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Effects Edge end Free-free Boundary Conditions for Analysis Free Vibration of Functionally Graded Cylindrical Shell with Ring based on Third Order Shear Deformation Theory using Hamilton's Principle

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***Abstract*—**In this paper a study on the vibration of thin cylindrical shells with ring supports and made of functionally graded materials (FGMs) composed of stainless steel and nickel is presented. Material properties vary along the thickness direction of the shell according to volume fraction power law. The cylindrical shells have ring supports which are arbitrarily placed along the shell and impose zero lateral deflections. The study is carried out based on third order shear deformation shell theory (T.S.D.T). The analysis is carried out using Hamilton's principle. The governing equations of motion of FGM cylindrical shells are derived based on shear deformation theory. Results are presented on the frequency characteristics, influence of ring support position and the influence of boundary conditions. The present analysis is validated by comparing results with those available in the literature.

***Keywords*—**Vibration, FGM, Cylindrical shell, Hamilton's principle, Ring support.

I. INTRODUCTION

CYLINDRICAL shells have found many applications in the industry. They are often used as load bearing structures for aircrafts, ships and buildings. Understanding of vibration behavior of cylindrical shells is an important aspect for the successful applications of cylindrical shells. Researches on free vibrations of cylindrical shells have been carried out extensively [1-5]. Recently, the present authors presented studies on the influence of boundary conditions on the frequencies of a multi–layered cylindrical shell [6]. In all the above works, different thin shell theories based on Love–hypothesis were used. Vibration of cylindrical shells with ring support is considered by Loy and Lam [7]. The concept of functionally graded materials (FGMs) was first introduced in 1984 by a group of materials scientists in Japan [8-9] as a means of preparing thermal barrier materials. Since then, FGMs have attracted much interest as heat-shielding materials. FGMs are made by combining different materials using power metallurgy methods [10]. They possess variations in constituent volume fractions that lead to continuous change in the composition, microstructure, porosity, etc., resulting in gradients in the mechanical and thermal properties [11-12].

Vibration study of FGM shell structures is important. However, study of the vibration of FGM shells with ring supports is limited. In this paper a study on the vibration of FG cylindrical shells with ring supports is presented. The FGMs considered are composed of stainless steel and nickel where the volume fractions follow a power-law distribution. The study is carried out based on third order shear deformation shell theory. The analysis is carried out using Hamilton's principle. Studies are carried out for cylindrical shells with free–free F–F boundary conditions with an arbitrarily ring support along the axial direction of the cylindrical shell. Results presented include the frequency characteristics of cylindrical shells with ring supports, the influence of ring support position and the influence of boundary conditions. The present analysis is validated by comparing results with others in the literature.

II. FUNCTIONALLY GRADED MATERIALS

For the cylindrical shell made of FGM the material properties such as the modulus of elasticity E, Poisson ratio ν and the mass density ρ are assumed to be functions of the volume fraction of the constituent materials when the coordinate axis across the shell thickness is denoted by z and measured from the shell's middle plane. The functional relationships between E, ν and ρ with z for a stainless steel and nickel FGM shell are assumed as [13].

$$E = (E_1 - E_2)\left(\frac{2Z+h}{2h}\right)^N + E_2 \tag{1}$$

$$\nu = (\nu_1 - \nu_2)\left(\frac{2Z+h}{2h}\right)^N + \nu_2 \tag{2}$$

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$$\rho = (\rho_1 - \rho_2)\left(\frac{2Z+h}{2h}\right)^N + \rho_2 \qquad (3)$$

The strain-displacement relationships for a thin shell [14].

$$\epsilon_{11} = \frac{1}{A_1(1+\frac{\alpha_3}{R_1})}\left[\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} + U_3\frac{A_1}{R_1}\right] \qquad (4)$$

$$\epsilon_{22}\ \frac{1}{A_2(1+\frac{\alpha_3}{R_2})}\left[\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + U_3\frac{A_2}{R_2}\right] \qquad (5)$$

$$\epsilon_{33} = \frac{\partial U_3}{\partial \alpha_3} \qquad (6)$$

$$\epsilon_{12} = \frac{A_1(1+\frac{\alpha_3}{R_1})}{A_2(1+\frac{\alpha_3}{R_2})}\frac{\partial}{\partial \alpha_2}\left(\frac{U_1}{A_1(1+\frac{\alpha_3}{R_1})}\right) + \frac{A_2(1+\frac{\alpha_3}{R_2})}{A_1(1+\frac{\alpha_3}{R_1})}\frac{\partial}{\partial \alpha_1}\left(\frac{U_2}{A_2(1+\frac{\alpha_3}{R_2})}\right) \qquad (7)$$

$$\epsilon_{13} = A_1(1+\frac{\alpha_3}{R_1})\frac{\partial}{\partial \alpha_3}\left(\frac{U_1}{A_1(1+\frac{\alpha_3}{R_1})}\right) + \frac{1}{A_1(1+\frac{\alpha_3}{R_1})}\frac{\partial U_3}{\partial \alpha_1} \qquad (8)$$

$$\epsilon_{23} = A_2(1+\frac{\alpha_3}{R_2})\frac{\partial}{\partial \alpha_3}\left(\frac{U_2}{A_2(1+\frac{\alpha_3}{R_2})}\right) + \frac{1}{A_2(1+\frac{\alpha_3}{R_2})}\frac{\partial U_3}{\partial \alpha_2} \qquad (9)$$

