be easily comprehended with only a basic knowledge of advanced calculus. The method by help of Maple, is utter simplicity, and can be easily extended to all kinds of non-linear equations.

REFERENCES

- [1] J.H. He, *Variational iteration method-a kind of non-linear analytical technique: some examples*, Int. J. Non-linear Mech., 34(4) (1999) 699–708.
- [2] M.T. Darvishi, F. Khani, A.A. Soliman, *The numerical simulation for stiff systems of ordinary differential equations*, Comput. Math. Appl., 54(7-8) (2007) 1055–1063.
- [3] M.T. Darvishi, F. Khani, *Numerical and explicit solutions of the fifth-order Korteweg-de Vries equations*, Chaos, Solitons and Fractals, 39 (2009) 2484–2490.
- [4] S. Abbasbandy, M.T. Darvishi, *A numerical solution of Burgers' equation by modified Adomian method*, Appl. Math. Comput., 163 (2005) 1265–1272.
- [5] S. Abbasbandy, M.T. Darvishi, *A numerical solution of Burgers' equation by time discretization of Adomian's decomposition method*, Appl. Math. Comput., 170 (2005) 95–102.
- [6] J.H. He, *New interpretation of homotopy perturbation method*, Int. J. Mod. Phys. B, 20(18) (2006) 2561–2568.
- [7] M.T. Darvishi, F. Khani, *Application of He's homotopy perturbation method to stiff systems of ordinary differential equations*, Zeitschrift fur Naturforschung A, 63a (1-2) (2008) 19–23.
- [8] M.T. Darvishi, F. Khani, S. Hamed-Nezhad, S.W. Ryu, *New modification of the HPM for numerical solutions of the sine-Gordon and coupled sine-Gordon equations*, Int. J. Comput. Math., 87(4) (2010) 908–919.
- [9] M.T. Darvishi, A. Karami, B.C. Shin, *Application of He's parameter-expansion method for oscillators with smooth odd nonlinearities*, Phys. Lett. A, 372(33) (2008) 5381–5384.
- [10] B.C. Shin, M.T. Darvishi, A. Karami, *Application of He's parameter-expansion method to a nonlinear self-excited oscillator system*, Int. J. Nonlin. Sci. Num. Simul., 10(1) (2009) 137–143.
- [11] M.T. Darvishi, S. Kheybari, A. Yildirim, *Application of He's Parameter-expansion Method to a System of Two van der Pol Oscillators Coupled via a Bath*, Nonlin. Science Lett. A., 1(4) (2010) 399–405.
- [12] M.T. Darvishi, S. Kheybari, *Application of He's parameter-expansion method to a coupled van der Pol oscillators with two kinds of time-delay coupling*, International Journal of Mathematical and Computer Sciences, 7(4) (2011) 156–161.
- [13] M.T. Darvishi, *Preconditioning and domain decomposition schemes to solve PDEs*, International Journal of Pure and Applied Mathematics, 1(4) (2004) 419–439.
- [14] M.T. Darvishi, S. Kheybari, F. Khani, *A numerical solution of the Korteweg-de Vries equation by pseudospectral method using Darvishi's preconditionings*, Appl. Math. Comput., 182(1) (2006) 98–105.
- [15] M.T. Darvishi, *New algorithms for solving ODEs by pseudospectral method*, Journal of Applied Mathematics and Computing, 7(2) (2000) 319–331.
- [16] M.T. Darvishi, F. Khani, S. Kheybari, *Spectral collocation solution of a generalized Hirota-Satsuma KdV equation*, Int. J. Comput. Math., 84(4) (2007) 541–551.
- [17] M.T. Darvishi, F. Khani, S. Kheybari, *Spectral collocation method and Darvishi's preconditionings to solve the generalized Burgers-Huxley equation*, Commun. Nonlinear Sci. Numer. Simul., 13(10) (2008) 2091–2103.