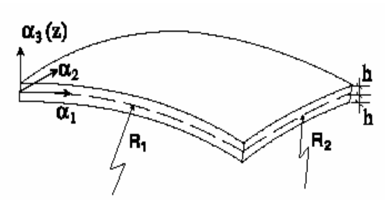

Fig. 1 Geometry of a generic shell

$$A_1 = \left|\frac{\partial \bar{r}}{\partial \alpha_1}\right| \quad , \quad A_2 = \left|\frac{\partial \bar{r}}{\partial \alpha_2}\right| \qquad (10)$$

where A_1 and A_2 are the fundamental form parameters or Lame parameters, U_1, U_2 and U_3 are the displacement at any point ($\alpha_1, \alpha_2, \alpha_3$), R_1 and R_2 are the radius of curvature related to α_1, α_2 and α_3 respectively. The third- order theory of Reddy used in the present study is based on the following displacement field:

$$\begin{cases} U_1 = u_1(\alpha_1,\alpha_2) + \alpha_3.\phi_1(\alpha_1,\alpha_2) + \alpha_3^2.\psi_1(\alpha_1,\alpha_2) + \alpha_3^3.\beta_1(\alpha_1,\alpha_2) \\ U_2 = u_2(\alpha_1,\alpha_2) + \alpha_3.\phi_2(\alpha_1,\alpha_2) + \alpha_3^2.\psi_2(\alpha_1,\alpha_2) + \alpha_3^3.\beta_2(\alpha_1,\alpha_2) \\ U_3 = u_3(\alpha_1,\alpha_2) \end{cases} \qquad (11)$$

These equations can be reduced by satisfying the stress-free conditions on the top and bottom faces of the laminates, which are equivalent to $\epsilon_{13} = \epsilon_{23} = 0$ at $Z = \pm\frac{h}{2}$ Thus,

$$\begin{cases} U_1 = u_1(\alpha_1,\alpha_2) + \alpha_3.\phi_1(\alpha_1,\alpha_2) - C_1.\alpha_3^3(-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial \alpha_1}) \\ U_2 = u_2(\alpha_1,\alpha_2) + \alpha_3.\phi_2(\alpha_1,\alpha_2) - C_1.\alpha_3^3(-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2\partial \alpha_2}) \\ U_3 = u_3(\alpha_1,\alpha_2) \end{cases} \qquad (12)$$

Where $C_1 = \frac{4}{3h^2}$. Substituting Eq. (12) into nonlinear strain-displacement relation (4) - (9), it can be obtained for the third-order theory of Reddy

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{11}^0 \\ \epsilon_{22}^0 \\ \epsilon_{12}^0 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} + \alpha_3^3 \begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} \qquad (13)$$

$$\begin{Bmatrix} \epsilon_{13} \\ \epsilon_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} + \alpha_3^2 \begin{Bmatrix} \gamma_{13}^2 \\ \gamma_{23}^2 \end{Bmatrix} + \alpha_3^3 \begin{Bmatrix} \gamma_{13}^3 \\ \gamma_{23}^3 \end{Bmatrix} \qquad (14)$$

where

$$\begin{Bmatrix} \epsilon_{11}^0 \\ \epsilon_{22}^0 \\ \epsilon_{12}^0 \end{Bmatrix} = \begin{Bmatrix} (\frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} + \frac{u_3}{R_1}) \\ (\frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} + \frac{u_3}{R_2}) \\ \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}(\frac{u_2}{A_2}) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}(\frac{u_1}{A_1}) \end{Bmatrix} \qquad (15)$$

$$\begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} = \begin{Bmatrix} (\frac{1}{A_1}\frac{\partial \phi_1}{\partial \alpha_1} + \frac{\phi_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}) \\ (\frac{1}{A_2}\frac{\partial \phi_2}{\partial \alpha_2} + \frac{\phi_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}) \\ (\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}(\frac{\phi_2}{A_2}) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}(\frac{\phi_1}{A_1})) \end{Bmatrix} \qquad (16)$$

$$\begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} = -C_1 \begin{Bmatrix} (\frac{1}{A_1}(\frac{\partial u_1}{R_1\partial \alpha_1} + \frac{\partial \phi_1}{\partial \alpha_1} + \frac{\partial^2 u_3}{A_1\partial \alpha_1^2} - \frac{\partial A_1}{A_1^2\partial \alpha_1}\frac{\partial u_3}{\partial \alpha_1}) + \frac{\partial A_1}{\partial \alpha_2}\frac{1}{A_1A_2}(\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2\partial \alpha_2})) \\ (\frac{1}{A_2}(\frac{\partial u_2}{R_2\partial \alpha_2} + \frac{\partial \phi_2}{\partial \alpha_2} + \frac{\partial^2 u_3}{A_2\partial \alpha_2^2} - \frac{\partial A_2}{A_2^2\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_2}) + \frac{\partial A_2}{\partial \alpha_1}\frac{1}{A_1A_2}(\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial \alpha_1})) \\ (\frac{A_2}{A_1}(\frac{\partial}{R_2\partial \alpha_1}(\frac{u_2}{A_2}) + \frac{\partial}{\partial \alpha_1}(\frac{\phi_2}{A_2}) + \frac{1}{A_2^2}\frac{\partial^2 u_3}{\partial \alpha_1\partial \alpha_2} - \frac{1}{A_2^4}\frac{\partial A_2^2}{\partial \alpha_1}\frac{\partial u_3}{\partial \alpha_2}) + \\ + \frac{A_1}{A_2}(\frac{\partial}{R_1\partial \alpha_2}(\frac{u_1}{A_1}) + \frac{\partial}{\partial \alpha_2}(\frac{\phi_1}{A_1}) + \frac{1}{A_1^2}\frac{\partial^2 u_3}{\partial \alpha_1\partial \alpha_2} - \frac{1}{A_1^4}\frac{\partial A_1^2}{\partial \alpha_2}\frac{\partial u_3}{\partial \alpha_1})) \end{Bmatrix} \qquad (17)$$

$$\begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} = \begin{Bmatrix} (\phi_1 - \frac{u_1}{R_1} + \frac{1}{A_1}\frac{\partial u_3}{\partial \alpha_1}) \\ (\phi_2 - \frac{u_2}{R_2} + \frac{1}{A_2}\frac{\partial u_3}{\partial \alpha_2}) \end{Bmatrix} \qquad (18)$$

$$\begin{Bmatrix} \gamma_{13}^2 \\ \\ \gamma_{23}^2 \end{Bmatrix} = 3C_1 \begin{Bmatrix} (-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1 \partial \alpha_1}) \\ (-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2 \partial \alpha_2}) \end{Bmatrix} \tag{19}$$

$$\begin{Bmatrix} \gamma_{13}^3 \\ \\ \gamma_{23}^3 \end{Bmatrix} = C_1 \begin{Bmatrix} \dfrac{(-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1 \partial \alpha_1})}{R_1} \\ \dfrac{(-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2 \partial \alpha_2})}{R_2} \end{Bmatrix} \tag{20}$$

Where $(\varepsilon^0, \gamma^0)$ are the membranes strains and $(k, k', \gamma^2, \gamma^3)$ are the bending strains, known as the curvatures.

III. Formulation

Consider a cylindrical shell with ring supports as shown in Figure 2, where R is the radius, L the length, h the thickness, and a the position of the ring support along the axial direction of the cylindrical shell. The reference surface is chosen to be the middle surface of the cylindrical shell where an orthogonal coordinate system x, θ, z is fixed. The displacements of the shell with reference to this coordinate system are denoted by U_1, U_2 and U_3 in the x, θ and z directions, respectively.

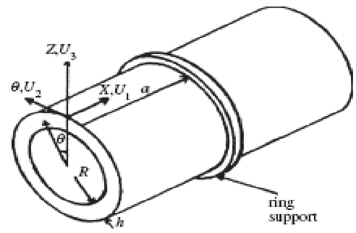

Fig. 2 Geometry of a cylindrical shell with ring support

For a thin cylindrical shell, the stress -strain relationship are defined as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \in_{11} \\ \in_{22} \\ \in_{23} \\ \in_{13} \\ \in_{12} \end{Bmatrix} \tag{21}$$

For a isotropic cylindrical shell the reduced stiffness Q_{ij} (i, j=1, 2 and 6) are defined as

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2} \quad , Q_{12} = \frac{\nu E}{1-\nu^2} \tag{22}$$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1+\nu)} \tag{23}$$

where E is the Young's modulus and ν is Poisson's ratio. Defining

$$\{A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}\} = \int_{-h/2}^{h/2} Q_{ij} \{1, \alpha_3, \alpha_3^2, \alpha_3^3, \alpha_3^4, \alpha_3^5, \alpha_3^6\} d\alpha_3 \tag{24}$$

where Q_{ij} are functions of z for functionally gradient materials. Here A_{ij} denote the extensional stiffness, D_{ij} the bending stiffness, B_{ij} the bending-extensional coupling stiffness and $E_{ij}, F_{ij}, G_{ij}, H_{ij}$ are the extensional, bending, coupling, and higher-order stiffness.

For a thin cylindrical shell the force and moment results are defined as

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} d\alpha_3 \ , \quad \begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3^3 d\alpha_3 \tag{25}$$

$$\begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3^3 d\alpha_3 \ , \begin{Bmatrix} P_{13} \\ \\ P_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \\ \sigma_{23} \end{Bmatrix} \alpha_3^3 d\alpha_3 \tag{26}$$

$$\begin{Bmatrix} Q_{13} \\ Q_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} d\alpha_3 \quad , \quad \begin{Bmatrix} R_{13} \\ R_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \alpha_3^2 d\alpha_3 \tag{27}$$

IV. The equations of motion for vibration of a generic shell

The equations of motion for vibration of a generic shell can be derived by using Hamilton's principle which is described by

$$\delta \int_{t_1}^{t_2} (\Pi - K) dt = 0 \quad , \quad \Pi = U - V \tag{28}$$

Where K, Π, U and V are the total kinetic, potential, strain and loading energies, t_1 and t_2 are arbitrary time. The kinetic, strain and loading energies of a cylindrical shell can be written as:

$$K = \frac{1}{2} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \rho (\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2) dV \tag{29}$$

$$U = \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} (\sigma_{11} \in_{11} + \sigma_{22} \in_{22} + \sigma_{12} \in_{12} + \sigma_{13} \in_{13} + \sigma_{23} \in_{23}) dV \tag{30}$$

$$V = \int_{\alpha_1} \int_{\alpha_2} (q_1 \delta U_1 + q_2 \delta U_2 + q_3 \delta U_3) A_1 A_2 d\alpha_1 d\alpha_2 \tag{31}$$