- [18] S.J. Liao, *A new branch of solutions of boundary-layer flows over an impermeable stretched plate*, Int. J. Heat Mass Transfer, 48 (2005) 2529–2539.
- [19] S.J. Liao, *A general approach to get series solution of non-similarity boundary-layer flows*, Commun. Nonlinear Sci. Numer. Simul., 14(5) (2009) 2144–2159.
- [20] M.T. Darvishi, F. Khani, *A series solution of the foam drainage equation*, Comput. Math. Appl., 58 (2009) 360–368.
- [21] A. Aziz, F. Khani, M.T. Darvishi, *Homotopy analysis method for variable thermal conductivity heat flux gage with edge contact resistance*, Zeitschrift fuer Naturforschung A, 65a(10) (2010) 771–776.
- [22] M.T. Darvishi, Maliheh Najafi, Mohammad Najafi, *Exact three-wave solutions for high nonlinear form of Benjamin-Bona-Mahony-Burgers equations*, International Journal of Mathematical and Computer Sciences, 6(3) (2010) 127–131.
- [23] M.T. Darvishi, Mohammad Najafi, *Some exact solutions of the (2+1)-dimensional breaking soliton equation using the three-wave method*, International Journal of Computational and Mathematical Sciences, 6(1) (2012) 13–16.
- [24] M.T. Darvishi, Maliheh Najafi, Mohammad Najafi, *New exact solutions for the (3+1)-dimensional breaking soliton equation*, International Journal of Information and Mathematical Sciences, 6(2) (2010) 134–137.
- [25] M.T. Darvishi, Maliheh Najafi, Mohammad Najafi, *New application of EHTA for the generalized (2+1)-dimensional nonlinear evolution equations*, International Journal of Mathematical and Computer Sciences, 6(3) (2010) 132–138.
- [26] M.T. Darvishi, Mohammad Najafi, *A modification of extended homoclinic test approach to solve the (3+1)-dimensional potential-YTSE equation*, Chin. Phys. Lett., 28(4) (2011) art. no. 040202.
- [27] M.T. Darvishi, Mohammad Najafi, *Some complexiton type solutions of the (3+1)-dimensional Jimbo-Miwa equation*, International Journal of Computational and Mathematical Sciences, 6(1) (2012) 25–27.
- [28] M.T. Darvishi, Maliheh Najafi, Mohammad Najafi, *Traveling wave solutions for the (3+1)-dimensional breaking soliton equation by $(\frac{G'}{G})$ -expansion method and modified F-expansion method*, International Journal of Computational and Mathematical Sciences, 6(2) (2012) 64–69.
- [29] J.H. He, M.A. Abdou, *New periodic solutions for nonlinear evolution equations using Exp-function method*, Chaos, Solitons and Fractals, 34 (2007) 1421–1429.
- [30] J.H. He, X.H. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Solitons and Fractals, 30(3) (2006) 700–708.
- [31] J.H. He, X.H. Wu, *Construction of solitary solution and compacton-like solution by variational iteration method*, Chaos, Solitons and Fractals, 29 (2006) 108–113.
- [32] F. Khani, S. Hamed-Nezhad, M.T. Darvishi, S.W. Ryu, *New solitary wave and periodic solutions of the foam drainage equation using the Exp-function method*, Nonlin. Anal.: Real World Appl., 10 (2009) 1904–1911.
- [33] B.C. Shin, M.T. Darvishi, A. Barati, *Some exact and new solutions of the Nizhnik-Novikov-Vesselov equation using the Exp-function method*, Comput. Math. Appl., 58(11/12) (2009) 2147–2151.
- [34] X.H. Wu, J.H. He, *Exp-function method and its application to nonlinear equations*, Chaos, Solitons and Fractals, 38(3) (2008) 903–910.
- [35] J.H. He, *Bookkeeping parameter in perturbation methods*, Int. J. Nonlinear Sci. Numer. Simul., 2 (2001) 257–264.
- [36] A. H. Nayfeh, *Perturbation methods*. JohnWiley and Sons, New York, NY, USA, 1973.