The infinitesimal volume is given by

$$dV = A_1 A_2 d\alpha_1 d\alpha_2 d\alpha_3 \tag{32}$$

with the use of Eqs. (11)-(20) and substituting into Eq. (28), we get the equations of motions a generic shell.

$$\begin{aligned}&-\frac{\partial(N_{11}A_2)}{\partial\alpha_1}+N_{22}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(N_{12}A_1^2)}{A_1\partial\alpha_2}-\frac{Q_{13}}{R_1}A_1A_2-\frac{\partial}{\partial\alpha_1}(\frac{P_{11}C_1A_2}{R_1})+\frac{P_{22}C_1}{R_1}\frac{\partial A_2}{\partial\alpha_1}-\\&\frac{\partial}{\partial\alpha_2}(\frac{P_{12}C_1A_1^2}{R_1})\frac{1}{A_1}+\frac{3C_1R_{13}}{R_1}A_1A_2-\frac{C_1P_{13}A_1A_2}{R_1^2}=-(\ddot{u}_1I_\circ+\ddot{\phi}_1I_1+[-C_1(-\frac{\ddot{u}_1}{R_1}+\\&\ddot{\phi}+\frac{\partial\ddot{u}_3}{A_1\partial\alpha_1})+\frac{C_1\ddot{u}_1}{R_1}]I_3+\frac{C_1\ddot{\phi}_1}{R_1}I_4-\frac{C_1^2}{R_1}(-\frac{\ddot{u}_1}{R_1}+\ddot{\phi}_1+\frac{\partial\ddot{u}_3}{A_1\partial\alpha_1})I_6)\end{aligned} \tag{33}$$

$$\begin{aligned}&\frac{\partial(N_{22}A_1)}{\partial\alpha_2}-N_{11}\frac{\partial A_1}{\partial\alpha_2}+\frac{\partial(N_{12}A_2^2)}{A_2\partial\alpha_1}+\frac{Q_{23}}{R_2}A_1A_2+\frac{\partial}{\partial\alpha_2}(\frac{P_{22}C_1A_1}{R_2})-\frac{P_{11}C_1}{R_2}\frac{\partial A_1}{\partial\alpha_2}+\\&\frac{\partial}{\partial\alpha_1}(\frac{P_{12}C_1A_2^2}{R_2})\frac{1}{A_2}-\frac{3C_1R_{23}}{R_2}A_1A_2+\frac{C_1P_{23}A_1A_2}{R_2^2}=(\ddot{u}_2I_\circ+\ddot{\phi}_2I_1+\frac{C_1\ddot{\phi}_2}{R_2}I_{42}+\\&\left[-c_1(-\frac{\ddot{u}_2}{R_2}+\ddot{\phi}_2+\frac{\partial\ddot{u}_3}{A_2\partial\alpha_2})+\frac{C_1\ddot{u}_2}{R_2}\right]I_3-\frac{C_1^2}{R_2}(-\frac{\ddot{u}_2}{R_2}+\ddot{\phi}_2+\frac{\partial u_3}{A_2\partial\alpha_2})I_6)\end{aligned} \tag{34}$$

$$\begin{aligned}&(-\frac{\partial^2(P_{11}C_1A_2/A_1)}{\partial\alpha_1^2}+N_{11}\frac{A_1A_2}{R_1}+\frac{\partial}{\partial\alpha_2}(\frac{C_1P_{11}}{A_2}\frac{\partial A_1}{\partial\alpha_2})+N_{22}\frac{A_1A_2}{R_2}-\frac{\partial^2(P_{22}A_1C_1/A_2)}{\partial\alpha_2^2}+\\&+\frac{\partial}{\partial\alpha_1}(\frac{P_{22}C_1}{A_1}\frac{\partial A_2}{\partial\alpha_1})-\frac{\partial^2(P_{12}C_1)}{\partial\alpha_1\partial\alpha_2}-\frac{\partial}{\partial\alpha_2}(\frac{P_{12}C_1}{A_2^2}\frac{\partial A_2^2}{\partial\alpha_1})-\frac{\partial^2(P_{12}C_1)}{\partial\alpha_1\partial\alpha_2}-\frac{\partial}{\partial\alpha_1}(\frac{P_{12}C_1}{A_1^2}\frac{\partial A_1^2}{\partial\alpha_2})\\&-\frac{\partial(Q_{13}A_2)}{\partial\alpha_1}+\frac{\partial(3C_1R_{13}A_2)}{\partial\alpha_1}-\frac{\partial}{\partial\alpha_1}(\frac{P_{13}C_1A_2}{R_1})-\frac{\partial(Q_{23}A_1)}{\partial\alpha_2}+\frac{\partial(3C_1R_{23}A_1)}{\partial\alpha_2}-\\&-\frac{\partial}{\partial\alpha_2}(\frac{C_1P_{23}A_1}{R_2})-\frac{\partial}{\partial\alpha_1}(\frac{P_{11}C_1A_2}{A_1^2}\frac{\partial A_1}{\partial\alpha_1})-\frac{\partial}{\partial\alpha_2}(P_{22}C_1\frac{A_1}{A_2^2}\frac{\partial A_2}{\partial\alpha_2}))=-\{\ddot{u}_3I_\circ+\\&+C_1\left[\frac{\partial}{\partial\alpha_1}(\frac{u_1}{A_1})+\frac{\partial}{\partial\alpha_2}(\frac{u_2}{A_2})\right]I_3+C_1\left[\frac{\partial}{\partial\alpha_1}(\frac{\ddot{\phi}_1}{A_1})+\frac{\partial}{\partial\alpha_2}(\frac{\ddot{\phi}_2}{A_2})\right]I_4-C_1^2I_6((-\frac{\partial}{R_2\partial\alpha_2}(\frac{\ddot{u}_2}{A_2})\\&+\frac{\partial}{\partial\alpha_2}(\frac{\ddot{\phi}_2}{A_2})+\frac{1}{A_2}\frac{\partial^2\ddot{u}_3}{\partial\alpha_2^2}-\frac{\partial A_2}{A_1^2\partial\alpha_2}\frac{\partial\ddot{u}_3}{\partial\alpha_2})+(-\frac{\partial}{R_1\partial\alpha_1}(\frac{\ddot{u}_1}{A_1})+\frac{\partial}{\partial\alpha_1}(\frac{\ddot{\phi}_1}{A_1})+\\&+\frac{1}{A_1}\frac{\partial^2\ddot{u}_3}{\partial\alpha_1^2}-\frac{\partial A_1}{A_1^2\partial\alpha_1}\frac{\partial\ddot{u}_3}{\partial\alpha_1}))\end{aligned} \tag{35}$$

$$\begin{aligned}&-\frac{\partial(M_{11}A_2)}{\partial\alpha_1}+\frac{\partial(C_1P_{11}A_2)}{\partial\alpha_1}+M_{22}\frac{\partial A_2}{\partial\alpha_1}-C_1P_{22}\frac{\partial A_2}{\partial\alpha_1}-\frac{\partial(M_{12}A_1^2)}{A_1\partial\alpha_2}+\frac{\partial(P_{12}C_1A_1^2)}{A_1\partial\alpha_2}-\\&-3C_1R_{13}A_1A_2+A_1A_2Q_{13}+\frac{C_1P_{13}}{R_1}A_1A_2=-[\ddot{u}_1I_1+\ddot{\phi}_1I_2-C_1\ddot{u}_1I_3+(-2C_1\ddot{\phi}_1+C_1\frac{\ddot{u}_1}{R_1}-\\&-\frac{C_1}{A_1}\frac{\partial\ddot{u}_3}{\partial\alpha_1})I_4+C_1^2(-\frac{\ddot{u}_1}{R_1}+\ddot{\phi}_1+\frac{\partial\ddot{u}_3}{A_1\partial\alpha_1})I_6]\end{aligned} \tag{36}$$

$$\begin{aligned}&-\frac{\partial(M_{22}A_1)}{\partial\alpha_2}+\frac{\partial(C_1A_1P_{22})}{\partial\alpha_2}+M_{11}\frac{\partial A_1}{\partial\alpha_2}-C_1P_{11}\frac{\partial A_1}{\partial\alpha_2}-\frac{\partial(M_{12}A_2^2)}{A_2\partial\alpha_1}+\frac{\partial(P_{12}C_1A_2^2)}{A_2\partial\alpha_1}-\\&-3C_1R_{23}A_1A_2+A_1A_2Q_{23}+\frac{C_1P_{23}}{R_2}A_1A_2=-[\ddot{u}_2I_1+\ddot{\phi}_2I_2-C_1\ddot{u}_2I_3+(-2C_1\ddot{\phi}_2+\\&+C_1\frac{\ddot{u}_2}{R_2}-\frac{C_1}{A_2}\frac{\partial\ddot{u}_3}{\partial\alpha_2})I_4+C_1^2(\frac{\ddot{u}_2}{R_2}+\ddot{\phi}_2+\frac{\partial\ddot{u}_3}{A_2\partial\alpha_2})I_6].\end{aligned} \tag{37}$$

For Eqs. (33) – (37) are defining as

$$I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\alpha_3^i d\alpha_3 \tag{38}$$

V. Equations of motion for vibration of cylindrical shell

The curvilinear coordinates and fundamental from parameters for a cylindrical shell are:

$$R_2 = a, \frac{1}{R} = 0, A_2 = a, A_1 = 0, \alpha_3 = \alpha_3, \alpha_2 = \theta, \alpha_1 = x \tag{39}$$

Substituting relationship (39) into Eqs. (33)-(37) the equations of motions for vibration of cylindrical shell with the third-order theory of Reddy are converted to

$$a\frac{\partial N_{11}}{\partial x}+\frac{\partial N_{12}}{\partial\theta}=I_0\ddot{u}_1+(I_1-C_1I_3)\ddot{\phi}_1-C_1I_3\frac{\partial\ddot{u}_3}{\partial x} \tag{40}$$

$$\begin{aligned}&\frac{\partial N_{22}}{\partial\theta}+C_1\frac{\partial P_{12}}{\partial x}+Q_{23}-3C_1R_{23}+C_1P_{23}=(I_0+2\frac{C_1}{a}I_3+\frac{C_1^2}{a^2}I_6)\ddot{u}_2+\\&(I_1-C_1I_3+\frac{C_1}{a}I_4-\frac{C_1^2}{a}I_6)\ddot{\phi}_2-(\frac{C_1}{a}I_3-\frac{C_1^2}{a^2}I_6)\frac{\partial\ddot{u}_3}{\partial\theta}\end{aligned} \tag{41}$$

$$\begin{aligned}&-C_1a\frac{\partial^2P_{11}}{\partial x^2}+N_{22}-\frac{C_1}{a}\frac{\partial^2P_{22}}{\partial\theta^2}-2C_1\frac{\partial^2P_{12}}{\partial x\partial\theta}-a\frac{\partial Q_{13}}{\partial x}+3C_1a\frac{\partial R_{13}}{\partial x}-\frac{\partial Q_{23}}{\partial\theta}+\\&+3C_1\frac{\partial R_{23}}{\partial\theta}-\frac{C_1}{a}\frac{\partial P_{23}}{\partial\theta}=-C_1I_3\frac{\partial u_1}{\partial x}-\frac{C_1}{a}I_3\frac{\partial u_2}{\partial\theta}+(-C_1I_4+C_1^2I_6)\frac{\partial\ddot{\phi}_1}{\partial x}+\\&(-\frac{C_1}{a}I_4+\frac{C_1^2}{a}I_6)\frac{\partial\ddot{\phi}_2}{\partial\theta}-\frac{C_1^2}{a^2}I_6\frac{\partial\ddot{u}_2}{\partial\theta}+C_1^2I_6\frac{\partial^2\ddot{u}_3}{\partial x^2}+\frac{C_1^2}{a}I_6\frac{\partial^2\ddot{u}_3}{\partial\theta^2}-\ddot{u}_3I_0\end{aligned} \tag{42}$$

$$\begin{aligned}&-a\frac{\partial M_{11}}{\partial x}+C_1a\frac{\partial P_{11}}{\partial x}-\frac{\partial M_{12}}{\partial\theta}+C_1\frac{\partial P_{12}}{\partial\theta}-3C_1R_{13}a+aQ_{13}=-I_1\ddot{u}_1+C_1I_3\ddot{u}_1+\\&(-I_2+2C_1I_4-C_1^2I_6)\ddot{\phi}_1+(C_1I_4-C_1^2I_6)\frac{\partial\ddot{u}_3}{\partial x}\end{aligned} \tag{43}$$

$$\begin{aligned}&-\frac{\partial M_{22}}{\partial\theta}-C_1\frac{\partial P_{22}}{\partial\theta}-a\frac{\partial M_{12}}{\partial x}+C_1a\frac{\partial P_{12}}{\partial x}-3C_1R_{23}a+aQ_{23}+C_1R_{23}=(-I_1\\&C_1I_3-\frac{C_1}{a}I_4)\ddot{u}_2+(-I_2+2C_1I_4)\ddot{\phi}_2-\frac{C_1}{a}I_4\frac{\partial\ddot{u}_3}{\partial\theta}\end{aligned} \tag{44}$$

The displacement fields for a FG cylindrical shell and the displacement fields which satisfy these boundary conditions can be written as

$$\begin{aligned}u_1 &= \overline{A}\frac{\partial\phi(x)}{\partial x}\cos(n\theta)\cos(\omega t)\\u_2 &= \overline{B}\phi(x)\sin(n\theta)\cos(\omega t)\\u_3 &= \overline{C}\phi(x)\cos(n\theta)\cos(\omega t)\\\phi_1 &= \overline{D}\frac{\partial\phi(x)}{\partial x}\cos(n\theta)\cos(\omega t)\\\phi_2 &= \overline{E}\phi(x)\sin(n\theta)\cos(\omega t)\end{aligned} \tag{45}$$

where, $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ and $\overline{E}$ are the constants denoting the amplitudes of the vibrations in the x,θ and z directions, ϕ_1 and ϕ_2 are the displacement fields for higher order deformation theories for a cylindrical shell, $\phi(x)$ is the axial function that satisfies the geometric boundary conditions. The axial function $\phi(x)$ is chosen as the beam function as

$$\phi(x)=\gamma_1\cosh(\frac{\lambda_m x}{L})+\gamma_2\cos(\frac{\lambda_m x}{L})-\zeta_m(\gamma_3\sinh(\frac{\lambda_m x}{L})+\gamma_4\sin(\frac{\lambda_m x}{L})) \tag{46}$$

The geometric boundary conditions for free boundary conditions can be expressed mathematically in terms of $\phi(x)$ as:

Free boundary condition

$$\phi''(0)=\phi'''(L)=0 \tag{47}$$

Substituting Eq. (45) into Eqs. (40) - (44) for third order theory we can be expressed

$$\det(C_{ij}-M_{ij}\,\omega^2)=0 \tag{48}$$

Expanding this determinant, a polynomial in even powers of ω is obtained

$$\beta_\circ\omega^{10}+\beta_1\omega^8+\beta_2\omega^6+\beta_3\omega^4+\beta_4\omega^2+\beta_5=\circ \tag{49}$$

where $\beta_i\,(i=0,1,2,3,4,5)$ are some constants. Eq. (49) is solved five positive and five negative roots are obtained. The five positive roots obtained are the natural angular frequencies of the cylindrical shell based third-order theory. The smallest of the five roots is the natural angular frequency studied in the present study.

VI. Results and discussion

To validate the present analysis, results for cylindrical shells are compared with Loy and Lam [15] and with M.R.Isvandzibaei [16]. The comparisons show that the present results agreed well with those in the literature.

TABLE I COMPARISON OF NATURAL FREQUENCY (HZ) FOR A SIMPLY SUPPORTED ISOTROPIC CYLINDRICAL SHELL

$L=20.3cm$, $R=5.08cm$, $h=0.25cm$, $E=2.07788*10^{11}Nm^{-2}$, $\nu=0.31775$
$\rho=8166kgm^{-3}$

n	m	Loy[15]	M.R.Isvandzibaei [16]	Present
2	1	2043.8	2043.6	2045.1
	2	5635.4	5635.2	5624.6
	3	8932.5	8932.1	8821.5
	4	11407.5	11407.2	11437
	5	13253.2	13252.8	13197.5
	6	14790.0	14789.8	14790.6

In this paper, studied are presented for vibration of FG cylindrical shell. Free-Free (F-F) boundary conditions, are considered in the study. Figure 3 shows the variation of the natural frequency with the circumferential wave number n for a FG cylindrical shell with a ring support at $a=0.3L$. The frequencies for free-free boundary conditions increased with the circumferential wave number. This increase in frequencies is most significant when n increased from 1 to 2 and for n greater than 2 the frequencies increase gradually with the circumferential wave number.

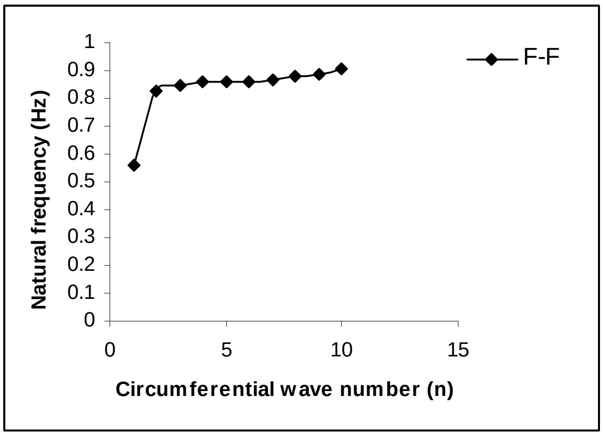

Fig. 3 The natural frequencies (Hz) with circumferential wave number n for a FG cylindrical shell with a ring support

Figure 4 shows the natural frequencies with position of the ring support. For a FG cylindrical shell with ring support with same end-conditions applied in both edges, such F-F boundary conditions, the natural frequencies are the greatest when the ring support is in the middle of the cylindrical shell. The natural frequencies decreased as the ring support moved away from center towards either end of the shell. Thus the natural frequencies curve is symmetrical about the centre of the shell.

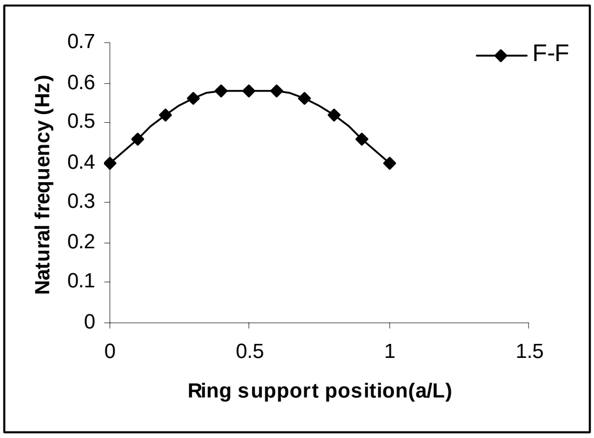

Fig. 4 The natural frequencies (Hz) versus position of the ring support a/L for F-F boundary conditions

Figures 5 show the variation of the natural frequencies cylindrical shell with position of the ring support a/L at different L/R ratios for F-F boundary conditions. From the figure, the influence of the ring support position on the natural frequencies is generally significant at large L/R ratio. It can be seen that boundary conditions have some effects on this influence.

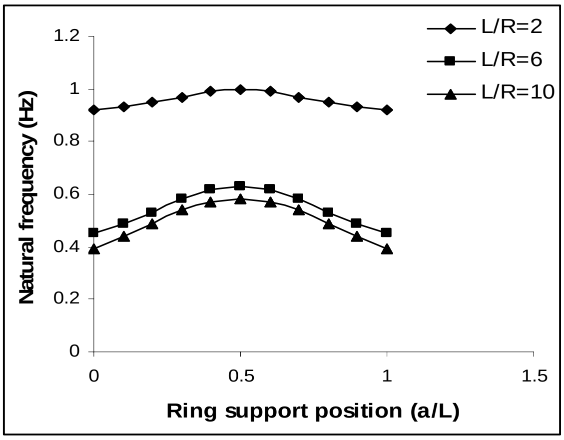

Fig. 5 Variation of the natural frequencies FG cylindrical shell with the position of the ring support a/L at different L/R ratios for F-F boundary conditions (m=1, n=1, h/R=0.01)

Figure 6 show the variation of the natural frequencies FG cylindrical shell with position of the ring support a/L at different h/R ratios for F-F boundary conditions. From the figure it is apparent that the frequencies are higher at larger h/R ratios. The influence of the ring support position is significant at small h/R ratios. The frequencies are also higher at large h/R ratios.

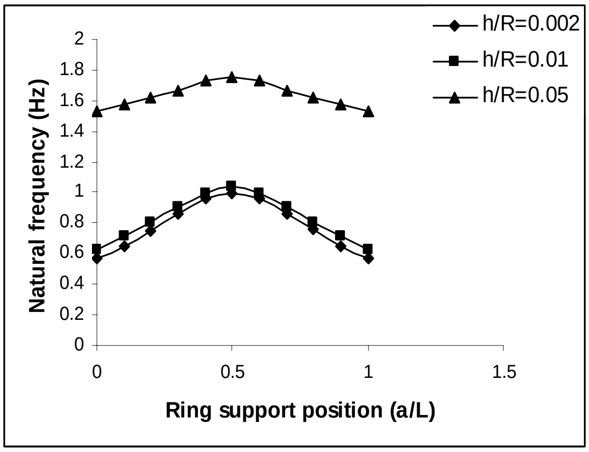

Fig. 6. Variation of the natural frequencies FG cylindrical shell with the position of the ring support a/L at different h/R ratios for F-F boundary conditions. (m=1, n=10, L/R=20)

VII. Conclusions

A study on the vibration of functionally graded (FG) cylindrical shell with a ring support arbitrarily placed along the shell composed of stainless steel and nickel has been presented. Material properties are graded in the thickness direction of the shell according to volume fraction power law distribution. The study is carried out using third shear deformation shell theory with Hamilton's principle. Studies are carried out for cylindrical shells with free–free F–F boundary conditions with an arbitrarily ring support along the axial direction of the cylindrical shell.

Studied were made on the frequency characteristics, the influence of ring support position and the influence of boundary conditions. The study showed that a ring support has significant influence on the frequencies and the extent of this influence depends on the position of the ring support and the boundary conditions of the functionally graded cylindrical shell. However, because of the functionally graded cylindrical shells exhibit interesting frequency characteristics when the constituent volume fractions are varied. This is done by varying the power law exponent N. The study showed that For a functionally graded cylindrical shell with ring support with same end-conditions applied in both edges, such F-F boundary conditions, the natural frequencies are the greatest when the ring support is in the middle of the functionally graded cylindrical shell and natural frequencies decreased as the ring support moved away from center towards either end of the shell, Thus the natural frequencies curve is symmetrical about the centre of the shell, This symmetry of the frequency curve is as expected since the end conditions are symmetrical about the ring support. The present analysis is validated by comparing results with those available in the literature.

References

[1] Arnold, R.N., Warburton, G.B., 1948. Flexural vibrations of the walls of thin cylindrical shells. Proceedings of the Royal Society of London A; 197:238-256.
[2] Ludwig, A., Krieg, R., 1981.An analysis quasi-exact method for calculating eigen vibrations of thin circular shells. J. Sound vibration; 74,155-174.
[3] Chung, H., 1981. Free vibration analysis of circular cylindrical shells. J. Sound vibration; 74, 331-359.
[4] Soedel, W., 1980.A new frequency formula for closed circular cylindrical shells for a large variety of boundary conditions. J. Sound vibration; 70,309-317.
[5] Forsberg, K., 1964. Influence of boundary conditions on modal characteristics of cylindrical shells. AIAA J; 2, 182- 189.
[6] Lam, K.L., Loy, C.T., 1995. Effects of boundary conditions on frequencies characteristics for a multi- layered cylindrical shell. J. Sound vibration; 188, 363-384
[7] Loy, C.T., Lam, K.Y., 1996.Vibration of cylindrical shells with ring support. I.Joumal of Impact Engineering; 1996; 35:455.
[8] Koizumi, M., 1993. The concept of FGM Ceramic Transactions, Functionally Gradient Materials.
[9] Makino A, Araki N, Kitajima H, Ohashi K. Transient temperature response of functionally gradient material subjected to partial, stepwise heating. Transactions of the Japan Society of Mechanical Engineers, Part B 1994; 60:4200-6(1994).
[10] Anon, 1996.FGM components: PM meets the challenge. Metal powder Report. 51:28-32.
[11] Zhang, X.D., Liu, D.Q., Ge, C.C., 1994. Thermal stress analysis of axial symmetry functionally gradient materials under steady temperature field. Journal of Functional Materials; 25:452-5.
[12] Wetherhold, R.C., Seelman, S., Wang, J.Z., 1996. Use of functionally graded materials to eliminate or control thermal deformation. Composites Science and Technology; 56:1099-104.
[13] Najafizadeh, M.M., Hedayati, B. Refined Theory for Thermoelastic Stability of Functionally Graded Circular Plates. Journal of thermal stresses; 27:857-880.

[14] Soedel, W., 1981. Vibration of shells and plates. MARCEL DEKKER, INC, New York.
[15] Loy, C.T., Lam, K.Y., Reddy, J.N., 1999.Vibration of functionally graded cylindrical shells; 41:309-324.
[16] Najafizadeh, M.M., Isvandzibaei, M.R., 2007. Vibration of functionally graded cylindrical shells based on higher order shear deformation plate theory with ring support. Acta Mechanica; 191:75-91.